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V. Volpert (Lyon)
B. Kaźmierczak (Warszawa)
M. Massot (Lyon)
Z. Peradzyński (Warszawa)

SOLVABILITY CONDITIONS FOR ELLIPTIC PROBLEMS WITH NON-FREDHOLM OPERATORS

Abstract. The paper is devoted to solvability conditions for linear elliptic problems with non-Fredholm operators. We show that the operator becomes normally solvable with a finite-dimensional kernel on properly chosen subspaces. In the particular case of a scalar equation we obtain necessary and sufficient solvability conditions. These results are used to apply the implicit function theorem for a nonlinear elliptic problem; we demonstrate the persistence of travelling wave solutions to spatially periodic perturbations.

1. Introduction. Consider the elliptic operator

$$Lu = a(x)\Delta u + \sum_{j=1}^{n} b_j(x)\,\frac{\partial u}{\partial x_j} + c(x)u \tag{1.1}$$

acting from $C^{2+\alpha}(\mathbb{R}^n)$ to $C^{\alpha}(\mathbb{R}^n)$. Here $u = (u_1, \ldots, u_p)$, $a(x), b_j(x), c(x)$ are $p \times p$ matrices with $C^{\alpha}(\mathbb{R}^n)$ entries, and $a(x)$ is symmetric positive definite,

$$(a(x)\xi, \xi) \ge a_0|\xi|^2$$

for any vector $\xi \in \mathbb{R}^p$, $x \in \mathbb{R}^n$ with a constant $a_0 > 0$.

The space $C^{k+\alpha}(\mathbb{R}^n)$, $\alpha > 0$, is the space of functions bounded in $\mathbb{R}^n$ together with their derivatives up to order k, and the senior derivatives satisfying the Hölder condition with exponent α uniformly in x. To simplify

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the presentation, we consider the case $n = 2$ with the independent variables x and y.

An important role in what follows is played by the location of the essential spectrum. To determine it explicitly, we make some simplifying assumptions. We assume the existence of the limits

$$a(x,y) \to a^{\pm}(y), \quad b_j(x,y) \to b_j^{\pm}(y), \quad c(x,y) \to c^{\pm}(y)$$

as $x \to \pm\infty$. Here the convergence is uniform in y on every bounded set in $\mathbb{R}^1$. If $y \to \pm\infty$, then

$$a(x,y) \to a_{\pm}^0, \quad b_j(x,y) \to b_{j\pm}^0, \quad c(x,y) \to c_{\pm}^0$$

uniformly on every bounded set. Here $a_{\pm}^0, b_{j\pm}^0$, and $c_{\pm}^0$ are constant matrices,

$$a_{\pm}^0 = \lim_{y\to\pm\infty} a^{\pm}(y), \quad b_{j\pm}^0 = \lim_{y\to\pm\infty} b_j^{\pm}(y), \quad c_{\pm}^0 = \lim_{y\to\pm\infty} c^{\pm}(y).$$

These assumptions allow us to define the limiting operators

$$L^{\pm}u = a^{\pm}(y)\Delta u + b_1^{\pm}(y)\,\frac{\partial u}{\partial x} + b_2^{\pm}(y)\,\frac{\partial u}{\partial y} + c^{\pm}(y)u, \tag{1.2}$$

$$L_{\pm}^0 u = a_{\pm}^0 \Delta u + b_{1\pm}^0\,\frac{\partial u}{\partial x} + b_{2\pm}^0\,\frac{\partial u}{\partial y} + c_{\pm}^0 u. \tag{1.3}$$

Consider the problems

$$L^{\pm}u = \lambda u, \quad L_{\pm}^0 u = \lambda u. \tag{1.4}$$

If one of them has a nonzero solution in $C^{2+\alpha}(\mathbb{R}^2)$, then the corresponding value of λ belongs to the essential spectrum of the operator L, i.e., the operator $L - \lambda I$ is not Fredholm [6], [2].

We suppose that the last problem in (1.4) does not have nonzero solutions for any λ with nonnegative real part and that there exists a nonzero solution of at least one of the limiting problems

$$L^{+}u = 0, \tag{1.5}$$

$$L^{-}u = 0, \tag{1.6}$$

in $C^{2+\alpha}(\mathbb{R}^2)$. Then the operator L is not Fredholm. Therefore the usual solvability conditions for Fredholm operators are not applicable here. In this work we study the solvability conditions for non-Fredholm operators. In the next section we construct a reduction of the operator to a subspace where its image is closed and the kernel is finite-dimensional. This allows us to localize the non-Fredholm properties of the operator in a complementary subspace.

In Section 3 we consider a scalar equation. We verify the conditions imposed on the operators in Section 2 and we obtain more complete solvability conditions.

In the last section we consider some applications to nonlinear elliptic problems.

2. Normal solvability. We suppose for convenience that both problems (1.5) and (1.6) have nonzero solutions. We impose the following conditions.

CONDITION 1. Problems (1.5) and (1.6) have unique nonzero solutions $v^+ \in C^{2+\alpha}(\mathbb{R}^1)$ and $v^- \in C^{2+\alpha}(\mathbb{R}^1)$, respectively.

CONDITION 2. The coefficients of the limiting operators are sufficiently smooth, and the formally adjoint problems

$$\widehat{L}^{\pm} u \equiv \Delta(\widehat{a}^{\pm}(y)u) - \frac{\partial(\widehat{b}_1^{\pm}(y)u)}{\partial x} - \frac{\partial(\widehat{b}_2^{\pm}(y)u)}{\partial y} + \widehat{c}^{\pm}(y)u = 0 \tag{2.1}$$

have nonzero solutions $\widehat{v}^{\pm} \in C^{2+\alpha}(\mathbb{R}^1)$ such that

$$\int_{\mathbb{R}^1} |\widehat{v}^{\pm}(y)|\, dy < \infty$$

and

$$\int_{\mathbb{R}^1} (v^+(y), \widehat{v}^+(y))\, dy \neq 0, \qquad \int_{\mathbb{R}^1} (v^-(y), \widehat{v}^-(y))\, dy \neq 0.$$

Here $\widehat{a}^{\pm}, \widehat{b}_j^{\pm}, \widehat{c}^{\pm}$ are the matrices transposed to $a^{\pm}, b_j^{\pm}, c^{\pm}$, respectively and $(\ ,\ )$ denotes the inner product in $\mathbb{R}^2$.

CONDITION 3. Let

$$a_{11} = \int_{\mathbb{R}^1} (v^+(y), \widehat{v}^+(y))\, dy, \quad a_{12} = \int_{\mathbb{R}^1} (v^-(y), \widehat{v}^+(y))\, dy,$$
$$a_{21} = \int_{\mathbb{R}^1} (v^+(y), \widehat{v}^-(y))\, dy, \quad a_{22} = \int_{\mathbb{R}^1} (v^-(y), \widehat{v}^-(y))\, dy.$$

Then $a_{11}a_{22} \neq a_{12}a_{21}$.

CONDITION 4. Let $y^k \to \pm\infty$,

$$a(x, y + y^k) \to a^0_{\pm}, \qquad b_j(x, y + y^k) \to b^0_{j\pm}, \qquad c(x, y + y^k) \to c^0_{\pm}$$

uniformly on every bounded set, where $a^0_{\pm}, b^0_{j\pm}$, and $c^0_{\pm}$ are constant matrices. The limiting problems

$$L_0 u \equiv a^0_{\pm} \Delta u + b^0_{1\pm} \frac{\partial u}{\partial x} + b^0_{2\pm} \frac{\partial u}{\partial y} + c^0_{\pm} u = 0$$

do not have nonzero solutions in $C^{2+\alpha}(\mathbb{R}^2)$.

Introducing the operators

$$L_1^{\pm} u = a^{\pm}(y) \frac{\partial^2 u}{\partial y^2} + b_2^{\pm}(y) \frac{\partial u}{\partial y} + c^{\pm}(y) u$$

acting from $C^{2+\alpha}(\mathbb{R}^1)$ to $C^{\alpha}(\mathbb{R}^1)$, we note that

$$L_1^{\pm} v^{\pm} = 0, \qquad (L_1^{\pm})^* \widehat{v}^{\pm} = 0.$$

The essential spectrum $\lambda(\xi)$ of these operators, which is also a part of the essential spectrum of the operator L, is a set of all complex numbers λ satisfying the following algebraic equation:

$$\det(-a_{\pm}^0 \xi^2 + i b_{2\pm}^0 \xi + c_{\pm}^0 - \lambda) = 0, \qquad \xi \in \mathbb{R}^1.$$

If it lies in the left half-plane, then the operators $L_1^{\pm}$ are Fredholm with zero index [3]. According to Condition 1 they have a zero eigenvalue. Condition 2 means that it is simple.

Set $E = C^{2+\alpha}(\mathbb{R}^2)$, $E' = C^{\alpha}(\mathbb{R}^2)$ and

$$E_0 = \Big\{ u \in E : \int_{\mathbb{R}^1} u(x) \widehat{v}^{+}(y)\, dy = \int_{\mathbb{R}^1} u(x) \widehat{v}^{-}(y)\, dy = 0,\ \forall x \in \mathbb{R}^1 \Big\}.$$

Lemma 2.1. *For any $u \in E$ the following representation holds*:

$$u(x,y) = u_0(x,y) + c^{+}(x) v^{+}(y) + c^{-}(x) v^{-}(y),$$

where $u_0 \in E_0$, $c^{\pm} \in C^{2+\alpha}(\mathbb{R}^1)$.

Proof. Let $c^{+}(x)$, $c^{-}(x)$ be a solution of the system

$$a_{11} c^{+}(x) + a_{12} c^{-}(x) = \int_{\mathbb{R}^1} u(x,y) \widehat{v}^{+}(y)\, dy, \tag{2.2}$$

$$a_{21} c^{+}(x) + a_{22} c^{-}(x) = \int_{\mathbb{R}^1} u(x,y) \widehat{v}^{-}(y)\, dy. \tag{2.3}$$

The integrals on the right-hand sides of (2.2), (2.3) are well defined, and they belong to $C^{2+\alpha}(\mathbb{R}^1)$ as functions of x.

By Condition 3 we can find $c^{\pm} \in C^{2+\alpha}(\mathbb{R}^1)$ from (2.2), (2.3). The function

$$u_0(x,y) = u(x,y) - c^{+}(x) v^{+}(y) - c^{-}(x) v^{-}(y)$$

belongs to E_0. The lemma is proved.

Thus we can represent the space E as a direct sum of E_0 and the complementary subspace

$$\widehat{E} = \{ u \in E :\ u = c^{+}(x) v^{+}(y) + c^{-}(x) v^{-}(y),\ c^{\pm} \in C^{2+\alpha}(\mathbb{R}^1) \}.$$

Remark. If $v^{+}(y) \equiv v^{-}(y)$, then Condition 3 is not satisfied. Instead of the representation in Lemma 2.1, in this case we put

$$u(x,y) = u_0(x,y) + c(x) v(y).$$

Lemma 2.2. *Let a sequence $u_k \in E_0$ be bounded in the norm of E. If $u_k \to u_0$ uniformly on every bounded set, then $u_0 \in E_0$.*

The proof is obvious.

LEMMA 2.3. *The kernel of the operator $L : E_0 \to E'$ is finite-dimensional.*

Proof. Consider a sequence $u_k \in E_0$ with $\|u_k\| \le 1$ and

$$Lu_k = 0. \tag{2.4}$$

We will show that it has a converging subsequence. From this we will conclude that the unit sphere in the kernel of the operator is compact and, consequently, the kernel is finite-dimensional.

Since u_k is bounded in $C^{2+\alpha}(\mathbb{R}^2)$, there exists a subsequence, still denoted by u_k, converging to some $u_0 \in E$ in C^2 uniformly on every bounded set. Passing to the limit in (2.4), we obtain $Lu_0 = 0$. Set $v_k = u_k - u_0$. Then $Lv_k = 0$.

We show that the convergence $v_k \to 0$ is uniform in $\mathbb{R}^2$. Suppose that it is not and there exists a sequence (x_k, y_k) such that

$$|v_k(x_k, y_k)| \ge \varepsilon > 0.$$

Then $x_k^2 + y_k^2 \to \infty$.

Consider first the case where y_k are uniformly bounded. Then we can assume that $y_k \to y_0$ and x_k converges to $+\infty$. Put

$$w_k(x, y) = v_k(x + x_k, y + y_k).$$

We have

$$a(x + x_k, y + y_k)\Delta w_k + b_1(x + x_k, y + y_k)\,\frac{\partial w_k}{\partial x} + b_2(x + x_k, y + y_k)\,\frac{\partial w_k}{\partial y} + c(x + x_k)w_k = 0,$$

$|w_k(0)| \ge \varepsilon$, and w_k converges to some w_0 in C^2 uniformly on every bounded set. Therefore

$$a^+(y + y_0)\Delta w_0 + b_1^+(y + y_0)\,\frac{\partial w_0}{\partial x} + b_2^+(y + y_0)\,\frac{\partial w_0}{\partial y} + c^+(y + y_0)w_0 = 0.$$

By Condition 1, $w_0(x, y) \equiv v^+(y + y_0)$.

On the other hand

$$\int_{\mathbb{R}^1} v_k(x + x_k, y + y_k)\widehat{v}^+(y + y_k)\,dy = \int_{\mathbb{R}^1} (u_k(x + x_k, y + y_k) - u_0(x + x_k, y + y_k))\widehat{v}^+(y + y_k)\,dy = 0,\ \forall x \in \mathbb{R}^1,$$

for all k, because u_k and u_0 belong to E_0, and

$$\int_{\mathbb{R}^1} w_0(x, y)\widehat{v}^+(y + y_0)\,dy = \lim_{k\to\infty} \int_{\mathbb{R}^1} v_k(x + x_k, y + y_k)\widehat{v}^+(y + y_k)\,dy.$$

We obtain a contradiction with Condition 2.

Therefore, $v_k \to 0$ uniformly in $\mathbb{R}^2$. The Schauder estimate [1] implies the convergence $v_k \to 0$ in $C^{2+\alpha}(\mathbb{R}^2)$. Hence the unit sphere in the kernel of the operator is compact.

Suppose now that $|y_k|$ is unbounded. As above, we obtain a nonzero solution of one of the limiting problems

$$a^0_{\pm}\Delta u + b^0_{1\pm}\frac{\partial u}{\partial x} + b_2{}^0_{\pm}\frac{\partial u}{\partial y} + +c^0_{\pm}u = 0.$$

This contradicts Condition 4. The lemma is proved.

LEMMA 2.4. *The image of the operator $L : E_0 \to E'$ is closed.*

Proof. Let

$$Lu_k = f_k, \tag{2.5}$$

$f_k \in E'$, $f_k \to f_0$, $u_k \in E_0$. We will show that there exists $u_0 \in E_0$ such that $Lu_0 = f_0$.

Consider first the case where the sequence u_k is bounded in E. Then we can choose a subsequence converging to some $u_0 \in E$ in C^2 uniformly on every bounded set. Therefore $u_0 \in E_0$. Passing to the limit in (2.5), we obtain $Lu_0 = f_0$.

Suppose now that the sequence u_k is unbounded. Since the kernel of the operator L in E_0 is finite-dimensional, we can represent E_0 as a direct sum of $\operatorname{Ker} L$ and a complementary subspace $\widehat{E}_0$. Then

$$u_k = \widehat{u}_k + u^0_k,$$

where $\widehat{u}_k \in \widehat{E}_0$, $u^0_k \in \operatorname{Ker} L$. Then $L\widehat{u}_k = f_k$. If the sequence $\widehat{u}_k$ is bounded, we can proceed as above to obtain a function $\widehat{u}_0 \in E_0$ such that $L\widehat{u}_0 = f_0$.

Suppose now that $\widehat{u}_k$ is not bounded. Define

$$v_k = \widehat{u}_k/\|\widehat{u}_k\|_E, \qquad g_k = f_k/\|\widehat{u}_k\|_E.$$

Then

$$Lv_k = g_k. \tag{2.6}$$

We will show that there exists a subsequence of v_k converging to some $v_0 \in \widehat{E}_0$ in E and such that $Lv_0 = 0$. This will contradict the definition of $\widehat{E}_0$.

Since v_k is bounded, there exists a subsequence, denoted again by v_k, converging to some $v_0 \in E$ in C^2 uniformly on every bounded set. We have $v_0 \in E_0$. Let us show that this convergence is uniform in $\mathbb{R}^2$.

Passing to the limit in (2.6), we have

$$Lv_0 = 0$$

and for $w_k = v_k - v_0$,

$$Lw_k = g_k.$$

Suppose $w_k \to 0$ on every bounded set but not uniformly in $\mathbb{R}^2$. Then there exists a sequence (x_k, y_k) such that

$$|w_k(x_k, y_k)| \geq \varepsilon.$$

Obviously, $x_k^2 + y_k^2 \to \infty$. If y_k is unbounded, then we obtain a nonzero solution of one of the limiting problems

$$a_\pm^0 \Delta u + b_{1\pm}^0 \frac{\partial u}{\partial x} + b_{2\pm}^0 \frac{\partial u}{\partial y} + c_\pm^0 u = 0.$$

This contradicts Condition 4.

If y_k is bounded, then we can assume that $y_k \to y_0$, $x_k \to +\infty$. Put $\omega_k(x, y) = w_k(x + x_k, y + y_k)$. We can choose a subsequence ω_k converging to some ω_0 in C^2 uniformly on every bounded set. From the equation

$$\begin{gathered} a(x + x_k, y + y_k)\Delta\omega_k + b_1(x + x_k, y + y_k)\,\frac{\partial \omega_k}{\partial x} \\ b_2(x + x_k, y + y_k)\,\frac{\partial \omega_k}{\partial y} + c(x + x_k, y + y_k)\omega_k = g_k(x + x_k, y + y_k) \end{gathered}$$

we obtain

$$a^+(y + y_0)\Delta\omega_0 + b_1^+(y + y_0)\,\frac{\partial \omega_0}{\partial x} + b_2^+(y + y_0)\,\frac{\partial \omega_0}{\partial y} + c^+(y + y_0)\omega_0 = 0.$$

Hence $\omega_0(x, y) = v^+(y + y_0)$.

As in the previous lemma we have

$$\begin{aligned} &\int_{\mathbb{R}^1} \omega_0(x, y)\widehat{v}^+(y + y_0)\,dy \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^1} (v_k(x + x_k, y + y_k) - v_0(x + x_k, y + y_k))\widehat{v}^+(y + y_k)\,dy = 0. \end{aligned}$$

This contradicts Condition 2.

Thus we have shown that $v_k \to v_0$ uniformly on $\mathbb{R}^2$. From the Schauder estimate we obtain the convergence in E. Therefore $v_0 \in \widehat{E}_0$ and $Lv_0 = 0$. This contradiction proves the lemma.

Along with the subspace E_0 we can consider the subspaces

$$\begin{aligned} E_{r,s} = \Big\{u \in E : \int_{\mathbb{R}^1} u(x, y)\widehat{v}^+(y)\,dy = 0,\ \forall x \geq r, \\ \int_{\mathbb{R}^1} u(x, y)\widehat{v}^-(y)\,dy = 0,\ \forall x \leq s\Big\}. \end{aligned}$$

THEOREM 2.1. *For any r and s the operator $L : E_{r,s} \to E'$ has a finite-dimensional kernel and closed image.*

The proof is the same as above for the subspace E_0.

If we consider a sequence r_n converging to $+\infty$ and a sequence s_n converging to $-\infty$, we obtain a sequence of subspaces

$$\ldots \subset E_{r_n,s_n} \subset E_{r_{n+1},s_{n+1}} \subset \ldots$$

and the corresponding sequence of images

$$\ldots \subset L(E_{r_n,s_n}) \subset L(E_{r_{n+1},s_{n+1}}) \subset \ldots$$

Their union may not be closed. Thus the image of a non-Fredholm operator can be a countably normed space and not a normed space as for a Fredholm operator.

There are other possible ways to define subspaces of E where the image of the operator L is closed. Consider a subspace of functions periodic in x with a period τ and mean value 0:

$$E_\tau = \Big\{u \in E : u(x+\tau, y) = u(x,y),\ \int_0^\tau u(x,y)\,dx = 0,\ \forall y\Big\}.$$

Let

$$E_{\tau_1,\ldots,\tau_m} = \sum_{i=1}^m a_i u_i, \tag{2.7}$$

where a_i are constants and $u_i \in E_{\tau_i}$. Without loss of generality we can assume that the periods $\tau_1, \ldots, \tau_m$ are linearly independent. This means that no one of them can be represented as a linear combination with rational coefficients of other periods.

Consider a sequence $\{u^k\}$ of functions having the form (2.7). Suppose that it is uniformly bounded in the norm of $C^{2+\alpha}(\mathbb{R}^2)$. Then we can choose a subsequence converging to some $u_0 \in C^{2+\alpha}(\mathbb{R}^2)$ in C^2 uniformly on every bounded set. Then u_0 also has the form (2.7). Indeed, if each summand in the representation

$$u^k = \sum_{i=1}^m a_i^k u_i^k \tag{2.8}$$

is bounded independently of k, then the statement is obvious. Suppose that one of them is not uniformly bounded and all others are uniformly bounded. Then we obtain a contradiction with the assumption that the functions u^k are uniformly bounded. If more than one summand in (2.8) is not uniformly bounded, we deduce once again that the functions u^k are not uniformly bounded using the assumption that the periods are linearly independent.

As above we obtain the following theorem.

THEOREM 2.2. *The operator $L : E_{\tau_1,\ldots,\tau_m} \to E'$ has a finite-dimensional kernel and closed image.*

The proof uses the fact that the number of summands in (2.7) is finite. If it is infinite, then the image of the operator is not necessarily closed (see examples at the end of the next section). It can be closed under additional conditions on the coefficients a_i.

3. Scalar equation. In this section we consider a scalar problem in $\mathbb{R}^2$. We show that the conditions imposed on the operators in the previous section are satisfied, and we obtain more complete solvability conditions.

Consider the operator

$$Lu = \Delta u + b(y)u \tag{3.1}$$

acting from $C^{2+\delta}(\mathbb{R}^2)$ to $C^\delta(\mathbb{R}^2)$. According to Condition 1 the equation

$$u''(y) + b(y)u(y) = 0 \tag{3.2}$$

has a solution $v(y)$. We assume moreover that $b(\pm\infty) < 0$ and that 0 is the principal eigenvalue of the operator L. Then $v(y) > 0$, $y \in \mathbb{R}^1$.

Consider the equation

$$Lu = g, \quad g \in C^\delta(\mathbb{R}^2), \tag{3.3}$$

and put

$$k(x) = \int_{-\infty}^{\infty} g(x,y)v(y)\,dy. \tag{3.4}$$

The above integral is obviously well defined as $v(y)$ vanishes exponentially as $|y| \to \infty$ and one notes that $k \in C^\delta(\mathbb{R}^2)$. We can represent $g(x,y)$ in the form

$$g(x,y) = k(x)v(y) + g_0(x,y). \tag{3.5}$$

Without loss of generality we can assume that $v(y)$ has L^2 norm 1:

$$\int_{-\infty}^{\infty} v^2(y)\,dy = 1. \tag{3.6}$$

Then for all $x \in \mathbb{R}^1$,

$$\int_{-\infty}^{\infty} g_0(x,y)v(y)\,dy = 0. \tag{3.7}$$

Thus we can represent the space $E = C^{2+\delta}(\mathbb{R}^2)$ as a direct sum

$$E = E_0 + E_1,$$

where E_0 is the subspace of functions satisfying (3.7) and E_1 is the subspace of functions of the form $k(x)v(y)$. We will now consider the restriction of the operator L to E_0. Let $u \in E_0$. It is easy to note that

$$\int_{-\infty}^{\infty} Lu(x,y)v(y)\,dy = 0.$$

Indeed,

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} v(y)\, dy = \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} u(x,y) v(y)\, dy = 0,$$

$$\int_{-\infty}^{\infty} \left(\frac{\partial^2 u}{\partial y^2} + b(y) \right) v(y)\, dy = \int_{-\infty}^{\infty} u(x,y) \left(\frac{\partial^2}{\partial y^2} + b(y) \right) v(y)\, dy = 0.$$

Hence we can consider L as acting from E_0 to

$$\widehat{E}_0 = \Big\{ g \in C^\delta(\mathbb{R}^2) : \int_{-\infty}^{\infty} g(x,y) v(y)\, dy = 0 \Big\}.$$

Obviously L is a bounded operator.

The main result of this section is the following

THEOREM 3.1. *The operator $L : E_0 \to \widehat{E}_0$ has a bounded inverse.*

Before the proof of Theorem 3.1 we state some auxilliary results.

LEMMA 3.1. *The equation*

$$Lu - \sigma u = 0, \qquad \sigma \geq 0, \tag{3.8}$$

has only a zero solution in E_0.

Proof. It is sufficient to prove that $v(y)$ is a unique (up to a constant factor) solution of (3.8) in E. We will show that this factor is 0 for $\sigma > 0$. Suppose that $(L - \sigma)u_0 = 0$, $u_0 \in E$, $u_0 \neq cv(y)$ for any constant c. Let $r > 0$ be such that

$$b(y) < 0 \quad \text{ for } |y| \geq r. \tag{3.9}$$

We can choose a positive constant k such that

$$kv(y) \geq u_0(x,y), \qquad x \in (-\infty, \infty),\ |y| \leq r. \tag{3.10}$$

Then

$$kv(y) \geq u_0(x,y), \qquad x \in (-\infty, \infty),\ y \in (-\infty, \infty). \tag{3.11}$$

The proof of this fact will be given in Lemma 3.3.

Let k_0 be the infimum of the k for which (3.10) holds. We can assume that there are points (x,y) with $|y| \leq r$ where $u_0(x,y)$ is positive. Otherwise we could increase the value of r or change the sign of u_0. Then $k_0 > 0$. We have

$$k_0 v(y) \geq u_0(x,y), \qquad x, y \in (-\infty, \infty),$$

and for $k < k_0$ the function

$$w_k(x,y) = kv(y) - u_0(x,y)$$

is negative for some (x,y) with $|y| \leq r$.

Consider the function w_{k_0}. We have

$$(L-\sigma)w_{k_0} = f(y), \qquad w_{k_0}(x,y) \geq 0, \qquad x, y \in (-\infty, \infty),$$

where $f(y) = -\sigma k_0 v(y)$. Suppose that

$$\lim_{x\to\pm\infty} w_{k_0}(x,y) = 0. \tag{3.12}$$

We can choose a positive constant s such that

$$\omega_s(x,y) = sv(y) - w_{k_0}(x,y) \geq 0, \qquad |y| \leq r. \tag{3.13}$$

Hence

$$\omega_s(x,y) \geq 0, \qquad x, y \in (-\infty, \infty). \tag{3.14}$$

Let s_0 be the minimal value for which (3.13) holds, i.e. such that

$$\omega_{s_0}(x,y) \geq 0, \qquad x \in (-\infty,\infty),\ |y| \leq r. \tag{3.15}$$

Then

$$\omega_{s_0}(x,y) \geq \varepsilon > 0, \qquad |x| \geq N,\ |y| \leq r \tag{3.16}$$

for some ε and N; this follows from (3.12) and the fact that $w_{k_0} \in C^{2+\delta}(\mathbb{R}^2)$.

If we assume that

$$\omega_{s_0}(x,y) > 0, \qquad |x| \geq N,\ |y| \leq r, \tag{3.17}$$

then this inequality also holds for some $s < s_0$. Consequently, (3.14) is also valid for this s and we obtain a contradiction with the assumption that s_0 is the minimal value.

If (3.17) does not hold, then $\omega_{s_0}(x_0, y_0) = 0$ for some x_0, y_0. As $(L-\sigma)\omega_{s_0} \leq 0$ and $\omega_{s_0}(x,y) \geq 0$ for all x, y, we obtain a contradiction with the positiveness theorem. This contradiction shows that (3.12) cannot hold.

If we have the inequality

$$\lim_{x\to\pm\infty} w_{k_0}(x,y) \geq \eta > 0, \qquad |y| \leq r,$$

then the inequality

$$w_k(x,y) \geq 0, \qquad x, y \in (-\infty, \infty),$$

will remain valid for some $k < k_0$. This contradiction shows that there exist two sequences $\{(x_i^{(1)}, y_i^{(1)})\}$ and $\{(x_i^{(2)}, y_i^{(2)})\}$ such that

$$|x_i^{(j)}| \to \infty, \qquad |y_i^{(j)}| \leq r, \qquad j = 1, 2,$$
$$w_{k_0}(x_i^{(1)}, y_i^{(1)}) \to 0, \qquad w_{k_0}(x_i^{(2)}, y_i^{(2)}) \to \eta,$$

where η is a positive constant.

We can assume that $x_i^{(j)} \to +\infty$, $j = 1, 2$. Note that, since the coefficient b in L does not depend on x, the function $u_0(-x)$ is a solution to (3.8) whenever $u_0(x)$ is.

Consider the sequence

$$\widetilde{w}_i(x,y) = w_{k_0}(x + x_i^{(1)}, y), \qquad i = 1, 2, \ldots$$

Obviously, $(L-\sigma)\widetilde{w}_i = f(y) \le 0$ and the sequence $\{\widetilde{w}_i\}$ is uniformly bounded in $C^{2+\delta}(\mathbb{R}^2)$. Hence we can choose a subsequence $\{\widetilde{w}_{i_k}\}$ converging in C^2 to some $\widetilde{w}_0 \in C^{2+\delta}(\mathbb{R}^2)$ uniformly on every bounded subset. Moreover by choosing a subsequence we may assume that the sequence $\{y_i^{(1)}\}$ converges to some y_0 with $|y_0| \le r$. Therefore,

$$(L-\sigma)\widetilde{w}_0 = f(y), \qquad \widetilde{w}_0(x,y) \ge 0, \qquad x, y \in (-\infty, \infty),$$
$$\widetilde{w}_0(0, y_0) = 0.$$

Hence $\widetilde{w}_0(x,y) \equiv 0$. Otherwise we would obtain a contradiction with the positiveness theorem. Similarly, from the sequence

$$\widehat{w}_i(x,y) = w_{k_0}(x + x_i^{(2)}, y), \qquad i = 1, 2, \ldots,$$

we can choose a subsequence converging in C^2 uniformly on every bounded subset to a function $\widehat{w}_0 \in C^{2+\delta}(\mathbb{R}^2)$,

$$(L-\sigma)\widehat{w}_0 = f(y), \qquad \widehat{w}_0(x,y) > 0, \qquad x, y \in (-\infty, \infty).$$

If the function $\widehat{w}_0$ attains the value 0 somewhere then, according to the positiveness theorem, it is identically zero.

We note that $\widehat{w}_0$ does not belong to E_0 and it can be represented in the form

$$\widehat{w}_0(x,y) = k(x)v(y) + \widehat{w}^*(x,y),$$

where $k(x) = \int_{-\infty}^{\infty} \widehat{w}_0(x,y)v(y)\,dy$ is not identically zero and

$$\int_{-\infty}^{\infty} \widehat{w}^*(x,y)v(y)\,dy = 0.$$

It follows that w_{k_0} does not belong to E_0 and

$$w_{k_0}(x,y) = c(x)v(y) + w_{k_0}^*(x,y),$$

where $c(x) = \int_{-\infty}^{\infty} w_{k_0}(x,y)v(y)\,dy$ is not identically zero and

$$\int_{-\infty}^{\infty} w_{k_0}^*(x,y)v(y)\,dy = 0$$

identically with respect to x.

We have

$$c(x_i^{(1)}) \to 0, \qquad c(x_i^{(2)}) \to c_0 > 0. \tag{3.18}$$

Indeed, the first convergence follows from the convergence

$$w_{k0}(x + x_i^{(1)}, y) = \widetilde{w}_i(x,y) \to \widetilde{w}_0(x,y) \equiv 0$$

uniformly on every bounded set. The second convergence follows from the assumptions on the sequence $\{(x_i^{(2)}, y_i^{(2)})\}$ and from the positiveness of $v(y)$.

On the other hand, multiplying the equality

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + b(y) - \sigma\right) w_{k_0} = f(y)$$

by $v(y)$ and integrating, we obtain

$$c''(x) + c(x) \int_{-\infty}^{\infty} (v''(y) + b(y)v(y) - \sigma v(y))v(y)\,dy$$

$$+ \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} w_{k_0}^*(x,y)v(y)\,dy + \int_{-\infty}^{\infty} \left(\frac{\partial^2 w_{k_0}^*}{\partial x^2} + b(y)w_{k_0}^* - \sigma w_{k_0}^*\right) v(y)\,dy = -\sigma k_0,$$

or $c''(x) - \sigma c(x) = -\sigma k_0$. As $\sigma \geq 0$, the only bounded solution to the last equation is $c(x) \equiv \text{const}$ and we obtain a contradiction with (3.18). The lemma is proved.

In the proof of Lemma 3.1 we have used the following two lemmas.

Lemma 3.2. *Let $u : \mathbb{R}^2 \to \mathbb{R}$, $u \in C^2(\mathbb{R}^2)$, $u \not\equiv 0$, $u \geq 0$, satisfy*

$$\Delta u + c(x,y)u = \phi(x,y), \tag{3.19}$$

with $\phi \leq 0$, $|c|_{C^0(\mathbb{R}^2)} < \infty$. Then $u(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$.

The proof is based on the maximum principle, and it is standard.

Lemma 3.3. *Let $w(x,y)$ be a solution of the equation*

$$(L - \sigma)w = -\widetilde{f}(x,y), \qquad \widetilde{f}(x,y) \geq 0,$$

and

$$w(x,r) \geq 0, \qquad x \in (-\infty, \infty),$$

for some $r > 0$ such that $b(y) < 0$ for all $y \geq r$. Then

$$w(x,y) \geq 0, \qquad x \in (-\infty,\infty),\ y \geq r.$$

A similar lemma holds when the condition $y \geq r$ is replaced by $y \leq r$, where r is such that $b(y) < 0$ for all $y < r$.

Proof. Suppose that $w(x_0, y_0) < 0$ for some x_0 and $y_0 > r$. We can choose a constant $b_0 < 0$ such that

$$b(y) \leq b_0, \qquad y \geq r.$$

Let further w_0 be a negative constant such that

$$w(x,y) \geq w_0, \qquad x \in (-\infty,\infty),\ y \geq r.$$

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + b_0 - \sigma\right) u, \qquad u|_{t=0} = w_0.$$

Its solution is independent of x and y, it is negative and converges to 0 as $t \to \infty$.

On the other hand, $w(x,y)$ is a solution of the problem

$$\frac{\partial \widetilde{u}}{\partial t} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + b(y) - \sigma\right)\widetilde{u} + \widetilde{f}(x,y), \qquad \widetilde{u}|_{t=0} = w(x,y),$$

with $\widetilde{f}(x,y) \geq 0$. Set

$$\widehat{u}(x,y,t) = w(x,y) - u(x,y,t).$$

Then

$$\frac{\partial \widehat{u}}{\partial t} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + b(y) - \sigma\right)\widehat{u} + (-b_0 + b(y))u + \widetilde{f}(x,y),$$
$$\widehat{u}|_{y=r} > 0, \qquad \widehat{u}|_{t=0,\, y\geq r} \geq 0.$$

Since $(b(y) - b_0)u(x,y,t) \geq 0$ for all $t \geq 0$, $y \geq r$, we have $\widehat{u}(x,y,t) \geq 0$ for $t \geq 0$, $y \geq r$. Therefore $w(x,y) \geq 0$ for $y \geq r$. This contradiction proves the lemma.

LEMMA 3.4. *The image of the operator $L - \sigma : E_0 \to \widehat{E}_0$, $\sigma \geq 0$, is closed in $\widehat{E}_0$.*

The proof is given in Section 2 in a more general case.

Proof of Theorem 3.1. From the previous lemmas it follows that for all $\sigma \geq 0$ the operator $L_\sigma = L - \sigma : E_0 \to \widehat{E}_0$ is normally solvable, and its kernel is empty.

We show first of all that it is invertible for large positive σ. The operator L_σ considered from $C^{2+\alpha}(\mathbb{R}^2)$ to $C^\alpha(\mathbb{R}^2)$ is invertible for large positive σ. Therefore the equation

$$Lu - \sigma u = f \tag{3.20}$$

has a unique solution u for any $f \in C^\alpha(\mathbb{R}^2)$. Let $f \in \widehat{E}_0$. Then $u \in E_0$. Indeed, we can represent it in the form

$$u(x,y) = k(x)v(y) + u_0(x,y), \qquad u_0 \in E_0.$$

Multiplying (3.20) by $v(y)$ and integrating, we obtain $k'' - \sigma k = 0$. Hence $k(x) \equiv 0$.

We now consider the homotopy

$$L_{\tau\sigma} : E_0 \to \widehat{E}_0, \qquad \tau \in [0,1].$$

Since this operator is normally solvable with a finite-dimensional kernel for all τ, we can use stability of the index for Fredholm and semi-Fredholm operators (see Theorem IV.5.22 in [5]). We conclude that the index of L_0 equals the index of L_σ and both are 0. Since the equation $Lu = 0$ does not have nonzero solutions in E_0, the operator L is invertible. The theorem is proved.

This theorem allows us to obtain a solvability condition in the following form.

THEOREM 3.2. *The equation*

$$\Delta u + b(y)u = g(x,y), \qquad g \in C^{\alpha}(\mathbb{R}^2), \tag{3.21}$$

is solvable in $C^{2+\alpha}(\mathbb{R}^2)$ *if and only if the equation*

$$\phi'' = k, \tag{3.22}$$

where

$$k(x) = \int_{-\infty}^{\infty} g(x,y)v(y)\,dy,$$

is solvable in $C^{2+\alpha}(\mathbb{R}^1)$.

Proof. Let

$$g(x,y) = g_0(x,y) + k(x)v(y).$$

Then

$$\int_{-\infty}^{\infty} g_0(x,y)v(y)\,dy = 0.$$

We look for a solution of (3.21) in the form

$$u(x,y) = u_0(x,y) + \phi(x)v(y),$$

where

$$\phi(x) = \int_{-\infty}^{\infty} u(x,y)v(y)\,dy, \qquad \int_{-\infty}^{\infty} u_0(x,y)v(y)\,dy = 0.$$

From (3.21) we have

$$\Delta u_0 + b(y)u_0 + \phi''(x)v(y) = g_0(x,y) + k(x)v(y). \tag{3.23}$$

Multiplying this equation by $v(y)$ and integrating, we obtain (3.22). From (3.23) it now follows that

$$\Delta u_0 + b(y)u_0 = g_0(x,y). \tag{3.24}$$

By the previous theorem this equation is solvable in $C^{2+\alpha}(\mathbb{R}^2)$. Therefore, solvability of (3.21) is equivalent to solvability of (3.22). The theorem is proved.

Equation (3.22) can be easily solved explicitly. It provides a simple example to show the difficulties arising for non-Fredholm operators. The usual solvability condition applicable for Fredholm operators says that the equation is solvable if the right-hand side is orthogonal to the solution of the formally adjoint homogeneous equation. In this case $\phi(x) \equiv \text{const}$. So the equation would be solvable for any $k(x)$ such that

$$\int_{-\infty}^{\infty} k(x)\,dx = 0. \tag{3.25}$$

A priori $k(x)$ is not necessarily integrable. But even if the integral is well defined, (3.25) does not imply solvability. As an example we can take any odd function converging to 0 at infinity as $1/x^2$.

Consider some other examples.

Examples. 1. Let $k(x)$ be given as a Fourier series

$$k(x) = \sum_{j=1}^{\infty} a(\xi_j)\cos(\xi_j x).$$

Then

$$\phi(x) = -\sum_{j=1}^{\infty} \frac{a(\xi_j)}{\xi_j^2}\cos(\xi_j x).$$

If $\xi_j \to 0$, we can choose the coefficients $a(\xi_j)$ such that the first series converges and the second diverges. This example also shows that the image of the operator is not closed. For any partial sum

$$k_n(x) = \sum_{j=1}^{n} a(\xi_j)\cos(\xi_j x)$$

a solution u_n exists but the sequence u_n is not bounded.

2. Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u + b(y)u - g(x,y), \tag{3.26}$$

$$u(x,0) = 0. \tag{3.27}$$

We put

$$g(x,y) = g_0(x,y) + k(x)v(y)$$

and look for the solution in the form

$$u(x,y,t) = u_0(x,y,t) + \phi(x,t)v(y),$$

where

$$\int_{-\infty}^{\infty} u_0(x,y,t)v(y)\,dy = 0, \qquad \forall x,t.$$

Then we can reduce (3.26) to two equations

$$\frac{\partial u_0}{\partial t} = \Delta u_0 + b(y)u_0 - g_0(x,y) \quad \text{and} \quad \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} - k(x).$$

Since the operator $L : E_0 \to E_0'$ has a bounded inverse, the corresponding semigroup is well defined and $u_0(x, y, t)$ converges exponentially to the solution of (3.24).

If for example $k(x) = \varepsilon \cos(\varepsilon x)$, then $\phi(x, t)$ converges to $-(1/\varepsilon)\cos(\varepsilon x)$. Therefore even a small perturbation of the equation can give a growing perturbation of the solution if the period of the perturbation increases.

3. In the previous example we put $g = 0$ and

$$u(x, 0) = \sum_{j=1}^{\infty} a_j \cos(\xi_j x).$$

Then

$$\phi(x, t) = \sum_{j=1}^{\infty} a_j e^{-\xi_j^2 t} \cos(\xi_j x).$$

Let $a_j = \xi_j^m$. Then

$$\max_x \phi(x, t) = \sum_{j=1}^{\infty} \xi_j^m e^{-\xi_j^2 t}.$$

Consider the function

$$\psi(\xi, t) = \xi^m e^{-\xi^2 t}.$$

For each t fixed, ψ considered as a function of ξ has a maximum

$$\xi_m = \sqrt{\frac{m}{2t}}, \qquad \psi_m = \left(\frac{m}{2t}\right)^{m/2} e^{-m/2}.$$

Therefore the solution can converge to zero polynomially if the frequencies ξ_m converge to zero.

4. Application to a nonlinear problem. In this section we will use the results of the previous section to prove existence of solutions for the problem

$$\Delta u - c\frac{\partial u}{\partial y} + F(u) + \varepsilon S(x, y, u) = 0, \tag{4.1}$$

where ε is a small parameter.

We suppose that the perturbation $S(x, y, u)$ is a sufficiently smooth function periodic in x, i.e.

$$S(x, y, u) = S(x + \tau, y, u), \qquad \forall x, y, u \in \mathbb{R}^1.$$

We have already seen that periodicity of a perturbation is related to solvability conditions.

The unperturbed problem ($\varepsilon = 0$) describes travelling waves. If $F(u)$ is of the so-called bistable type,

$$F(0) = F(1) = F(u_0) = 0, \quad F(u) < 0,\ 0 < u < u_0, \quad F(u) > 0,\ u_0 < u < 1,$$

and $F'(0) < 0, F'(1) < 0$, then there exists a one-dimensional decreasing function $w(y)$ and a constant c_* satisfying

$$w'' - c_* w' + F(w) = 0, \quad w(-\infty) = 1, \quad w(+\infty) = 0.$$

The function $v(y) = -w'(y)$ is an eigenfunction corresponding to the zero eigenvalue of the problem

$$u'' - c_* u' + b(y)u = \lambda u, \quad u(\pm\infty) = 0,$$

where $b(y) = F'(w(y))$. The zero eigenvalue is principal and simple [9]. We suppose for simplicity that $c_* = 0$, i.e.,

$$\int_0^1 F(u)\, du = 0.$$

The operator

$$Lu = \Delta u + b(y)u$$

acting from $C^{2+\alpha}(\mathbb{R}^2)$ to $C^{\alpha}(\mathbb{R}^2)$ satisfies all conditions of Section 3. It is not Fredholm, and we cannot use directly the usual approaches to prove existence of solutions of the perturbed equation. We show in this section how to use the solvability conditions obtained above to apply the implicit function theorem.

Consider the operator

$$A(u, c, \varepsilon) = \Delta u - c\,\frac{\partial u}{\partial y} + F(u) + \varepsilon S(x, y, u)$$

acting from $C^{2+\alpha}(\mathbb{R}^2) \times \mathbb{R}^1 \times \mathbb{R}^1$ to $C^{\alpha}(\mathbb{R}^2)$.

Denote by $\mathcal{B}_k$, $k = 0, 1, 2$, the subspace of $C^{k+\alpha}(\mathbb{R}^2)$ consisting of the functions $u(x, y)$ which are τ-periodic with respect to x, with the $C^{k+\alpha}(\mathbb{R}^2)$ norm. Let further $\mathcal{B}_{20}$ be the subspace of $\mathcal{B}_2$ consisting of the functions $u(x, y)$ satisfying

$$\int_{-\infty}^{\infty} u(0, y) v(y)\, dy = 0. \tag{4.2}$$

Lemma 4.1. *The operator A is bounded and continuous from $\mathcal{B}_{20} \times \mathbb{R}^1 \times \mathbb{R}^1$ to $\mathcal{B}_0$. It has a Fréchet derivative $\mathcal{L}$ with respect to the variables (u, c). The operator $\mathcal{L}$ is continuous with respect to (u, c, ε) in some neigbourhood of the point $P \equiv (u, c, \varepsilon) = (w(y), 0, 0)$ in the operator norm. We have*

$$\mathcal{L}(\widetilde{u}, \widetilde{c}) = L\widetilde{u} - \widetilde{c} v(y). \tag{4.3}$$

In what follows the tildes over u and c are omitted. Let

$$u(x, y) = \phi(x) v(y) + u_0(x, y),$$
$$g(x, y) = k(x) v(y) + g_0(x, y),$$

where $u_0 \in E_0$ and $g_0 \in \widehat{E}_0$. Substituting these expressions into the equation

$$\mathcal{L}(u,c) = g(x,y)$$

we obtain

$$Lu_0(x,y) = g_0(x,y), \tag{4.4}$$

$$\phi''(x) - c = k(x), \tag{4.5}$$

where $g_0(x,y)$ and $k(x)$ are τ-periodic with respect to x. Due to Theorem 3.1, (4.4) is uniquely solvable in the space E_0 for any $g_0 \in \widehat{E}_0$.

LEMMA 4.2. *The solution of* (4.4) *is τ-periodic in x, i.e. it belongs to $\mathcal{B}_2$.*

Proof. The solution $u_0(x,y)$ is unique in E_0. Suppose that it is not τ-periodic. As g_0 is τ-periodic the function $u_0(x+\tau,y)$ is also a solution of (4.4). Hence, if $u_0(x+\tau,y) \neq u_0(x,y)$, then $u_0(x+\tau,y)$ would be another solution of (4.4) belonging to E_0 due to the fact that $v(y)$ is invariant with respect to translation in x. However, by Theorem 3.1 the solution is unique.

LEMMA 4.3. *Equation* (4.5) *has a unique solution* $(\phi(x),c)$, *where ϕ belongs to $\mathcal{B}_{20}$ and $c \in \mathbb{R}^1$.*

Proof. Integrating (4.5), we obtain

$$\phi'(x) = \int_0^x (k(y)+c)\,dy + \phi'(0).$$

The right-hand side is bounded (and τ-periodic) iff

$$\int_0^\tau (k(y)+c)\,dy = 0.$$

This equation allows us to find c. Integrating it we obtain

$$\phi(x) = \int_0^x\int_0^y ((k(z)+c)dz + \phi'(0))\,dy + \phi(0).$$

The right-hand side of the last equation is bounded for all $x \in \mathbb{R}^1$ and τ-periodic iff

$$\int_0^\tau\int_0^y ((k(z)+c)\,dz + \phi'(0))\,dy = 0.$$

From the above equation one can determine the constant $\phi'(0)$ uniquely. By the definition of $\mathcal{B}_{20}$, $\phi(0) = 0$, and so (4.5) can be uniquely solved in the space of τ-periodic functions. The lemma is proved.

According to Lemma 4.3 and Theorem 3.1 the equation $\mathcal{L}(u,c) = g$ is uniquely solvable in the space $\mathcal{B}_{20}\times\mathbb{R}^1$. Recall that $\mathcal{L}$ is the Fréchet derivative of the mapping A at the point $(u,c,\lambda) = (v(y),0,0)$.

Theorem 4.1. *For all ε sufficiently small there exists a unique solution (u,c) of (4.1) in $\mathcal{B}_{20}\times\mathbb{R}^1$ such that $(u,c)=(w,0)+\mathcal{O}(\varepsilon)$ as $\varepsilon\to 0$.*

This follows from the implicit function theorem (see e.g. [4]).

We have considered the case where τ is arbitrary but fixed. Let τ_n be a sequence converging to infinity. For each τ_n we can prove existence of a solution u_n of (4.1). However this sequence may be divergent. The operator $L: E_0\to E_0'$ is invertible but the norm of the inverse increases as $\tau_n\to\infty$ (cf. Section 3). So the sequence of solutions u_n may diverge in the $C^{2+\alpha}(\mathbb{R}^2)$ norm. Possibly, it can be convergent if the sequence $\varepsilon=\varepsilon(\tau_n)$ is properly chosen.

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Laboratoire de Mathématiques Appliquées
UMR 5585 CNRS, Université Lyon 1
69622 Villeurbanne, France
E-mail: volpert@maply.univ-lyon1.fr
massot@lan.univ-lyon1.fr

Institute of Fundamental Technological Research
Świętokrzyska 21
00-049 Warszawa, Poland
E-mail: bkazmier@ippt.gov.pl

Department of Mathematics,
Computer Science and Mechanics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: zperadz@ippt.gov.pl

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