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STABILITY OF CONSTANT SOLUTIONS TO THE NAVIER–STOKES SYSTEM IN $\mathbb{R}^3$

Abstract. The paper examines the initial value problem for the Navier–Stokes system of viscous incompressible fluids in the three-dimensional space. We prove stability of regular solutions which tend to constant flows sufficiently fast. We show that a perturbation of a regular solution is bounded in $W_r^{2,1}(\mathbb{R}^3 \times [k, k+1])$ for $k \in \mathbb{N}$. The result is obtained under the assumption of smallness of the L_2-norm of the perturbing initial data. We do not assume smallness of the $W_r^{2-2/r}(\mathbb{R}^3)$-norm of the perturbing initial data or smallness of the L_r-norm of the perturbing force.

Introduction. We consider the initial value problem for the Navier–Stokes system in $\mathbb{R}^3$. We show that a class of solutions which tend to constant flows is stable under perturbations of the initial data and of the external force. We restrict our attention to the case when the perturbing force is potential. In the proof we apply an L_r-estimate for the Stokes system; together with global-in-time energy estimates, this makes it possible to control the $W_r^{2,1}$-norm of the solutions in time. Similar methods have been used in [9].

The initial value problem for the Navier–Stokes system in $\mathbb{R}^2$ has been solved for regular data [5, 10]. In three space dimensions global-in-time existence of weak solutions is proved [3]. The problem of regularity is still open. Applying the theory of semigroups a unique regular solution for small data has been obtained [4]. A stability result for special solutions in the whole space has been proved in [1, 7].

We examine the motion of a viscous incompressible fluid described by the Navier–Stokes system in $\mathbb{R}^3$

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$$
(1)\qquad \begin{aligned}
&v_t + (v\nabla)v - \nu\Delta v + \nabla\widetilde{q} = h + f,\\
&\operatorname{div} v = 0,\\
&v|_{t=0} = w_0 + u_0,
\end{aligned}
$$

where $v = (v_1, v_2, v_3)$ is the velocity, $\widetilde{q}$ is the pressure, $h + f$ the external force and ν the constant positive viscosity coefficient.

Problem (1) is treated as a perturbation of the system

$$
(2)\qquad \begin{aligned}
&w_t + (w\nabla)w - \nu\Delta w + \nabla q = h,\\
&\operatorname{div} w = 0,\\
&w|_{t=0} = w_0.
\end{aligned}
$$

Put $u = v - w$ and $p = \widetilde{q} - q$. Then from (1) and (2) we obtain

$$
(3)\qquad \begin{aligned}
&u_t - \nu\Delta u + \nabla p = f - w\nabla u - u\nabla u - u\nabla w,\\
&\operatorname{div} u = 0,\\
&u|_{t=0} = u_0.
\end{aligned}
$$

We assume that system (2) has a unique global-in-time regular solution such that $w \in W^{1,0}_\infty(\mathbb{R}^3 \times [0,\infty))$, and moreover

$$
w = w_1 + w_2,
$$

where

$$
(4)\qquad \|\nabla w_1(\cdot,t)\|_{L_\infty(\mathbb{R}^3)} \in L_1(0,\infty), \qquad \|w_2(\cdot,t)\|_{L_\infty(\mathbb{R}^3)} \in L_2(0,\infty).
$$

Assumption (4) shows that w tends to a constant flow, but also we see that (4) gives no condition on integrability in time or smallness for $\|w_1(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}$ or $\|\nabla w_2(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}$.

The main result of the paper is the following theorem.

THEOREM. *Let* $r \ge 2$, $f = \nabla\varphi \in L_{r(\mathrm{loc})}(\mathbb{R}^3 \times (0,\infty))$, $u_0 \in W^{2-2/r}_r(\mathbb{R}^3) \cap L_2(\mathbb{R}^3)$, $\operatorname{div} u_0 = 0$ *and*

$$
\sup_{k\in\mathbb{N}} \|f\|_{L_r(\mathbb{R}^3\times(k,k+1))} + \|u_0\|_{W^{2-2/r}_r(\mathbb{R}^3)} + \|u_0\|_{L_2(\mathbb{R}^3)} \le M_0,
$$
$$
\|u_0\|_{L_2(\mathbb{R}^3)} \le \delta.
$$

If $\delta \le \delta_0(M_0)$, *where* $\delta_0(M_0)$ *tends to zero as* $M_0 \to \infty$, *then a perturbed solution* $(v, \widetilde{q})$ *to problem* (1) *exists globally in time and*

$$
(5)\qquad \|v - w\|_{W^{2,1}_r(\mathbb{R}^3\times[k,k+1])} + \|\nabla\widetilde{q} - \nabla q\|_{L_r(\mathbb{R}^3\times[k,k+1])} \le K(M_0)
$$

for $k \in \mathbb{N}$, *where* $K(M_0)$ *is a function independent of* k *and tends to zero as* $M_0 \to 0$. *Moreover*,

$$
(6)\qquad \begin{aligned}
\|v - w\|_{W^{2,1}_r(\mathbb{R}^3\times[k,k+1])} &+ \|\nabla\widetilde{q} - \nabla q\|_{L_r(\mathbb{R}^3\times[k,k+1])}\\
&\le c(\|u_0\|_{L_2(\mathbb{R}^3)} + \|f\|_{L_r(\mathbb{R}^3\times[k-1,k+1])})
\end{aligned}
$$

for $k \in \mathbb{N} \setminus \{0\}$.

REMARK. It is easily seen that we can reduce problem (3) to the case of $f \equiv 0$. Since $f = \nabla\varphi$ it is possible to make the transformation $p \mapsto p - \varphi$. But we will not use it in our considerations.

NOTATION. We will need the anisotropic Sobolev spaces $W_r^{m,n}(Q_T)$ where $m, n \in \mathbb{R}_+ \cup \{0\}$, $r \ge 1$ and $Q_T = Q \times (0, T)$ with the norm

$$\begin{aligned}\|u\|^r_{W_r^{m,n}(Q_T)} = & \int_0^T\int_Q |u(x,t)|^r\,dx\,dt \\ & + \sum_{0\le|m'|\le[|m|]} \int_0^T\int_Q |D_x^{m'} u(x,t)|^r\,dx\,dt \\ & + \sum_{|m'|=[|m|]} \int_0^T dt \int_Q\int_Q \frac{|D_x^{m'}u(x,t) - D_x^{m'}u(x',t)|^r}{|x-x'|^{s+r(|m|-[|m|])}}\,dx\,dx' \\ & + \sum_{0\le|n'|\le[|n|]} \int_0^T\int_Q |D_t^{n'} u(x,t)|^r\,dx\,dt \\ & + \int_Q dx \int_0^T\int_0^T \frac{|D_t^{[n]}u(x,t) - D_t^{[n]}(x,t')|^r}{|t-t'|^{1+r(n-[n])}}\,dt\,dt',\end{aligned}$$

where $s = \dim Q$, $[\alpha]$ is the integral part of α, and $D_x^l = \partial_{x_1}^{l_1}\ldots\partial_{x_s}^{l_s}$, where $l = (l_1, \ldots, l_s)$ is a multiindex.

For these spaces we have the following relations (see [2]). Let $u \in W_r^{m,n}(\Omega_T)$. If

$$\sum_{i=1}^3\left(\alpha_i + \frac{1}{r} - \frac{1}{q}\right)\frac{1}{m} + \left(\beta + \frac{1}{r} - \frac{1}{q}\right)\frac{1}{n} < 1$$

then

$$\|D_t^\beta D_x^\alpha u\|_{L_q(\Omega_T)} \le \varepsilon\|u\|_{W_r^{m,n}(\Omega_T)} + c(\varepsilon)\|u\|_{L_2(\Omega_T)}, \tag{7}$$

where $q \ge r \ge 2$ and $\varepsilon \in (0,1)$ and $c(\varepsilon) \to \infty$ with $\varepsilon \to 0$.

We use well known results such as the imbedding or trace theorems for Sobolev spaces. All constants are denoted by c. By $A, B, C, \ldots$ we denote constants which are fixed in each proof.

The main tool used in the proof is an estimate for solutions to the Stokes system

$$\begin{aligned} & u_t - \nu\Delta u + \nabla p = f, \\ & \operatorname{div} u = 0, \\ & u|_{t=0} = u_0. \end{aligned} \tag{8}$$

Lemma 1. *Let* $f \in L_r(\mathbb{R}^3 \times (0,\infty))$, $u_0 \in W_r^{2-2/r}(\mathbb{R}^3)$ *and* $\operatorname{div} u_0 = 0$. *Then there exists a unique solution of* (8) *such that* $u \in W_r^{2,1}(\mathbb{R}^3 \times (0,T))$, $p \in W_r^{1,0}(\mathbb{R}^3 \times (0,T))$ *and*

$$\|u\|_{W_r^{2,1}(\mathbb{R}^3\times(0,T))} + \|\nabla p\|_{L_r(\mathbb{R}^3\times(0,T))} \le C(T)(\|f\|_{L_r(\mathbb{R}^3\times(0,T))} + \|u_0\|_{W_r^{2-2/r}(\mathbb{R}^3)}), \tag{9}$$

where $C(T)$ *is an increasing positive function.*

The proof can be found in [6] or in [8].

Next we prove local existence of solutions to problem (3).

Lemma 2. *Let* $f \in L_r(\mathbb{R}^3 \times (0,T))$, $u_0 \in W_r^{2-2/r}(\mathbb{R}^3)$ *and* $\operatorname{div} u_0 = 0$. *Then there exists* $T_0 > 0$ *such that for all* $T \le T_0$ *system* (3) *has a unique solution such that* $u \in W_r^{2,1}(\mathbb{R}^3 \times (0,T))$, $p \in W_{r(\mathrm{loc})}^{1,0}(\mathbb{R}^3 \times (0,T))$ *and*

$$\|u\|_{W_r^{2,1}(\mathbb{R}^3\times(0,T))} + \|\nabla p\|_{L_r(\mathbb{R}^3\times(0,T))} \le C(T)(\|f\|_{L_r(\mathbb{R}^3\times(0,T))} + \|u_0\|_{W_r^{2-2/r}(\mathbb{R}^3)}). \tag{10}$$

Proof. We construct a sequence $\{u_m, p_m\}_{m=1}^{\infty}$ of approximations defined by

$$\begin{aligned} &u_{m,t} - \nu\Delta u_m + \nabla p_m = -u_{m-1}\nabla u_{m-1} + f - w\nabla u_{m-1} - u_{m-1}\nabla w,\\ &\operatorname{div} u_m = 0,\\ &u_m|_{t=0} = u_0, \end{aligned} \tag{11}$$

where $u_1 = 0$ and $p_1 = 0$.

Lemma 1 gives the following estimate for the solution of (11):

$$\begin{aligned} &\|u_m\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + \|\nabla p\|_{L_r(\mathbb{R}^3\times[0,T])}\\ &\quad \le A(\|f\|_{L_r(\mathbb{R}^3\times[0,T])} + \|u_0\|_{W_r^{2-2/r}(\Omega)} + \|u_{m-1}\nabla u_m\|_{L_r(\mathbb{R}^3\times[0,T])}\\ &\qquad + \|w\nabla u_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])} + \|u_{m-1}\nabla w\|_{L_r(\mathbb{R}^3\times[0,T])}). \end{aligned} \tag{12}$$

Since $r \ge 3$, we have the imbeddings $W_r^{2-2/r}(\mathbb{R}^3) \subset L_{3r}(\mathbb{R}^3)$ and $W_r^{1-2/r}(\mathbb{R}^3) \subset L_{(3/2)r}(\mathbb{R}^3)$ with the estimate

$$\begin{aligned} \sup_{t\le T} \|\nabla u_{m-1}(\cdot,t)\|_{L_{(3/2)r}(\mathbb{R}^3)} + \sup_{t\le T}\|u(\cdot,t)\|_{L_{3r}(\mathbb{R}^3)} &\le \sup_{t\le T}\|u(\cdot,t)\|_{W_r^{2-2/r}(\mathbb{R}^3)}\\ &\le \overline{\alpha}(\|u_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + \|u_0\|_{W_r^{2-2/r}(\mathbb{R}^3)}), \end{aligned} \tag{13}$$

where $\overline{\alpha}$ does not depend on T. By (13) and the Hölder inequality with $1/r = 1/(3r) + 2/(3r)$, we get

$$
\begin{aligned}
(14)\qquad &\|u_{m-1}\nabla u_m\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\qquad + \|w\nabla u_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])} + \|u_{m-1}\nabla w\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\le BT^{1/r}(\|u_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + \|u_{m-1}\|^2_{W_r^{2,1}(\mathbb{R}^3\times[0,T])}\\
&\qquad + \|u_0\|_{W_r^{2-2/r}(\mathbb{R}^3)} + \|u_0\|^2_{W_r^{2-2/r}(\mathbb{R}^3)}).
\end{aligned}
$$

If $2 \le r < 3$ to obtain (14) we have to use the parabolic imbedding.

Let $T \le 1$ and assume that

$$
\begin{aligned}
(15)\qquad &\|u_k\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + \|\nabla p_k\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\qquad \le 4A(\|f\|_{L_r(\mathbb{R}^3\times[0,T])} + \|u_0\|_{W_r^{2-2/r}(\mathbb{R}^3)}) \equiv M
\end{aligned}
$$

for $k = 1, \ldots, m-1$.

By (12) and (14) we see that we can choose T so small that (15) is satisfied for $k = m$. Since $u_1 = 0$ and $p_1 = 0$, by induction we obtain (15) for all $k \in \mathbb{N}$.

Next we prove convergence of the sequence.

From (11) we get the following system for $U_m = u_m - u_{m-1}$ and $P_m = p_m - p_{m-1}$:

$$
(16)\qquad
\begin{aligned}
&U_{m,t} - \nu\Delta U_m + \nabla P_m\\
&\qquad = -w\nabla U_{m-1} - U_{m-1}\nabla w - u_{m-1}\nabla U_m - U_{m-1}\nabla u_{m-1},\\
&\operatorname{div} U_m = 0,\\
&U_m|_{t=0} = 0.
\end{aligned}
$$

By Lemma 1 we obtain the estimate

$$
\begin{aligned}
(17)\qquad &\|U_m\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + \|\nabla P_m\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\le A(\|u_{m-1}\nabla U_m\|_{L_r(\mathbb{R}^3\times[0,T])} + \|U_{m-1}\nabla u_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\qquad + \|w\nabla U_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])} + \|U_{m-1}\nabla w\|_{L_r(\mathbb{R}^3\times[0,T])}).
\end{aligned}
$$

By the same argument as in (14) we have

$$
\begin{aligned}
(18)\qquad &\|w\nabla U_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])} + \|U_{m-1}\nabla w\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\qquad \le cT^{1/r}\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])}.
\end{aligned}
$$

By (7) with $r \ge 2$ we have $W_r^{2,1}(\mathbb{R}^3\times[0,T]) \subset\subset L_{3r}(\mathbb{R}^3\times[0,T])$ and $D_xW_r^{2,1}(\mathbb{R}^3\times[0,T]) \subset\subset L_{3r/2}(\mathbb{R}^3\times[0,T])$, hence applying the Hölder inequality ($1/r = 1/(3r) + 2/(3r)$) we get

$$
\begin{aligned}
(19)\qquad &\|U_{m-1}\nabla u_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])} + \|u_{m-1}\nabla U_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])}\\
&\le c(\varepsilon\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + c(\varepsilon)\|U_{m-1}\|_{L_r(\mathbb{R}^3\times[0,T])})\\
&\le c(\varepsilon\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + c(\varepsilon)T^{1/r}\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])})
\end{aligned}
$$

by (13). From (17)–(19) we get

$$(20)\qquad \|U_m\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])}+\|\nabla P_m\|_{L_r(\mathbb{R}^3\times[0,T])} \le C(\varepsilon\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])}+c(\varepsilon)T^{1/r}\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])}).$$

Taking ε and T small enough, we conclude that

$$\|U_m\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])}\le \tfrac{1}{2}\|U_{m-1}\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])},$$

which gives

$$U_m\to 0 \text{ in } W_r^{2,1}(\mathbb{R}^3\times[0,T]) \quad\text{and}\quad \nabla P_m\to 0 \text{ in } L_r(\mathbb{R}^3\times[0,T]).$$

Thus $\{(u_m,p_m)\}_{m=1}^{\infty}$ is convergent to a solution of (3), and (10) comes from (15). The proof of Lemma 2 is complete.

The next result enables us to control in time the L_2-norm of solutions only by the L_2-norm of initial data.

LEMMA 3. *The solution of problem* (3) *on* $[0,T]$ *satisfies the estimate*

$$\|u\|_{L_\infty(0,T;L_2(\mathbb{R}^3))}\le \bar{A}\|u_0\|_{L_2(\mathbb{R}^3)},$$

where $\bar{A}$ *does not depend on* T.

Proof. Multiplying $(3)_1$ by u and integrating over $\mathbb{R}^3$ we get

$$(21)\qquad \frac{1}{2}\frac{d}{dt}\int u^2\,dx+\nu\int|\nabla u|^2\,dx=-\int\nabla q\cdot u+\int f\cdot u\,dx -\int(u\nabla)w\cdot u\,dx-\int(u\nabla)u\cdot u\,dx-\int(w\nabla)u\cdot u\,dx.$$

The first, fourth and fifth terms vanish by $(2)_2$ and $(3)_2$. The second term vanishes by the assumption that f is potential ($f=\nabla\varphi$). Thus, since $w=w_1+w_2$ (see (4)), we obtain

$$\frac{d}{dt}\int u^2\,dx+\nu\int|\nabla u|^2\,dx\le A_1\|\nabla w_1\|_{L_\infty(\mathbb{R}^3)}\int u^2\,dx+\int u\nabla w_2\cdot u\,dx,$$

but

$$\Big|\int u\nabla w_2\cdot u\,dx\Big|=\Big|\int u\nabla u\cdot w_2\,dx\Big|\le \nu\int|\nabla u|^2\,dx+A_2\|w_2(\cdot,t)\|^2_{L_\infty(\mathbb{R}^3)}\int u^2\,dx.$$

Hence we get

$$(22)\qquad \frac{d}{dt}\int u^2\,dx\le (A_1\|\nabla w_1(\cdot,t)\|_{L_\infty(\mathbb{R}^3)}+A_2\|w_2(\cdot,t)\|^2_{L_\infty(\mathbb{R}^3)})\int u^2\,dx.$$

By the Gronwall inequality we obtain

$$\|u(\cdot,t)\|^2_{L_2(\mathbb{R}^3)} \le \exp\Big\{\int_0^t(A_1\|\nabla w_1(\cdot,s)\|_{L_\infty(\mathbb{R}^3)}+A_2\|w_2(\cdot,s)\|^2_{L_\infty(\mathbb{R}^3)})\,ds\Big\}\|u_0\|^2_{L_2(\mathbb{R}^3)}.$$

By assumption (4) we get the assertion of the lemma.

LEMMA 4. *The solution of problem* (3) *on* $[0,T]$ *satisfies the estimate*

$$\|u\|_{W_r^{2,1}(O_k)} + \|\nabla p\|_{L_r(O_k)} \le C(\|f\|_{L_r(O_{k-1}\cup O_k)} + \|u_0\|_{L_2(\mathbb{R}^3)} + \|u_0\|_{W_r^{2-2/r}(\mathbb{R}^3)}), \tag{23}$$

where $O_k = \mathbb{R}^3 \times [kQ,(k+1)Q]$, $Q \le T$ *and* C *is independent of* T, *if* $\|u_0\|_{L_2(\mathbb{R}^3)}$ *is small enough.*

Proof. We introduce a smooth function $\zeta_k : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\zeta_k(t) = \begin{cases} 1 & \text{for } t \ge kQ, \\ 0 & \text{for } t \le (k-1)Q, \end{cases}$$

for $k \in \mathbb{N} \setminus \{0\}$, $0 \le \zeta \le 1$, $\zeta' \ge 0$ and $|\zeta'| \le 2/Q$.

Multiplying (3) by ζ_k and setting $U^k = \zeta_k u$, $P^k = \zeta_k p$, we get

$$\begin{aligned} &U_t^k - \nu\Delta U^k - \nabla P^k = \zeta_k f - U^k\nabla w - w\nabla U^k - u\nabla U^k + \zeta_k' u, \\ &\operatorname{div} U^k = 0, \\ &U^k|_{t=(k-1)Q} = 0. \end{aligned} \tag{24}$$

Applying Lemma 1 with $T = 2Q$ we obtain

$$\begin{aligned} &\|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} + \|\nabla P^k\|_{L_r(O_{k-1}\cup O_k)} \\ &\le C(2Q)(\|f\|_{L_r(O_{k-1}\cup O_k)} + \|U^k\nabla w\|_{L_r(O_{k-1}\cup O_k)} \\ &\quad + \|w\nabla U^k\|_{L_r(O_{k-1}\cup O_k)} + \|u\nabla U^k\|_{L_r(O_{k-1}\cup O_k)} + \|\zeta_k' u\|_{L_r(O_{k-1}\cup O_k)}). \end{aligned} \tag{25}$$

Using (7) we estimate the unknown terms of the r.h.s. of (25):

$$\begin{aligned} &\|U^k\nabla w\|_{L_r(O_{k-1}\cup O_k)} \\ &\qquad \le \|\nabla w\|_{L_\infty}(\varepsilon\|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon)\|U^k\|_{L_2(O_{k-1}\cup O_k)}), \\ &\|w\nabla U^k\|_{L_r(O_{k-1}\cup O_k)} \\ &\qquad \le \|w\|_{L_\infty}(\varepsilon\|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} + c(\varepsilon)\|U^k\|_{L_2(O_{k-1}\cup O_k)}), \\ &\|u\nabla U^k\|_{L_r(O_{k-1}\cup O_k)} \le \|U^{k-1}\|_{L_{3r}(O_{k-1})}(\varepsilon\|U^k\|_{W_r^{2,1}(O_{k-1}\cup O_k)} \\ &\qquad + c(\varepsilon)\|U^k\|_{L_2(O_{k-1}\cup O_k)}) + \|U^k\nabla U^k\|_{L_r(O_k)}, \\ &\|\zeta_k' u\|_{L_r(O_{k-1}\cup O_k)} \le \varepsilon|\zeta_k'|\cdot\|U^{k-1}\|_{W_r^{2,1}(O_{k-1})} + |\zeta_k'|c(\varepsilon)\|u\|_{L_2(O_{k-1})}. \end{aligned} \tag{26}$$

Assumption (4) gives an estimate on $\|\nabla w\|_{L_\infty(\mathbb{R}^3\times[0,T])}$ and $\|w\|_{L_\infty(\mathbb{R}^3\times[0,T])}$. Choosing sufficiently small ε and applying (26) to (25), we obtain

$$\begin{aligned} &\|U^k\|_{W_r^{2,1}(O_k)} + \|\nabla P^k\|_{L_r(O_k)} \\ &\quad \le A(\|U^k\nabla U^k\|^2_{L_r(O_k)} + \|f\|_{L_r(O_{k-1}\cup O_k)} + \varepsilon|\zeta_k'|\cdot\|U^{k-1}\|_{W_r^{2,1}(O_{k-1})} \\ &\qquad + (c(\varepsilon)\|U^{k-1}\|_{W_r^{2,1}(O_{k-1})}|\zeta_k'| + c(\varepsilon)W)\|u\|_{L_2(O_{k-1}\cup O_k)}), \end{aligned} \tag{27}$$

where $W = \|\nabla w\|_{L_\infty(\mathbb{R}^3\times[0,T])} + \|w\|_{L_\infty(\mathbb{R}^3\times[0,T])}$.

By the same argument as in (19) we obtain

$$\|U^k\nabla U^k\|_{L_r(O_k)} \le \|U^k\|_{L_{3r}}\|\nabla U^k\|_{L_{(3/2)r}(O_k)} \le \sigma\|U^k\|^2_{W^{2,1}_r(O_k)} + c(\sigma)\|u\|^2_{L_\infty((k-1)Q,kQ;L_2(\mathbb{R}^3))}. \tag{28}$$

Let

$$X_m = \|U^m\|_{W^{2,1}_r(O_m)} + \|\nabla P^m\|_{L_r(O_m)}, \qquad F = \sup_m B\|f\|_{L_r(O_{m-1}\cup O_m)}.$$

We prove estimate (23) by induction. For $k=0$ the functions U^0 and P^0 are defined as the local solution given by Lemma 2. We assume that $X_{k-1} \le M$, where M is so large that $M \ge 16F$.

By (27) and (28) we get

$$X_k \le F + B\varepsilon M + c(\varepsilon)(M+W)\|u\|_{L_2(O_{k-1}\cup O_k)} + c(\sigma)\|u\|^2_{L_\infty((k-1)Q,kQ;L_2(\mathbb{R}^3))} + B\sigma X_k^2 \equiv \alpha + \beta X_k^2. \tag{29}$$

To obtain a uniform estimate for X_k we need two relations:

$$1 - 4\alpha\beta > 0 \quad \text{and} \quad \frac{1-\sqrt{1-4\alpha\beta}}{2\beta} \le M, \tag{30}$$

which give $X_k \le M$. Taking σ, ε and $\|u_0\|_{L_2(\mathbb{R}^3)}$ sufficiently small, by Lemma 3, we get the first condition of (30). To obtain the second one we note that $(1-\sqrt{1-4\alpha\beta})/(2\beta) \le 2\alpha$. So we have to prove that $2\alpha \le M$; but this is satisfied, because

$$F \le \frac{1}{16}M, \qquad c(\sigma)\|u\|^2_{L_\infty((k-1)Q,kQ;L_2(\mathbb{R}^3))} \le \frac{1}{4}M,$$
$$B\varepsilon \le \frac{1}{16}, \qquad c(\varepsilon)(M+W)\|u\|_{L_2(O_{k-1}\cup O_k)} \le \frac{1}{8}M,$$

assuming that $\|u_0\|_{L_2(\mathbb{R}^3)}$ is small enough (see Lemma 3). This way we get $X_k \le M$. Lemma 4 has been proved.

Lemma 4 gives estimates which enable one to continue the solution using the local existence result from Lemma 2. Thus we get the first part of the Theorem concerning the existence of global-in-time solutions. Inequality (5) comes from (23).

To finish the proof of the Theorem we need the following lemma.

LEMMA 5. *Solutions of* (3) *satisfy the estimate*

$$\|v-w\|_{W^{2,1}_r(\mathbb{R}^3\times[k,k+1])} + \|\nabla\widetilde{q} - \nabla q\|_{L_r(\mathbb{R}^3\times[k,k+1])} \le c(\|u_0\|_{L_2(\mathbb{R}^3)} + \|f\|_{L_r(\mathbb{R}^3\times[k,k+1])})$$

for $k \in \mathbb{N}\setminus\{0\}$.

Proof. We take a smooth function $\eta_l : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\eta_l(t) = \begin{cases} 1 & \text{for } t \geq t_0 + l, \\ 0 & \text{for } t \leq t_0 + l/2, \end{cases}$$

and $0 \leq \eta_l \leq 1$, $|D\eta_l| \leq 3/l$, $l \in (0,1]$.

Now we repeat the considerations from the proof of Lemma 4. We multiply (3) by η_l, and applying Lemma 1 we find estimates on the functions $U_l = \eta_l u$ and $P_l = \eta_l p$:

$$\begin{aligned} \|U_l\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} &+ \|\nabla P_l\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} \\ &\leq A\big(\|f\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} + B\|u_0\|_{L_2(\mathbb{R}^3)} + \tfrac{3}{l}\|U_{l/2}\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])}\big). \end{aligned} \tag{31}$$

To obtain (31) we repeat all steps ((25)–(27)) of the considerations for (24) omitting $(26)_3$.

From [2, Chap. 18] we have the interpolation inequality

$$\|u\|_{L_r(\mathbb{R}^3)} \leq \varepsilon\|u\|_{W_r^2(\mathbb{R}^3)} + c\varepsilon^{-(3/2)(1/2-1/r)}\|u\|_{L_2(\mathbb{R}^3)}, \tag{32}$$

where $\varepsilon \in (0,1)$. From (32) and Lemma 3 we get

$$\begin{aligned} \tfrac{3}{l}\|U_{l/2}\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} &\leq \varepsilon_0\|U_{l/2}\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} \\ &\quad + Cl^{-1}(\varepsilon_0 l)^{-(3/2)(1/2-1/r)}\|u\|_{L_\infty(t_0,t_0+2;L_2(\mathbb{R}^3))}. \end{aligned} \tag{33}$$

Using (33) in (31) we get

$$\begin{aligned} \|U_l\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} &\leq \varepsilon_0\|U_{l/2}\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} \\ &\quad + Dl^{-1}(\varepsilon_0 l)^{-(3/2)(1/2-1/r)}\|u_0\|_{L_2(\mathbb{R}^3)} \\ &\quad + A(\|f\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} + B\|u_0\|_{L_2(\mathbb{R}^3)}). \end{aligned} \tag{34}$$

Putting $l = 1$ and repeating the same method as for (34) we conclude that

$$\begin{aligned} \|U_1\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} &+ \|\nabla P_1\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} \\ &\leq \varepsilon_0^k\|U_{1/2^k}\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} \\ &\quad + E\|u_0\|_{L_2(\mathbb{R}^3)}(1 + \varepsilon_0 2^{7/4-3/(2r)} + \ldots + (\varepsilon_0 2^{7/4-3/(2r)})^{k-1}) \\ &\quad + A(\|f\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} + B\|u_0\|_{L_2(\mathbb{R}^3)})(1 + \varepsilon_0 + \ldots + \varepsilon_0^{k-1}). \end{aligned} \tag{35}$$

Taking ε so small that $\varepsilon_0 \leq 2^{-(7/4-3/(2r))}$ and letting $k \to \infty$ in (35), we get

$$\begin{aligned} \|U_1\|_{W_r^{2,1}(\mathbb{R}^3\times[t_0,t_0+2])} &+ \|\nabla P_1\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} \\ &\leq c(\|f\|_{L_r(\mathbb{R}^3\times[t_0,t_0+2])} + \|u_0\|_{L_2(\mathbb{R}^3)}). \end{aligned} \tag{36}$$

Inequality (36) gives (6). The proof of the Theorem is finished.

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