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SEMI-MARKOV CONTROL PROCESSES WITH NON-COMPACT ACTION SPACES AND DISCONTINUOUS COSTS

Abstract. We establish the average cost optimality equation and show the existence of an (ε-)optimal stationary policy for semi-Markov control processes without compactness and continuity assumptions. The only condition we impose on the model is the V-geometric ergodicity of the embedded Markov chain governed by a stationary policy.

1. Introduction and preliminaries. In this paper we deal with the ratio-average cost optimality criterion for semi-Markov control processes on a Borel space. We only assume that the one-step cost function is lower semianalytic, and the transition probability function satisfies certain ergodicity conditions. For such a model, we show the existence of a lower semianalytic solution to the optimality equation. Moreover, as a consequence we obtain an ε-optimal universally measurable stationary policy for the decision maker. This result was stated in [8] without proof. In this paper we give its proof as well as some examples not satisfying the commonly used compactness-continuity assumptions.

The idea of solving the optimality equation via a fixed point argument under V-geometric ergodicity assumptions goes back to Vega-Amaya [15]. He has established the optimality equation under additional requirements. Namely, he assumes that the one-step cost function is lower semicontinuous, and the transition law is setwise continuous. By setwise continuity of the transition probability q, we mean that $q(D \mid x, a)$ is continuous in a for each x and any Borel set D of X. His idea combined with a "regularizing" technique was applied by Jaśkiewicz [7] to derive the optimality equation for semi-

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Markov models with weakly continuous transition laws. For the details of the aforementioned approaches the reader is referred to [8].

Whereas certain continuity assumptions imposed on the transition law are satisfied in many cases arising from practical situations, the compactness condition on the admissible action sets sometimes turns out to be restrictive, e.g., for inventory models (see [3, 4, 14, 16]). Moreover, for these models even the setwise convergence of the transition probability function is too strong. A natural assumption is weak continuity of the transition law [3, 4]. However, one can imagine examples of (semi-)Markov control processes with transition probabilities which are neither weakly continuous nor setwise continuous (see Section 4). In addition, there are models for which transition laws are continuous (in some topology) and the admissible action sets are compact, but the cost functions need not be lower (or upper) semicontinuous. Nevertheless, in all these cases the optimality equation can still be derived using a fixed point argument. This fact is of crucial importance, since it allows one to deal with another optimality cost criterion, the so-called time-average cost criterion. Having established the optimality equation with respect to the ratio-average cost we are able to show the existence of an (ε-)optimal stationary policy and optimal cost with respect to the latter criterion (see Theorem 3 in [8]).

We start with the definitions of analytic and universally measurable sets and functions. Recall that by a *Borel space* X we mean a non-empty Borel subset of some Polish space. We assume that it is endowed with the σ-algebra $\mathcal{B}(X)$ of all its Borel subsets.

Let $\mathbb{N}^{\mathbb{N}}$ be the set of sequences of positive integers, equipped with the product topology. So $\mathbb{N}^{\mathbb{N}}$ is a Polish space. Let Y be a separable metric space. We say that Y is an *analytic set* or *analytic space* provided there is a continuous function g from $\mathbb{N}^{\mathbb{N}}$ onto Y. There are other equivalent definitions of analytic sets in a Borel space X (consult [1] and references therein).

Now let E be an analytic subset of an analytic space X and let p be any probability measure on the Borel subsets of X. Then E is *universally measurable* if E is in the completion of the Borel σ-algebra with respect to every probability measure p.

From now on let X and Y be Borel spaces. Let $\mathcal{A}(X)$ be the analytic σ-algebra and $\mathcal{U}(X)$ be the σ-algebra of all universally measurable subsets of X. We say that a function $f : X \to Y$ is *analytically measurable* [*universally measurable*] if $f^{-1}(B) \in \mathcal{A}(X)$ [$f^{-1}(B) \in \mathcal{U}(X)$] for every $B \in \mathcal{B}(Y)$. We have $\mathcal{B}(X) \subset \mathcal{A}(X) \subset \mathcal{U}(X)$.

Let $B \subset X$ and $f : B \to \mathbb{R}$. If B is analytic and $\{x \in B : f(x) < c\}$ is analytic for each $c \in \mathbb{R}$, then f is said to be *lower semianalytic* (l.s.a.).

Now we state some basic results on l.s.a. functions and universally measurable selectors.

LEMMA A (Proposition 7.48 in [1]). *Let $f : X \times Y \to \mathbb{R}$ be l.s.a., and $q(dy \mid x)$ a Borel measurable stochastic kernel on Y given X. Then the function $\bar{f} : X \to \mathbb{R}$ defined by*

$$\bar{f}(x) = \int_Y f(x,y)\, q(dy \mid x)$$

is l.s.a.

LEMMA B (Jankov–von Neumann's theorem). *If $K \subset X \times Y$ is analytic, then there exists an analytically measurable function $\phi : \mathrm{proj}_X(K) \to Y$ such that*

$$\mathrm{Gr}(\phi) := \{(x,y) : y = \phi(x),\, x \in \mathrm{proj}_X(K)\} \subset K.$$

For the proof the reader is referred to [1, p. 182]. This lemma brings us to the following selection theorem for l.s.a. functions.

LEMMA C. *Let $K \subset X \times Y$ be analytic and $f : K \to \mathbb{R}$ be l.s.a. Define $f^* : \mathrm{proj}_X(K) \to \mathbb{R}$ by*

$$f^*(x) = \inf_{y \in Y(x)} f(x,y),$$

with $Y(x) := \{y \in Y : (x,y) \in K\}$. Then:

(a) *f^* is l.s.a.,*

(b) *the set*

$$I = \{x \in \mathrm{proj}_X(K) | : \textit{for some } y_x \in Y(x),\ f(x,y_x) = f^*(x)\}$$

is universally measurable, and for every $\varepsilon > 0$ there exists a universally measurable function $\phi : \mathrm{proj}_X(K) \to Y$ such that $\mathrm{Gr}(\phi) \subset K$ and for all $x \in \mathrm{proj}_X(K)$,

$$f(x,\phi(x)) = f^*(x) \ \textit{if } x \in I, \qquad f(x,\phi(x)) \le f^*(x) + \varepsilon \quad \textit{if } x \notin I.$$

Part (a) follows from the proof of Proposition 7.47 in [1], whilst (b) is a consequence of Proposition 7.50 in [1].

2. The model. A *semi-Markov control process* is described by the following objects:

(i) The *state space* X is a standard Borel space.

(ii) A is a Borel *action space.*

(iii) K is a non-empty analytic subset of $X \times A$. We assume that for each $x \in X$, the non-empty x-section

$$A(x) = \{a \in A : (x,a) \in K\}$$

of K represents the *set of actions available* in state x.

(iv) $Q(\cdot \mid x,a)$ is a regular transition measure from $X \times A$ into $\mathbb{R}_+ \times X$, where $\mathbb{R}_+ = [0,\infty)$. It is assumed that $Q(D \mid x,a)$ is a Borel function on

$X \times A$ for any Borel subset $D \subset \mathbb{R}_+ \times X$, and $Q(\cdot \mid x, a)$ is a probability measure on $\mathbb{R}_+ \times X$ for any $x \in X$ and $a \in A(x)$. Define

$$Q(t, \hat{X} \mid x, a) := Q([0, t] \times \hat{X} \mid x, a)$$

for any Borel set $\hat{X} \subset X$. If $a \in A(x)$ is selected in state x, then $Q(t, \hat{X} \mid x, a)$ is the joint probability that the sojourn time is not greater than $t \in \mathbb{R}_+$ and the next state y is in $\hat{X}$. Denote by $H(\cdot \mid x, a)$ the distribution of the sojourn time when the process is in state x and the action $a \in A(x)$ is selected, that is, $H(t \mid x, a) = Q(t, X \mid x, a)$. Let $\tau(x, a)$ be the mean holding time, i.e.,

$$\tau(x, a) = \int_0^\infty t\, H(dt \mid x, a).$$

Put $q(\cdot \mid x, a) := Q(\mathbb{R}_+, \cdot \mid x, a)$. Then q is called the *transition law of the embedded Markov process*. Moreover, the distribution of the sojourn time and the next state is conditionally independent of (x, a), i.e.,

$$Q(t, \hat{X} \mid x, a) = q(\hat{X} \mid x, a) H(t \mid x, a).$$

(v) Let $c_i : K \to \mathbb{R}$, $i = 1, 2$. Then the expected one-step *cost function* $c : K \to \mathbb{R}$ equals

$$c(x, a) = c_1(x, a) + \tau(x, a) c_2(x, a).$$

Here c_1 is an immediate cost paid by the decision maker at the transition time, and the cost c_2 is incurred until the next transition occurs.

Let $\{T_n\}$ denote a sequence of random decision (jump) epochs with $T_0 := 0$ and set $x := x_0$. If the action $a_0 \in A(x)$ is selected, then the immediate cost $c_1(x, a_0)$ is incurred for the decision maker and the process remains in state x up to time $T = T_1 - T_0 = T_1$. The cost $c_2(x, a_0)$ per unit time is incurred until the next transition occurs. Afterwards the system moves to state x_1 according to the probability measure $q(\cdot \mid x, a_0)$. The procedure repeats itself yielding a trajectory $(x_0, a_0, t_1, x_1, a_1, t_2, \ldots)$ of some stochastic process, where x_n and a_n describe the state and the action chosen, respectively, on the nth step of the process. Obviously, t_n is a realization of the random variable T_n, and the distribution function of the random holding time $T_{n+1} - T_n$ is $H(\cdot \mid x_n, a_n)$.

A *policy* is a sequence $\pi = \{\pi_n\}$ where π_n $(n \geq 0)$ is a universally measurable stochastic kernel on A given $(X \times A \times \mathbb{R}_+)^n \times X$ satisfying $\pi_n(A(x_n) \mid h_n) = 1$ for any history $h_n = (x_0, a_0, t_1, \ldots, x_n)$ of the process (clearly, $h_0 = x_0$). We denote by Π the class of all policies. Let F be the set of all universally measurable transition probabilities f from X to A such that $f(x) \in A(x)$ for each $x \in X$. A *stationary policy* π is of the form $\pi = \{f, f, \ldots\}$, where $f \in F$. Thus, every stationary policy $\pi = \{f, f, \ldots\}$ can be identified with the mapping $f \in F$. Since K is analytic, the Jankov–

von Neumann theorem guarantees that there exists at least one $f \in F$. Therefore, F and Π are non-empty.

Let $\Omega = (K \times \mathbb{R}_+)^\infty$ be the space of all infinite histories of the process, endowed with $\mathcal{U}$ (the σ-algebra of universally measurable sets in Ω). According to Proposition 7.45 in [1], for any $\pi \in \Pi$ and any initial state $x_0 = x \in X$ there exists a unique probability measure P_x^π defined on Ω. We denote by E_x^π the expectation operator with respect to P_x^π.

We shall consider the following *ratio-average cost*:

$$J(x,\pi) := \limsup_{n\to\infty} \frac{E_x^\pi(\sum_{k=0}^{n-1} c(x_k,a_k))}{E_x^\pi(\sum_{k=0}^{n-1} \tau(x_k,a_k))}.$$

Clearly, $J(x) := \inf_{\pi\in\Pi} J(x,\pi)$ is an optimal cost. A policy π^ε is called *ε-optimal* if

$$J(x,\pi^\varepsilon) - \varepsilon \le J(x)$$

for all $x \in X$. Now we present our assumptions.

(**B**) *Basic assumptions*:

(i) there exist a constant $B > 0$ and a Borel measurable function $V : X \to [1,+\infty)$ such that

$$|c(x,a)| \le BV(x) \quad \text{and} \quad \tau(x,a) \le BV(x)$$

for every $(x,a) \in K$;

(ii) the function τ is Borel measurable, whilst c is l.s.a. on K.

(**GE**) *V-geometric ergodicity assumptions*:

(i) there exists a Borel set $C \subset X$ such that for some $\lambda \in (0,1)$ and $\eta > 0$, we have

$$\int_X V(y)\, q(dy \,|\, x,a) \le \lambda V(x) + \eta 1_C(x)$$

for all $(x,a) \in K$, where V is the function introduced in (**B**);

(ii) the function V is bounded on C, i.e.,

$$v_C := \sup_{x\in C} V(x) < \infty;$$

(iii) there exist some $\delta \in (0,1)$ and a probability measure μ concentrated on the Borel set C with the property that

$$q(D \,|\, x,a) \ge \delta\mu(D)$$

for each Borel set $D \subset C$, $x \in C$ and $a \in A(x)$.

For any function $u : X \to \mathbb{R}$ define the *V-norm*

$$\|u\|_V := \sup_{x\in X} \frac{|u(x)|}{V(x)}.$$

Under (**GE**) the embedded state process $\{x_n\}$ governed by a stationary policy $f \in F$ is a positive recurrent aperiodic Markov chain and there exists a unique invariant probability measure π_f (see Theorem 11.3.4 on p. 116 in [9]). Moreover, by Theorem 2.3 in [10], $\{x_n\}$ is V-*ergodic*, that is, there exist $\theta > 0$ and $\alpha \in (0,1)$ such that

$$\Big|\int_X u(y)\, q^n(dy\,|\,x, f(x)) - \int_X u(y)\,\pi_f(dy)\Big| \le V(x)\|u\|_V \theta\alpha^n \tag{1}$$

for every u with $\|u\|_V < \infty$, and $x \in X$, $n \ge 1$. Here $q^n(\cdot\,|\,x, f(x))$ denotes the n-stage transition probability induced by q and a stationary policy f. As an immediate consequence of (1), one can easily get

$$J(f) := J(x,f) = \frac{\int_X c(x,f(x))\,\pi_f(dx)}{\int_X \tau(x,f(x))\,\pi_f(dx)} \tag{2}$$

for every $f \in F$.

(**R**) *Regularity condition*: there exist $\kappa > 0$ and $\beta < 1$ such that

$$H(\kappa\,|\,x,a) \le \beta$$

for all $x \in C$ and $a \in A(x)$.

Assumptions (**R**) and (**GE**) ensure that an infinite number of transitions do not occur in a finite time interval [6]. Moreover, (**R**) implies that

$$\tau(x,a) \ge \kappa(1-\beta) \quad \text{for } x \in C \text{ and } a \in A(x). \tag{3}$$

For a further discussion of the assumptions the reader is referred to [5, 9, 14].

3. Main results. We begin with two auxiliary lemmas.

LEMMA 1. *Let* (**GE**) *hold. Then*

(a) $\inf_{f\in F} \pi_f(C) \ge (1-\lambda)/\eta$;

(b) $\sup_{f\in F} \int_X V(y)\,\pi_f(dy) \le \eta/(1-\lambda)$.

Proof. Let the process be governed by a stationary policy $f \in F$. Integrating both sides of (**GE**)(i) with respect to the invariant probability measure π_f we get

$$\int_X V(y)\,\pi_f(dy) \le \lambda \int_X V(y)\,\pi_f(dy) + \eta\pi_f(C).$$

Now part (a) easily follows from the fact that $V \ge 1$, whilst (b) is a consequence of $\pi_f(C) \le 1$. ■

LEMMA 2. *Let assumption* (**GE**) *hold. Then for*

$$W(x) := V(x) + \frac{\eta}{\delta} \quad \textit{and} \quad \lambda' := \frac{\lambda + \eta/\delta}{1 + \eta/\delta} < 1,$$

the following inequality holds:

$$\int_X W(y)\, q(dy \mid x,a) \le \lambda' W(x) + \delta 1_C(x) \int_C W(y)\, \mu(dy).$$

For the proof the reader is referred to Lemma 3.1 in [7].

For any function $u : X \to \mathbb{R}$ we define the W-*norm* as

$$\|u\|_W := \sup_{x\in X} \frac{|u(x)|}{W(x)}.$$

We note that

$$\|u\|_V < \infty \quad \text{iff} \quad \|u\|_W < \infty.$$

Let L_W denote the set of all l.s.a. functions whose W-norm is finite. Note that L_W is a complete metric space.

For any $(x,a) \in K$ set

$$p(\cdot \mid x,a) := q(\cdot \mid x,a) - 1_C(x)\delta\mu(\cdot).$$

Observe that from Lemma 2 we have

$$\int_X W(y)\, p(dy \mid x,a) \le \lambda' W(x). \tag{4}$$

Put

$$g := \inf_{f\in F} J(f).$$

From (**B**), (**GE**)(i) and (**R**) we conclude that $g < \infty$. Indeed, by (2) and (3),

$$|J(f)| \le \frac{\int_X |c(x,f(x))|\, \pi_f(dx)}{\int_X \tau(x,f(x))\, \pi_f(dx)} \le \frac{B\int_X V(x)\, \pi_f(dx)}{\kappa(1-\beta)\pi_f(C)},$$

and Lemma 1 yields

$$g \le \frac{B\frac{\eta}{1-\lambda}}{\frac{1-\lambda}{\eta}\kappa(1-\beta)}. \tag{5}$$

We finally arrive at the following result.

Theorem 1. *Assume* (**B**, **GE**, **R**).

(a) *There exist a constant g^* and function $h \in L_W$ such that*

$$h(x) = \inf_{a\in A(x)} \Big(c(x,a) - g^*\tau(x,a) + \int_X h(y)\, q(dy \mid x,a) \Big) \tag{6}$$

for all $x \in X$.

(b) *For any $\varepsilon > 0$ there exists a universally measurable function $f^\varepsilon \in F$ such that*

$$h(x) \ge c(x,f^\varepsilon(x)) - g^*\tau(x,f^\varepsilon(x)) + \int_X h(y)\, q(dy \mid x, f^\varepsilon(x)) - \varepsilon \tag{7}$$

for all $x \in X$.

(c) $g^* = g = \inf_{\pi\in\Pi} J(x,\pi)$ *and* $g^* \ge J(f^\varepsilon) - \varepsilon$.

Proof. For any $u \in L_W$ define

$$(Tu)(x) := \inf_{a\in A(x)}\Big(c(x,a) - g\tau(x,a) + \int_X u(y)p(dy\,|\,x,a)\Big) \tag{8}$$

for all $x \in X$. We claim that T has a fixed point $h \in L_W$. Indeed, observe that by (**B**), (4), and (5) there exists a constant $L^* > 0$ such that

$$|(Tu)(x)| \le L^* W(x), \qquad x \in X. \tag{9}$$

This fact together with Lemmas A and C(a) shows that T maps L_W into itself. Now fix $x \in X$ and $u_1, u_2 \in L_W$ and note that by (4) we have

$$\begin{aligned}|(Tu_1)(x) - (Tu_2)(x)| &\le \sup_{a\in A(x)}\Big|\int_X u_1(y)\,p(dy\,|\,x,a) - \int_X u_2(y)\,p(dy\,|\,x,a)\Big| \\ &\le \|u_1 - u_2\|_W \lambda' W(x).\end{aligned}$$

Since $\lambda' < 1$, it follows that T is a contraction operator. Hence, from the Banach fixed point theorem there exists $h \in L_W$ such that

$$\begin{aligned}h(x) = \inf_{a\in A(x)}\Big(&c(x,a) - g\tau(x,a) \\ &+ \int_X h(y)\,q(dy\,|\,x,a) - 1_C(x)\delta\int_X h(y)\,\mu(dy)\Big).\end{aligned} \tag{10}$$

Note that if $x \notin C$ then (6) holds. The case of $x \in C$ requires more delicate handling. Put

$$d := -\delta\int_C h(x)\,\mu(dx).$$

We shall show that $d = 0$. Suppose that, on the contrary, $d \ne 0$.

Case 1: Let $d < 0$. Let f be an arbitrary stationary policy. Then from (10) we get

$$h(x) \le c(x,f(x)) - g\tau(x,f(x)) + \int_X h(y)\,q(dy\,|\,x,f(x)) + d1_C(x). \tag{11}$$

Integrating both sides of (11) with respect to the invariant probability measure π_f, we obtain

$$g \le \frac{\int_X c(x,f(x))\,\pi_f(dx)}{\int_X \tau(x,f(x))\,\pi_f(dx)} + \frac{d\pi_f(C)}{\int_X \tau(x,f(x))\,\pi_f(dx)}.$$

Note that by Lemma 1(a) we have

$$\inf_{f\in F}\pi_f(C) \ge (1-\lambda)/\eta.$$

Hence, taking into account that $d < 0$ and applying (2) and (**B**) we conclude

$$g \le J(f) + \frac{(1-\lambda)d/\eta}{\int_X \tau(x,f(x))\,\pi_f(dx)} \le J(f) + \frac{(1-\lambda)d/\eta}{B\int_X V(x)\,\pi_f(dx)}.$$

Now making use of Lemma 1(b), we see that

$$g \le J(f) + \frac{(1-\lambda)d/\eta}{B\eta/(1-\lambda)}.$$

Furthermore, since f is arbitrary, we may write

$$g \le \inf_{f\in F} J(f) + \frac{(1-\lambda)d/\eta}{B\eta/(1-\lambda)} = g + \frac{(1-\lambda)d/\eta}{B\eta/(1-\lambda)}.$$

However, this contradicts the fact that $d < 0$.

Case 2: $d > 0$. From Lemma C(b), for any $\varepsilon > 0$ there exists a universally measurable function f^ε such that from (10) we have

$$\begin{aligned}
(12)\qquad h(x) &= \inf_{a\in A(x)} \Big(c(x,a) - g^*\tau(x,a) + \int_X h(y)\, q(dy \,|\, x,a)\Big) + d1_C(x)\\
&\ge c(x, f^\varepsilon(x)) - g^*\tau(x, f^\varepsilon(x)) + \int_X h(y)\, q(dy \,|\, x, f^\varepsilon(x)) + d1_C(x) - \varepsilon.
\end{aligned}$$

Further, we proceed as above, that is, we first integrate both sides of (12) with respect to π_{f^ε}. By Lemma 1(a) and (2) we obtain

$$\begin{aligned}
g &\ge \frac{\int_X c(x, f^\varepsilon(x))\, \pi_{f^\varepsilon}(dx)}{\int_X \tau(x, f^\varepsilon(x))\, \pi_{f^\varepsilon}(dx)} + \frac{d\pi_{f^\varepsilon}(C) - \varepsilon}{\int_X \tau(x, f^\varepsilon(x))\, \pi_{f^\varepsilon}(dx)}\\
&\ge J(f^\varepsilon) + \frac{d(1-\lambda)/\eta - \varepsilon}{\int_X \tau(x, f^\varepsilon(x))\, \pi_{f^\varepsilon}(dx)}.
\end{aligned}$$

Since ε is arbitrary, we may choose $\varepsilon < d(1-\lambda)/\eta$. Therefore, from Lemma 1(b) we conclude

$$g \ge J(f^\varepsilon) + \frac{d(1-\lambda)/\eta - \varepsilon}{B\eta/(1-\lambda)}.$$

Moreover,

$$g \ge \inf_{f\in F} J(f) + \frac{d(1-\lambda)/\eta - \varepsilon}{B\eta/(1-\lambda)}.$$

Since $\varepsilon > 0$ is arbitrarily small, we get

$$g \ge \inf_{f\in F} J(f) + \frac{d(1-\lambda)/\eta}{B\eta/(1-\lambda)} = g + \frac{d(1-\lambda)/\eta}{B\eta/(1-\lambda)}.$$

This contradicts our assumption that $d > 0$.

We have proved that $d = 0$. Hence, the optimality equation (6) is satisfied with $g^* := g$ and $h^* = h$. Part (b) follows directly from Lemma C(b). Part (c) is an immediate consequence of a standard dynamic programming argument [5]. ∎

4. Examples. In this section we give two examples of Markov controlled processes arising from economics. Here, we do not deal with semi-Markov controlled models, since randomness between successive jumps does not change the crux of the problem. The most important issue we wish to focus on is non-compactness of the admissible action sets and a merely Borel measurable transition law.

Our first example concerns the economic growth model (see, e.g., [2, 13]) in which the admissible action sets are compact, but the transition law induced by a certain recurrence equation is not necessarily weakly or setwise continuous. In our second example, neither compactness nor continuity conditions are satisfied.

We first recall that q is called *weakly continuous* if the function

$$(x,a) \mapsto \int w(y)\, q(dy \mid x,a)$$

is continuous for any bounded continuous function w. On the other hand, we say that q is *setwise continuous* if the function

$$a \mapsto q(B \mid x,a)$$

is continuous for each $x \in X$ and any Borel set $B \subset X$.

EXAMPLE 1. Let $X = A = [0,\infty)$. Let $x_n \in X$ denote a capital available for consumption in period n. By restricting x_n to be non-negative, we assume the economy to be debt free. The decision maker chooses the level of consumption $a_n \in A(x_n) := [0, x_n] \subset A$, and invests the remaining capital $x_n - a_n$. The consumption generates the immediate utility $U(a_n)$, where $U : [0,\infty) \to \mathbb{R}$ is some Borel measurable (or continuous) function. The investment produces capital at the next decision epoch according to the dynamic equation

$$x_{n+1} = \Psi(x_n - a_n)\xi_n, \tag{13}$$

where the function $\Psi : [0,\infty) \to [0,\infty)$ represents the existing technology and anticipated inflation, and the quantity ξ_n denotes a random disturbance which accounts for unanticipated inflation and random shocks to the system. We impose the following assumptions:

- (A1) $\{\xi_n\}$ is an i.i.d. sequence of nonnegative random variables with a distribution G; $E\xi_0 = \theta < 1$ and $P(\xi_0 = 0) = \varepsilon > 0$;
- (A2) $\Psi(0) = 0$, Ψ is non-decreasing, and there exists an $s > 0$ such that $\Psi(s) > 1$;
- (A3) there is $\beta > 0$ such that $\Psi(y) > y$ for $0 < y < \beta$, and $\Psi(y) < y$ for $y > \beta$.

Assume for the moment that

$$G(z) := \begin{cases} 0, & z \le 0, \\ \varepsilon, & 0 < z \le 1, \\ 1, & z > 1. \end{cases}$$

First we show that the transition law induced by the recurrence equation (13) need not be setwise continuous. With this in view, we put $\Psi(x) := \sqrt{x}$. Let $x^* = 1$ and $a^* = 0$, and define $a^k := 1/k$ for $k = 2, 3, \ldots$. Obviously, $a^k \in A(1)$, and $a^k \to a^*$ as $k \to \infty$. If $B = (0, 1)$, then

$$q(B \,|\, x^*, a^*) = \int_0^\infty 1_B((\sqrt{1-0})z)\, G(dz) = \varepsilon 1_B(0) + (1-\varepsilon)1_B(1) = 0$$

and

$$\begin{aligned} q(B \,|\, x^*, a^k) &= \int_0^\infty 1_B\left(\left(\sqrt{1 - \frac{1}{k}}\right) z\right) G(dz) = \varepsilon 1_B(0) + (1-\varepsilon) 1_B\left(\sqrt{1 - \frac{1}{k}}\right) \\ &= 1 - \varepsilon. \end{aligned}$$

Hence, we see that

$$q(B \,|\, x^*, a^k) \nrightarrow q(B \,|\, x^*, a^*) \quad \text{as } k \to \infty.$$

Now we turn to discussing weak continuity of q. We do not presume that Ψ is a continuous function, and, for instance, Ψ can only be left continuous, say at 1. This and (A2) imply that

$$\Psi(1) = \lim_{y \to 1^-} \Psi(y) < \lim_{y \to 1^+} \Psi(y).$$

Let $x^k := 1 + 1/k$ and $a^k := 1/k^2$ with $k = 2, 3, \ldots$. Clearly, $a^k \in A(x^k)$, and $x^k \to x^* := 1$ and $a^k \to a^* := 0$ as $k \to \infty$. If $w(x) := \arctan(x)$, we easily notice that

$$\int_0^\infty w(\Psi(x^* - a^*)z)\, G(dz) = (1-\varepsilon)\arctan(\Psi(1)),$$

whilst

$$\begin{aligned} \int_0^\infty w(\Psi(x^k - a^k)z)\, G(dz) &= (1-\varepsilon)\arctan\left(\Psi\left(1 + \frac{1}{k} - \frac{1}{k^2}\right)\right) \\ &\nrightarrow (1-\varepsilon)\arctan(\Psi(1)) \quad \text{as } k \to \infty. \end{aligned}$$

Thus, q need not be weakly continuous.

Finally, we are going to prove that our assumptions (**GE**) are satisfied with $V(x) = x + 1$. Let now G be any distribution of ξ_0 satisfying (A1). We begin by showing inequality (**GE**)(i):

$$\int_X V(y)\,q(dy\,|\,x,a) = \int_0^\infty (\Psi(x-a)z+1)\,G(dz) = \theta\Psi(x-a)+1 \tag{14}$$
$$\leq \theta\Psi(x)+1 \leq \frac{1+\theta}{2}\,(\Psi(x)+1),$$

where the last inequality holds when $\Psi(x)\geq 1$. We consider two cases:

(I) $\Psi(\beta)\leq 1$ and $C := \{x\in[0,\infty) : \Psi(x)\leq 1\}$ (note that by (A2), $C=[0,\beta^*]$ or $C=[0,\beta^*)$ for some constant $\beta^*\geq\beta$);

(II) $\Psi(\beta)>1$ and $C:=[0,\beta]$.

(**GE**)(i) Let (I) hold true. For $x\notin C$, from (14) by (A3) we get

$$\int_X V(y)\,q(dy\,|\,x,a) \leq \frac{\theta+1}{2}\,(x+1). \tag{15}$$

On the other hand, for $x\in C$,

$$\int_X V(y)\,q(dy\,|\,x,a) \leq \theta\Psi(x)+1 \leq \frac{\theta+1}{2}\,(\Psi(x)+1)+1 \tag{16}$$
$$\leq \frac{\theta+1}{2}\,(x+1)+\theta+2.$$

Combining (15) and (16), we obtain

$$\int_X V(y)\,q(dy\,|\,x,a) \leq \frac{\theta+1}{2}\,(x+1)+(\theta+2)1_C(x).$$

Now, assume (II). We observe that inequality (15) is satisfied for $x\notin C$. If $x\in C$, then

$$\int_X V(y)\,q(dy\,|\,x,a) \leq \theta\Psi(x)+1 \leq \frac{\theta+1}{2}\,(\Psi(x)+1)+1 \tag{17}$$
$$\leq \frac{\theta+1}{2}\,(x+1)+\frac{\theta+1}{2}\,(\Psi(\beta)+1).$$

Combining (15) and (17), we obtain

$$\int_X V(y)\,q(dy\,|\,x,a) \leq \frac{\theta+1}{2}\,(x+1)+\frac{\theta+1}{2}\,(\Psi(\beta)+1)1_C(x).$$

Thus, in both cases $\lambda := (\theta+1)/2$, whilst $\eta := \theta+2$ in case (I) and $\eta := (\theta+1)(\Psi(\beta)+1)/2$ in case (II).

(**GE**)(ii) This condition holds in both cases, since C is a finite interval.

(**GE**)(iii) We note that for any $D \subset X$, $x \in X$, and $a \in A(x)$, from (A1) it follows that

$$q(D \mid x, a) = \int_0^\infty 1_D(\Psi(x-a)z)\, G(dz) \geq \varepsilon 1_D(\Psi(x-a)0) = \varepsilon 1_D(0).$$

Hence, this condition is satisfied with $\mu(D) := 1_D(0)$ and $\delta := \varepsilon$.

Example 2. This example has its roots in economic games (see [11, 12] and references therein). The special feature of these models is that the transition probability functions are convex combinations of finitely many probability measures on the state space. More precisely, we assume that $X := [0, \infty)$, $A(x) = A := [0, \infty)$, and

$$q(\cdot \mid x, a) = \alpha(x, a)\nu_1(\cdot) + (1 - \alpha(x, a))\nu_2(\cdot),$$

where ν_1, ν_2 are probability measures on X, and $\alpha : X \times A \to [0, 1]$. Further, we assume that

$$\text{(A)} \qquad \delta' := \sup_{x \in X, a \in A} \alpha(x, a) < 1.$$

Obviously, if α is continuous, then q is both weakly and setwise continuous. But, if

$$\alpha(x, a) := \begin{cases} a^2, & a \leq 1/4, \\ 1 - e^{-a-0.1}, & a > 1/4, \end{cases}$$

then q is neither weakly nor setwise continuous: just take $a^* := 1/4$, and a sequence $a^k := a^* + 1/k$ with $k = 1, 2, \ldots$, and use the fact that $\alpha(\cdot, x)$ is discontinuous at a^*, for each $x \in X$.

We observe that in this case assumptions (**GE**) are immediately satisfied. Indeed, (**GE**)(i) holds with $V \equiv 1$, any $\lambda < 1$ and $C := X$, since

$$\int_X V(y)\, q(dy \mid x, a) = 1 = \lambda V(x) + (1 - \lambda)1_C(x).$$

Condition (**GE**)(ii) is clearly satisfied. As for (**GE**)(iii), by (A) it follows that

$$\begin{aligned} q(D \mid x, a) &= \alpha(x, a)\nu_1(D) + (1 - \alpha(x, a))\nu_2(D) \\ &\geq (1 - \alpha(x, a))\nu_2(D) \geq (1 - \delta')\nu_2(D) \end{aligned}$$

for any $D \subset X$, $x \in X$, and $a \in A$. Hence, (**GE**)(iii) holds with $\mu := \nu_2$ and $\delta := 1 - \delta'$.

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