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OPTIMAL CONTROL FOR 2-D NONLINEAR CONTROL SYSTEMS

Abstract. Necessary conditions for some optimal control problem for a nonlinear 2-D system are considered. These conditions can be obtained in the form of a quasimaximum principle.

1. Introduction. Most popular models of two-dimensional (2-D) discrete systems have been introduced by Attasi, Roesser, Fornasini and Marchesini. These systems have been studied in particular by Kaczorek, Klamka and Kurek. Detailed investigations of Roesser's and other models have been treated in many articles reported in [7]–[9].

For at least two decades a great deal of attention has been given to the linear control problem for 2-D systems. The most important contributions deal with controllability, observability, stability and stabilizability.

Some 2-D optimal control problems have been considered by several researchers (Bisiacco, Dymkov, Klamka, Gaishun and others [1]–[6]).

In the present paper some optimal control problems for a nonlinear discrete 2-D system are formulated and discussed. The local constrained controllability problems for nonlinear 2-D systems have been considered by Klamka in [10], [11].

2. Problem formulation. Consider a nonlinear 2-D system described by the following difference equations (Kaczorek [8], Klamka [11]):

$$x(i+1,j+1)=f(x(i,j),x(i+1,j),x(i,j+1),u(i,j)) \tag{2.1}$$

where

- $(i,j)\in\mathbb{Z}^+\times\mathbb{Z}^+$, $\mathbb{Z}^+$ is the set of nonnegative integers,
- $x(i,j)\in\mathbb{R}^n$ is the state vector at the point (i,j),

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- $u(i,j) \in \mathbb{R}^m$ is a control vector at the point (i,j),
- $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given function.

Let $U \subset \mathbb{R}^m$ be a given arbitrary set. The sequence of controls

$$u = \{u(i,j) : (0,0) \le (i,j),\ u(i,j) \in U\}$$

is called an *admissible sequence of controls*. The set of all such admissible sequences of controls forms the so-called *admissible set of controls*.

The boundary conditions for the nonlinear difference equation (2.1) are given by

$$x(i,0) = x_{i0} \in \mathbb{R}^n \quad \text{and} \quad x(0,j) = x_{0j} \in \mathbb{R}^n \quad \text{for } (i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \tag{2.2}$$

where x_{i0} and x_{0j} are known vectors.

We assume that:

1) The function $f(x(i,j), x(i+1,j), x(i,j+1), u(i,j)) = f(x,y,z,u,i,j)$ is continuous in the product space $\Omega = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$.

2) The second partial derivatives of $f(x,y,z,u,i,j)$ with respect to x,y,z are continuous in Ω.

For any given boundary conditions (2.2) and for an arbitrary admissible sequence of controls there exists a unique solution of the nonlinear difference equation (2.1), which may be computed by successive iterations.

For the model (2.1) and (2.2) we will consider the following optimal problem: Choose the sequence of admissible controls $u(i,j)$, $0 \le i \le M-1$, $0 \le j \le N-1$, so as to minimize the cost function index

$$J(u) = c^T x(M,N). \tag{2.3}$$

3. Problem solution. Assume that the control vector $u(i,j)$ corresponds to the state vector $x(i,j)$ and $u(i,j) + \Delta u(i,j)$ corresponds to $x(i,j) + \Delta x(i,j)$. For any sequence of n-dimensional vectors $p(i,j)$ we have

$$\begin{aligned} &\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i,j)\Delta x(i+1,j+1) \\ &= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i-1,j-1)\Delta x(i,j) + \sum_{i=0}^{M-2} p^T(i,N-1)\Delta x(i+1,N) \\ &\quad + \sum_{j=0}^{N-2} p^T(M-1,j)\Delta x(M,j+1) + p^T(M-1,N-1)\Delta x(M,N) \\ &\quad - \sum_{i=0}^{M-1} p^T(i-1,-1)\Delta x(i,0) - \sum_{j=0}^{N-1} p^T(-1,j-1)\Delta x(0,j) \end{aligned} \tag{3.1}$$

where $\Delta x(i,0) = \Delta x(0,j) = 0$ for $i,j \in \mathbb{Z}^+$. From (3.1) we can express $p^T(M-1,N-1)\Delta x(M,N)$ as a function of $p(i,j)$ and $\Delta x(i,j)$. Let us introduce the notations:

$$\begin{aligned}(3.2)\qquad & \Delta x(i+1,j+1)\\ &\qquad = f(x+\Delta x, y+\Delta y, z+\Delta z, u+\Delta u, i, j) - f(x,y,z,u,i,j)\end{aligned}$$

where $\Delta x = \Delta x(i,j)$, $\Delta y = \Delta x(i+1,j)$, $\Delta z = \Delta x(i,j+1)$. Hence we have

$$\begin{aligned}(3.3)\qquad & -p^T(M-1,N-1)\Delta x(M,N)\\ &= -\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i,j)[f(x+\Delta x, y+\Delta y, z+\Delta z, u+\Delta u, i, j)\\ &\quad - f(x,y,z,u,i,j)]\\ &\quad + \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i-1,j-1)\Delta x(i,j)\\ &\quad - \sum_{i=0}^{M-2} p^T(i,N-1)\Delta x(i+1,N) - \sum_{j=0}^{N-2} p^T(M-1,j)\Delta x(M,j+1).\end{aligned}$$

On the right-hand side of (3.3) we can add and subtract the sum

$$-\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i,j) f(x,y,z,u+\Delta u,i,j).$$

The first sum in (3.3) can be written in the form

$$\begin{aligned}&-\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i,j)[f(x+\Delta x, y+\Delta y, z+\Delta z, u+\Delta u, i, j)\\ &- f(x,y,z,u+\Delta u,i,j)]\\ &-\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} p^T(i,j)[f(x,y,z,u+\Delta u,i,j) - f(x,y,z,u,i,j)].\end{aligned}$$

Define

$$(3.4)\qquad H(x,y,z,u,i,j) = p^T(i,j) f(x,y,z,u,i,j).$$

We assume the vectors $p(i,j)$ satisfy the equalities

$$\begin{aligned}(3.5)\qquad p(i-1,j-1) &= \frac{\partial H(x,y,z,u,i,j)}{\partial x} + \frac{\partial H(x,y,z,u,i-1,j)}{\partial y}\\ &\quad + \frac{\partial H(x,y,z,u,i,j-1)}{\partial z}\end{aligned}$$

for $i = 0, 1, \ldots, M-1$ and $j = 0, 1, \ldots, N-1$ with boundary conditions

$$
\begin{aligned}
&p(M-1, N-1) = -c, \\
(3.6)\quad &p(i, N-1) = \frac{\partial H(x,y,z,u,i+1,N-1)}{\partial z}, \qquad i = 0, 1, \ldots, M-2, \\
&p(M-1, j) = \frac{\partial H(x,y,z,u,M-1,j+1)}{\partial z}, \qquad j = 0, 1, \ldots, N-2.
\end{aligned}
$$

Hence, using (3.4)–(3.6), the sum (3.3) can be rewritten in the form

$$
\begin{aligned}
(3.7)\quad c^T \Delta(M,N) = &- \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}[H(x,y,z,u+\Delta u,i,j) - H(x,y,z,u,i,j)] \\
&- \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}[H(x+\Delta x, y+\Delta y, z+\Delta z, u+\Delta u, i, j) \\
&- H(x,y,z,u+\Delta u,i,j)] \\
&- \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i,j)\frac{\partial H(x,y,z,u,i,j)}{\partial x} \\
&+ \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i,j)\frac{\partial H(x,y,z,u,i-1,j)}{\partial y} \\
&+ \sum_{j=0}^{N-2} \Delta x^T(M,j+1)\frac{\partial H(x,y,z,u,M-1,j+1)}{\partial y} \\
&+ \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i,j)\frac{\partial H(x,y,z,u,i,j-1)}{\partial z} \\
&+ \sum_{i=0}^{M-2} \Delta x^T(i+1,N)\frac{\partial H(x,y,z,u,i+1,N-1)}{\partial z}.
\end{aligned}
$$

We can transform (3.7) as follows:

$$
\begin{aligned}
(3.8)\quad &\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i,j)\frac{\partial H(x,y,z,u,i-1,j)}{\partial y} \\
&\qquad + \sum_{j=0}^{N-2} \Delta x^T(M,j+1)\frac{\partial H(x,y,z,u,M-1,j+1)}{\partial y}
\end{aligned}
$$

$$= \sum_{i=0}^{M-2}\sum_{j=0}^{N-1} \Delta x^T(i+1,j)\frac{\partial H(x,y,z,u,i,j)}{\partial y}$$
$$+ \sum_{j=0}^{N-1} \Delta x^T(M,j)\frac{\partial H(x,y,z,u,M-1,j)}{\partial y}$$
$$= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i+1,j)\frac{\partial H(x,y,z,u,i,j)}{\partial y}$$

where $\Delta x(0,j)=0$ and $\Delta x(M,0)=0$.

Similarly, we have

$$\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i,j)\frac{\partial H(x,y,z,u,i,j-1)}{\partial z} \tag{3.9}$$
$$+ \sum_{i=0}^{M-2} \Delta x^T(i+1,N)\frac{\partial H(x,y,z,u,i+1,N-1)}{\partial z}$$
$$= \sum_{i=0}^{M-1}\sum_{j=0}^{N-2} \Delta x^T(i,j+1)\frac{\partial H(x,y,z,u,i,j)}{\partial z}$$
$$+ \sum_{i=0}^{M-1} \Delta x^T(i,N)\frac{\partial H(x,y,z,u,i,N-1)}{\partial z}$$
$$= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} \Delta x^T(i,j+1)\frac{\partial H(x,y,z,u,i,j)}{\partial z}$$

where $\Delta x(i,0)=0$ and $\Delta x(0,N)=0$.

To simplify notation, the elements x,y,z will be combined in a $3n$-dimensional vector $w=[x,y,z]^T$. Then Taylor's formula yields

$$\begin{aligned} H(x+\Delta x, y+\Delta y, z+\Delta z, u+\Delta u, i,j) &- H(x,y,z,u+\Delta u,i,j) \\ &= H(w+\Delta w, u+\Delta u, i,j) - H(w,u+\Delta u,i,j) \\ &= \Delta w^T(i,j)\,\frac{\partial H(w,u+\Delta u,i,j)}{\partial w} \\ &\quad + \frac{1}{2}\Delta w^T(i,j)\,\frac{\partial^2 H(w+\theta\Delta w, u+\Delta u,i,j)}{\partial w^2}\Delta w(i,j) \end{aligned} \tag{3.10}$$

where $0 \le \theta = \theta(i,j) \le 1$. Finally, we have

$$
\begin{aligned}
(3.11)\quad \Delta J &= c^T \cdot \Delta x(M,N) \\
&= -\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}[H(x,y,z,u+\Delta u,i,j) - H(x,y,z,u,i,j)] \\
&\quad -\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\Bigg[\Delta w^T(i,j)\,\frac{\partial H(w,u+\Delta u,i,j)}{\partial w} \\
&\quad + \frac{1}{2}\Delta w^T(i,j)\,\frac{\partial^2 H(w+\theta\Delta w,u+\Delta u,i,j)}{\partial w^2}\,\Delta w(i,j)\Bigg] \\
&\quad + \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\Delta w^T(i,j)\,\frac{\partial H(w,u,i,j)}{\partial w} \\
&= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}[H(w,u+\Delta u,i,j) - H(w,u,i,j)] - \eta
\end{aligned}
$$

where

$$
\begin{aligned}
(3.12)\quad \eta &= \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\Delta w^T(i,j)\left[\frac{\partial H(w,u+\Delta u,i,j)}{\partial w} - \frac{\partial H(w,u,i,j)}{\partial w}\right] \\
&\quad + \frac{1}{2}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\Delta w^T(i,j)\,\frac{\partial^2 H(w+\theta\Delta w,u+\Delta u,i,j)}{\partial w^2}\,w(i,j).
\end{aligned}
$$

Consider the following increment:

$$
(3.13)\qquad \Delta u^*(i,j) = \begin{cases} u^* - u(k,l), & i=k, j=l, \\ 0, & i\neq k, j\neq l. \end{cases}
$$

Let $\Delta w^*(i,j)$ be an increment corresponding to $\Delta u^*(i,j)$. Therefore, (3.11) has the form

$$
(3.14)\qquad \Delta^* J = c^T \cdot \Delta x^*(M,N) = -[H(w,u^*,k,l) - H(w,u,k,l) - \eta^*]
$$

where

$$
(3.15)\qquad \eta^* = \frac{1}{2}\sum_{i=k+1}^{M-1}\sum_{j=l+1}^{N-1}\Delta w^T(i,j)\,\frac{\partial^2 H(w+\theta\Delta^* w,u^*,i,j)}{\partial w^2}\,w(i,j).
$$

Under the above assumption we can estimate

$$
\begin{aligned}
(3.16)\quad |\eta^*| &\le \frac{1}{2}\|f(w,u^*,k,l) - f(u,w,k,l)\|^2 \sum_{i=k+1}^{M-1}\sum_{j=l+1}^{N-1} C(i,j)\left\|\frac{\partial^2 H}{\partial w^2}\right\| \\
&= \overline{\eta^*}
\end{aligned}
$$

where $C(i,j) = \text{const}$ and $\overline{\eta^*}$ is a majorant of η^*. Define

$$U_\varepsilon(k,l) = \{u \in U : |\overline{\eta^*}| \le \varepsilon\}, \qquad \varepsilon > 0. \tag{3.17}$$

THEOREM 1. *The sequence of optimal controls for problem* (2.1) *satisfies*:

(a) *the quasimaximum conditions*

$$H(w,u,k,l) \ge H(w,u^*,k,l) - \varepsilon \tag{3.18}$$

for all $u^* \in U_\varepsilon(k,l)$, $k = 0,1,\ldots,M-2$, $l = 0,1,\ldots,N-2$,

(b) *the maximum conditions*

$$H(w,u,k,l) \ge H(w,u^*,k,l) \tag{3.19}$$

for all $u^* \in U$ *and* $k = M-1$ *for* $l = 0,1,\ldots,N-2$, *and* $k = 0,1,\ldots,M-1$ *for* $l = N-1$.

Proof. (a) Let $u(i,j)$ be the optimal control and assume the existence of (k,l), $\varepsilon > 0$ and $u^* \in U_\varepsilon(k,l)$ such that $H(w,u,k,l) < H(w,u^*,k,l) - \varepsilon$. Hence $\Delta^* J < -\varepsilon + \varepsilon = 0$. This contradicts the optimality of the control.

(b) If $\eta^* = 0$ then $\Delta^* J \ge 0$, which implies $H(w,u,k,l) \ge H(w,u^*,k,l)$.

In particular, we consider the linear 2-D system (2.1) with constant coefficients described by the following difference equation:

$$x(i+1,j+1) = A_0 x(i,j) + A_1 x(i+1,j) + A_2 x(i,j+1) + Bu(i,j) \tag{3.20}$$

where A_0, A_1, A_2 are constant $n \times n$ matrices and B is an $n \times m$ constant matrix. From (3.15) it follows that $\eta = 0$ for (3.20) and we immediately obtain the following theorem:

THEOREM 2. *The sequence of optimal controls for the system* (3.20) *satisfies the maximum conditions*

$$H(w,u,k,l) \ge H(w,u^*,k,l)$$

where $i = 0,1,\ldots,M-1$, $j = 0,1,\ldots,N-1$.

An optimal sequence of input vectors $u(i,j)$ can be found by using the following algorithm:

STEP 1. For a given system (2.1), (2.2) and performance index (2.3) define the hamiltonian function (3.4).

STEP 2. Write the conditions for vectors $p(i,j)$ using (3.5), (3.6).

STEP 3. Find the maximum of the function $H(\ldots,i,j)$ in U.

STEP 4. From the conditions for $p(i,j)$ and $u(i,j)$ find the control vectors.

EXAMPLE 1. Consider the linear system

$$x(i+1,j+1) = A_0 x(i,j) + A_1 x(i+1,j) + A_2 x(i,j+1) + Bu(i,j)$$

with

$$
\begin{gathered}
n = m = 1, \quad (M, N) = (2, 3), \\
A_0 = A_2 = 1, \quad A_1 = -1, \quad B = 2, \quad u \in U = [-1, 1], \\
x_{00} = x_{10} = x_{20} = x_{01} = x_{02} = x_{03} = 1,
\end{gathered}
$$

and the performance index $J(u) = c^T x(M, N)$ where $c = 2$. From (3.4)–(3.6) we have

$$
\begin{aligned}
&H(x, y, z, u, i, j) \\
&\qquad = p^T(i, j)[A_0 x(i, j) + A_1 x(i+1, j) + A_2 x(i, j+1) + Bu(i, j)] \\
&\qquad = p^T(i, j)[x - y + z + 2u], \\
&p(0, 0) = \frac{\partial H(\ldots, 1, 1)}{\partial x} + \frac{\partial H(\ldots, 0, 1)}{\partial y} + \frac{\partial H(\ldots, 1, 0)}{\partial z}, \\
&p(0, 1) = \frac{\partial H(\ldots, 1, 2)}{\partial x} + \frac{\partial H(\ldots, 0, 2)}{\partial y} + \frac{\partial H(\ldots, 1, 1)}{\partial z}, \\
&p(1, 2) = -c = -2, \\
&p(0, 2) = \frac{\partial H(\ldots, 1, 2)}{\partial z} = p(1, 2), \\
&p(1, 0) = \frac{\partial H(\ldots, 1, 1)}{\partial y} = -p(1, 1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
p(0, 0) &= p(1, 1) - p(0, 1) + p(1, 0), \\
p(0, 1) &= p(1, 2) - p(0, 2) + p(1, 1), \\
p(1, 1) &= \frac{\partial H(\ldots, 1, 2)}{\partial y} = p(1, 2)(-1).
\end{aligned}
$$

Finally

$$
\begin{array}{lll}
p(1, 2) = -2, & p(0, 2) = -2, & p(1, 1) = 2, \\
p(1, 0) = -2, & p(0, 0) = -2, & p(0, 1) = 2.
\end{array}
$$

By Theorem 2, if u is an optimal control vector then

$$
H(w, u, k, l) = \max_{u^* \in U} H(w, u^*, k, l).
$$

Hence

$$
\begin{aligned}
H(w, u(0, 0), 0, 0) &= \max_{-1 \le u^* \le 1} [p(0, 0)(x - y + z + 2u^*)] \\
&= \max_{-1 \le u^* \le 1} [-2(x - y + z + 2u^*)].
\end{aligned}
$$

The maximum is achieved for $u(0, 0) = 1$.

In a similar way, we obtain

$$\begin{aligned} H(w, u(0,1), 0, 1) &= \max_{-1 \le u^* \le 1} [p(0,1)(x - y + z + 2u^*)] \\ &= \max_{-1 \le u^* \le 1} [2(x - y + z + 2u^*)]. \end{aligned}$$

The maximum is achieved for $u(0,1) = 1$.

Analogous calculations for other points yield the following results:

$$u(0,2) = -1, \quad u(1,0) = -1, \quad u(1,1) = 1, \quad u(1,2) = -1.$$

The value of the performance index is equal to

$$J(u) = 2 \cdot (-11) = -22.$$

Example 2. Consider the nonlinear system

$$x(i+1, j+1) = \frac{1}{2} x^2(i,j) - x(i+1, j) + x(i, j+1) + 2u(i,j)$$

with

$$(M, N) = (2, 3), \; u \in U = [-1, 1], \; x_{00} = x_{10} = x_{20} = x_{01} = x_{02} = x_{03} = 1$$

and the performance index $J(u) = c^T x(M, N)$ where $c = 2$. From (3.4)–(3.6) we have

$$\begin{aligned} H(x, y, z, u, i, j) &= p^T(i,j) \left[\frac{1}{2} x^2(i,j) - x(i+1, j) + x(i, j+1) + 2u(i,j) \right] \\ &= p^T(i,j) \left[\frac{1}{2} x^2 - y + z + 2u \right], \\ p(0,0) &= \frac{\partial H(\ldots, 1, 1)}{\partial x} + \frac{\partial H(\ldots, 0, 1)}{\partial y} + \frac{\partial H(\ldots, 1, 0)}{\partial z} \\ &= p(1,1) x(1,1) - p(0,1) + p(1,0), \\ p(0,1) &= \frac{\partial H(\ldots, 1, 2)}{\partial x} + \frac{\partial H(\ldots, 0, 2)}{\partial y} + \frac{\partial H(\ldots, 1, 1)}{\partial z} \\ &= p(1,2) x(1,2) - p(0,2) + p(1,1), \\ p(1,2) &= -c = -2, \\ p(0,2) &= \frac{\partial H(\ldots, 1, 2)}{\partial z} = p(1,2), \\ p(1,0) &= \frac{\partial H(\ldots, 1, 1)}{\partial y} = p(1,1)(-1). \end{aligned}$$

These conditions for $p(i,j)$ imply

$$\begin{aligned} p(0,0) &= 2x(1,1) - p(0,1) + p(1,0), \\ p(0,1) &= -2x(1,2) - p(0,2) + p(1,1), \\ p(1,1) &= \frac{\partial H(\ldots,1,2)}{\partial y} = p(1,2)(-1). \end{aligned}$$

Finally

$$\begin{gathered} p(1,2) = -2, \quad p(0,2) = -2, \quad p(1,1) = 2, \quad p(1,0) = -2, \\ p(0,0) = 2x(1,1) + 2x(1,2), \qquad p(0,1) = 4 - 2x(1,2). \end{gathered}$$

Next we have

$$\begin{aligned} &H(w, u(0,0), 0, 0) \\ &\qquad = \max_{-1 \le u^* \le 1} p(0,0)\left[\frac{1}{2}x^2(0,0) - x(1,0) + x(0,1) + 2u^*(0,0)\right] \\ &\qquad = \max_{-1 \le u^* \le 1} p(0,0)\left(\frac{1}{2} + 2u^*\right). \end{aligned}$$

The maximum is achieved for $u(0,0) = 1$ when $p(0,0) > 0$, and for $u(0,0) = -1$ when $p(0,0) < 0$. Next

$$x(1,1) = \frac{1}{2}x^2(0,0) - x(1,0) + x(0,1) + 2u^*(0,0) = \frac{1}{2} + 2u^*.$$

In a similar way we obtain

$$H(w, u(0,1), 0, 1) = \max_{-1 \le u^* \le 1} p(0,0)\left[\frac{1}{2}x^2(0,1) - x(1,1) + x(0,2) + 2u^*(0,1)\right].$$

The maximum is achieved for $u(0,1) = 1$ when $p(0,1) > 0$, and for $u(0,1) = -1$ when $p(0,1) < 0$. Analogous calculations for other points yield the following results:

$$\begin{gathered} u(0,2) = -1, \quad u(1,0) = -1, \quad u(1,1) = 1, \quad u(1,2) = -1, \\ p(0,0) = -7, \quad u(0,0) = -1, \quad p(0,1) = 2, \quad u(0,1) = 1. \end{gathered}$$

We can compute the state vectors:

$$\begin{aligned} x(1,1) &= -\frac{3}{2}, \qquad x(1,2) = 1, \qquad x(1,3) = -\frac{3}{2}, \\ x(2,1) &= -4, \qquad x(2,2) = 8\frac{1}{8}, \qquad x(2,3) = -11\frac{1}{8}. \end{aligned}$$

The value of the performance index is

$$J(u) = 2 \cdot \left(-11\frac{1}{8}\right) = -22\frac{1}{4}.$$

Conclusions. In the paper nonlinear discrete 2-D dynamical systems with the performance index as a function of the final state have been considered. Necessary conditions for the maximum have been formulated and proved. Two simple numerical examples illustrate the presented method.

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