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ON SECOND ORDER BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

Abstract. We investigate the existence of solutions on a compact interval to second order boundary value problems for a class of functional differential inclusions in Banach spaces. We rely on a fixed point theorem for condensing maps due to Martelli.

1. Introduction. In this paper we shall prove a theorem which assures the existence of solutions defined on a compact real interval for the boundary value problem (BVP for short) of the second order functional differential inclusion

$$y'' \in F(t, y_t), \quad t \in J = [0,1], \tag{1}$$

$$y_0 = \phi, \quad y(1) = \eta, \tag{2}$$

where $F : J \times C(J_0, E) \to 2^E$ (here $J_0 = [-r, 0]$) is a bounded, closed, convex valued multivalued map, $\phi \in C(J_0, E)$, $\eta \in E$, and E is a real Banach space with the norm $|\cdot|$.

For any continuous function y defined on the interval $J_1 = [-r, 1]$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

The method we are going to use is to reduce the existence of solutions to problem (1)–(2) to the search for fixed points of a suitable multivalued map on the Banach space $C(J_1, E)$. In order to prove the existence of fixed

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points, we shall rely on a fixed point theorem for condensing maps due to Martelli [10].

For recent results on BVP for functional differential equations we refer, for instance, to the books of Erbe, Kong and Zhang [4] and Henderson [5], the survey paper of Ntouyas [12], the papers of Nieto, Jiang and Jurang [11], Liz and Nieto [9] and the references cited therein. The methods used are usually the topological transversality of Granas [3] and the monotone iterative method combined with upper and lower solutions [7].

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J_0, E)$ is the Banach space of all continuous functions from J_0 into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

A measurable function $y : J \to E$ is *Bochner integrable* if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [13].)

$L^1(J, E)$ denotes the Banach space of functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)|\, dt \quad \text{for all } y \in L^1(J, E).$$

Finally $W^{2,1}(J, E)$ denotes the Sobolev class of functions $y : J \to E$ such that y' is absolutely continuous and $y'' \in L^1(J, E)$.

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $N : X \to 2^X$ is *convex* (resp. *closed*) *valued* if $N(x)$ is convex (resp. closed) for all $x \in X$. N is *bounded on bounded sets* if $N(B) = \bigcup_{x\in B} N(x)$ is bounded in X for any bounded subset B of X (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

N is called *upper semicontinuous* (u.s.c.) on X if for each $x_* \in X$ the set $N(x_*)$ is a nonempty, closed subset of X, and if for each open subset B of X containing $N(x_*)$, there exists an open neighbourhood V of x_* such that $N(V) \subseteq B$.

N is said to be *completely continuous* if $N(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map N is completely continuous with nonempty compact values, then N is u.s.c. if and only if N has a closed graph (i.e. $x_n \to x_*,\ y_n \to y_*,\ y_n \in N(x_n)$ imply $y_* \in N(x_*)$).

N has a *fixed point* if there is $x \in X$ such that $x \in N(x)$.

In the following BCC(X) denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $N : J \to \mathrm{BCC}(E)$ is said to be *measurable* if for each $x \in E$ the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(x, N(t)) = \inf\{|x - z| : z \in N(t)\}$$

is measurable.

An upper semicontinuous map $N : X \to 2^X$ is said to be *condensing* if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(N(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banaś and Goebel [1].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps and the proof of known results cited in this section we refer to the books of Deimling [2] and Hu and Papageorgiou [6].

Definition 2.1. A multivalued map $F : J \times C(J_0, E) \to E$ is said to be L^1-*Carathéodory* if

(i) $t \mapsto F(t, u)$ is measurable for each $u \in C(J_0, E)$;
(ii) $u \mapsto F(t, u)$ is upper semicontinuous for almost all $t \in J$;
(iii) for each $k > 0$, there exists $m_k \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq m_k(t)$$

for all $\|u\| \leq k$ and for almost all $t \in J$.

Let us introduce the following hypotheses:

(H1) $F : J \times C(J_0, E) \to \mathrm{BCC}(E)$ is an L^1-Carathéodory map and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \{g \in L^1(J, E) : g(t) \in F(t, u) \text{ for a.e. } t \in J\}$$

is nonempty.

(H2) There exists a function $H \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| := \sup\{|v| \in F(t, u)\} \leq H(t)$$

for almost all $t \in J$ and all $u \in C(J_0, E)$.

(H3) For each bounded set $B \subset C(J_1, E)$ and $t \in J$ the set

$$\left\{\phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t, s)g(s)\, ds : g \in S_{F,B}\right\}$$

is relatively compact in E, where $S_{F,B} = \bigcup\{S_{F,y} : y \in B\}$.

Remark 2.1. (i) If $\dim E < \infty$, then $S_{F,u} \neq \emptyset$ for each $u \in C(J_0, E)$ (see Lasota and Opial [8]).

(ii) For each $u \in C(J_0, E)$ the set $S_{F,u}$ is nonempty if and only if $\inf\{|g| : g \in F(t,u)\}$ belongs to $L^1(J, \mathbb{R}_+)$.

(iii) We note that (H3) is trivially satisfied if for each $t \in J$ the multivalued map $F_t : C(J_0, E) \to 2^E : u \mapsto F(t,u)$ is completely continuous or if $\dim E$ is finite.

DEFINITION 2.2. A function $y : J_1 \to E$ is called a *solution* for the BVP (1)–(2) if $y \in C(J_1, E) \cap W^{2,1}(J, E)$ and y satisfies the differential inclusion (1) a.e. on J and the boundary conditions (2).

Our considerations are based on the following lemmas.

LEMMA 2.1 [8]. *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H1) *and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \to \mathrm{BCC}(C(I, X)), \qquad y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}),$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

LEMMA 2.2 [10]. *Let X be a Banach space and $N : X \to \mathrm{BCC}(X)$ be a u.s.c. condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main result. Now, we are able to state and prove our main theorem.

THEOREM 3.1. *Assume that Hypotheses* (H1)–(H3) *hold. Then the BVP* (1)–(2) *has at least one solution on J_1.*

Proof. Let $C(J_1, E)$ be the Banach space provided with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r, 1]\} \quad \text{for } y \in C(J_1, E).$$

We transform the problem into a fixed point problem. Consider the multivalued map $N : C(J_1, E) \to 2^{C(J_1, E)}$ defined by

$$N(y) := \left\{ h \in C(J_1, E) : h(t) = \begin{cases} \phi(t) & \text{if } t \in J_0, \\ \phi(0) + t(\eta - \phi(0)) & \\ + \int_0^1 G(t,s) g(s)\, ds & \text{if } t \in J, \end{cases} \right\}$$

where G is the Green's function for the BVP

$$y''(t) = 0, \qquad y(0) = 0, \qquad y(1) = 0,$$

which is given by the formula

$$G(x,s) = \begin{cases} (1-x)s & \text{if } 0 \le s \le x \le 1, \\ (1-s)x & \text{if } 0 \le x \le s \le 1, \end{cases}$$

and

$$g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

REMARK 3.1. It is clear that the fixed points of N are solutions to problem (1)–(2).

We shall show that N satisfies the assumptions of Lemma 2.2. The proof will be given in several steps.

STEP 1: *$N(y)$ is convex for each $y \in C(J,E)$.* Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = \phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t,s)g_i(s)\,ds, \qquad i = 1,2.$$

Let $0 \le \alpha \le 1$. Then for each $t \in J$ we have

$$(\alpha h_1 + (1-\alpha)h_2)(t) = \phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t,s)[\alpha g_1(s) + (1-\alpha)g_2(s)]\,ds.$$

Since $S_{F,y}$ is convex (because F has convex values) we see that

$$\alpha h_1 + (1-\alpha)h_2 \in N(y).$$

STEP 2: *N is bounded on bounded sets of $C(J,E)$.* Indeed, it is enough to show that there exists a positive constant c such that for each $h \in N(y)$ with $y \in B_q = \{y \in C(J,E) : \|y\|_\infty \le q\}$ one has $\|h\|_\infty \le c$.

If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t,s)g(s)\,ds.$$

By (H1) for each $t \in J$ we have

$$\begin{aligned}|h(t)| &\le |2\phi(0)| + |\eta| + \int_0^1 \|G(t,s)g(s)\|\,ds\\ &\le 2\|\phi\| + |\eta| + \int_0^1 |G(t,s)|m_q(s)\,ds.\end{aligned}$$

Thus

$$\|h\|_\infty \le 2\|\phi\| + |\eta| + \sup_{t\in[0,1]} \Big(\int_0^1 G(t,s)m_q(s)\,ds\Big) =: c.$$

STEP 3: *N sends bounded sets of $C(J,E)$ into equicontinuous sets.* Let $t_1, t_2 \in J$, $t_1 < t_2$ and B_q be a bounded set of $C(J_1,E)$. For each $y \in B_q$

and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = \phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t,s)g(s)ds, \qquad t \in J.$$

Thus we obtain

$$|h(t_2) - h(t_1)| \leq (t_2 - t_1)|\eta - \phi(0)| + \int_0^1 |G(t_2, s) - G(t_1, s)| m_q(s)\, ds.$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ follows from the uniform continuity of ϕ on the interval J_0 and from the relation

$$|h(t_2) - h(t_1)| = |h(t_2) - \phi(t_1)| \leq |h(t_2) - h(0)| + |\phi(0) - \phi(t_1)|$$

respectively.

As a consequence of Step 2, Step 3 and (H3) together with the Ascoli–Arzelà theorem we can conclude that N is completely continuous, and therefore a condensing map.

STEP 4: *N has a closed graph.* Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We shall prove that $h_* \in N(y_*)$. Now, $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = \phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t,s)g_n(s)\, ds, \qquad t \in J.$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$h_*(t) = \phi(0) + t(\eta - \phi(0)) + \int_0^1 G(t,s)g_*(s)\, ds, \qquad t \in J.$$

Clearly we have

$$\|(h_n - (\phi(0) + t(\eta - \phi(0)))) - (h_* - (\phi(0) + t(\eta - \phi(0))))\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Now, we consider the linear continuous operator

$$\Gamma : L^1(J, E) \to C(J, E), \qquad g \mapsto \Gamma(g)(t) = \int_0^1 G(t,s)g(s)\, ds.$$

From Lemma 2.1, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have

$$h_n(t) - (\phi(0) + t(\eta - \phi(0))) \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.1 that

$$h_*(t) - (\phi(0) + t(\eta - \phi(0))) = \int_0^1 G(t,s)g_*(s)\, ds$$

for some $g_* \in S_{F,y_*}$.

STEP 5: *The set*

$$\Omega := \{y \in C(J_1, E) : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}\phi(0) + \lambda^{-1}t(\eta - \phi(0)) + \lambda^{-1}\int_0^1 G(t,s)g(s)\,ds, \qquad t \in J.$$

This implies by (H2) that for each $t \in J$ we have

$$|y(t)| \le 2\|\phi\| + |\eta| + \int_0^1 |G(t,s)|H(s)\,ds.$$

Thus

$$\|y\|_\infty \le 2\|\phi\| + |\eta| + \sup_{(t,s)\in J\times J} |G(t,s)| \int_0^1 H(s)\,ds.$$

This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.2 we deduce that N has a fixed point which is a solution of (1)–(2) on J_1. ∎

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