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A UNIFORM CENTRAL LIMIT THEOREM FOR DEPENDENT VARIABLES

Abstract. Niemiro and Zieliński (2007) have recently obtained uniform asymptotic normality for the Bernoulli scheme. This paper concerns a similar problem. We show the uniform central limit theorem for a sequence of stationary random variables.

1. Introduction. We consider a strictly stationary sequence of random variables $X_1, X_2, \ldots$ defined on a statistical space $(\Omega, \mathcal{F}, \{P_\theta : \theta \in \Theta\})$, where P_θ is a marginal distribution of the sequence $X_1, X_2, \ldots$ with $\mathbb{E}_\theta X_i = \mu(\theta)$ and finite variance $\operatorname{Var}_\theta X_i = \sigma^2(\theta)$.

We assume that there exist a function $\sigma^2_{\rm as}(\theta) > 0$ and a sequence $a_n \to 0$ such that

$$\text{(A1a)} \qquad \sup_{\theta\in\Theta} \left| \frac{1}{n} \operatorname{Var}_\theta\Big(\sum_{i=1}^n X_i\Big) - \sigma^2_{\rm as}(\theta) \right| \le a_n,$$

$$\text{(A1b)} \qquad \inf_{\theta\in\Theta} \sigma^2_{\rm as}(\theta) > M_1 \quad \text{for some } M_1 > 0.$$

Define

$$S_n^* := \frac{S_n - n\mu(\theta)}{\sigma_{\rm as}(\theta)\sqrt{n}},$$

where $S_n := \sum_{i=1}^n X_i$.

Let Φ be the c.d.f. of $N(0,1)$. We say that the sequence S_n^* is *uniformly asymptotically normal* (UAN) over Θ if

$$\text{(1)} \qquad \sup_{\theta\in\Theta} \sup_{x\in\mathbb{R}} |P_\theta(S_n^* \le x) - \Phi(x)| = o(1) \quad \text{as } n \to \infty.$$

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Clearly, (1) implies that $\bar{X}_n$ is $\mathrm{UAN}(\mu(\theta), \sigma_{\mathrm{as}}(\theta)/\sqrt{n})$, i.e.

$$\sup_{\theta\in\Theta}\sup_{x\in\mathbb{R}}\left|P_\theta\left(\sqrt{n}\,\frac{\bar{X}_n-\mu(\theta)}{\sigma_{\mathrm{as}}(\theta)}\le x\right)-\Phi(x)\right|=o(1).$$

This fact is useful for example when constructing the asymptotic confidence interval for $\mu(\theta)$ or θ for dependent statistical data.

In Section 2 we show UAN for dependent random variables together with some necessary lemmas. In Section 3 we give applications of our results to linear processes and AR(1) processes.

2. Main results. Now, we present a basic lemma to obtain UAN for dependent random variables.

LEMMA 1. *If there exists a sequence $c_n \to 0$ such that, for every $t \in \mathbb{R}$,*

$$\sup_{\theta\in\Theta}|\mathbb{E}_\theta \exp(itS_n^*)-\exp(-t^2/2)|\le c_n(|t|+t^2+|t|^3), \tag{2}$$

then there exists an absolute constant $C>0$ such that

$$\sup_{\theta\in\Theta}\sup_{x\in\mathbb{R}}|P_\theta(S_n^*\le x)-\Phi(x)|\le C\sqrt{c_n}. \tag{3}$$

Proof. The main tool is the following well-known inequality:

$$\sup_{x\in\mathbb{R}}|P_\theta(S_n^*\le x)-\Phi(x)|\le C_1\int_{-T}^{T}\left|\frac{\varphi_{n,\theta}(t)-\varphi(t)}{t}\right|dt+\frac{C_2}{T} \tag{4}$$

for every $T>0$, for some absolute constants C_1, C_2, where $\varphi_{n,\theta}(t) := \mathbb{E}_\theta \exp(itS_n^*)$ and $\varphi(t) := \exp(-t^2/2)$. Using (2), we have

$$\begin{aligned}\sup_{\theta\in\Theta}\sup_{x\in\mathbb{R}}|P_\theta(S_n^*\le x)-\Phi(x)| &\le C_1\int_{-T}^{T} c_n\,\frac{|t|+t^2+|t|^3}{|t|}\,dt+\frac{C_2}{T}\\ &\le C_1'c_n(T+T^2+T^3)+C_2T^{-1},\end{aligned}$$

where C_1' is an absolute constant. Putting $T=c_n^{-\alpha}$ with $\alpha=1/2$ we get

$$\sup_{\theta\in\Theta}\sup_{x\in\mathbb{R}}|P_\theta(S_n^*\le x)-\Phi(x)|\le C'c_n^\alpha$$

for some absolute constant C'. ∎

Now, we formulate some assumptions which imply (2). We will use Bernstein's "large block - small block" technique. Let $p=p(n)$ and $q=q(n)$ be sequences of positive integers such that $p\to\infty$, $q\to\infty$, $q/p\to\infty$ as $n\to\infty$, and let $k=[n/(p+q)]$. Moreover,

$$B_j=((p+q)(j-1)+1,\dots,(p+q)(j-1)+p]\cap\mathbb{N}$$

is a block of size p and B_j' is the block between B_j and B_{j+1} of size q.

Set
$$\tilde{X}_j := \frac{X_j - \mu(\theta)}{\sigma_{\mathrm{as}}(\theta)}, \qquad U_j := \sum_{i\in B_j} \tilde{X}_i.$$
We consider the following assumptions:

(A3) there exists a sequence $b_n \to 0$ such that, for every $t \in \mathbb{R}$,
$$\sup_{\theta\in\Theta} \sum_{j=2}^{k} \Big|\mathrm{Cov}_\theta\Big\{\exp\Big(\frac{it}{\sqrt{n}}\sum_{s=1}^{j-1} U_s\Big), \exp\Big(\frac{it}{\sqrt{n}} U_j\Big)\Big\}\Big| \le |t| b_n;$$

(A4) we have
$$\sup_{\theta\in\Theta} \sum_{j=0}^{\infty} |\mathrm{Cov}_\theta(X_1, X_{1+j})| < M_2$$
for some $M_2 > 0$;

(A5) for every $n \in \mathbb{N}$ there exists an absolute constant C'' such that
$$\sup_{\theta\in\Theta} \mathbb{E}_\theta \Big|\sum_{i=1}^{n} \tilde{X}_i\Big|^3 \le C'' n^{3/2}. \tag{5}$$

THEOREM 2. *The assumptions* (A1a)–(A1b) *and* (A3)–(A5) *imply* (2).

Proof. For fixed $t \in \mathbb{R}$ we define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = \exp(itx)$ and set
$$S := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{X}_i, \quad Z := \frac{1}{\sqrt{n}} \sum_{j=1}^{k} U_j, \quad Z^* := \frac{1}{\sqrt{n}} \sum_{j=1}^{k} U_j^*,$$
where the sequence (U_j^*) is i.i.d., and U_1^* has the same distribution as U_1. Moreover, let $Y := \frac{1}{\sqrt{n}} \sum_{j=1}^{k} N_j$, where $N_j \sim \mathcal{N}(0, \mathrm{Var}(U_j))$ and (N_j) is i.i.d. Then, similarly to Doukhan and Wintenberger (2007), we have
$$\begin{aligned}
&\mathbb{E}_\theta \exp(itS_n^*) - \exp(-t^2/2)\\
&\quad= \mathbb{E}_\theta(f(S) - f(Z)) + \mathbb{E}_\theta(f(Z) - f(Z^*)) + \mathbb{E}_\theta(f(Z^*) - f(Y))\\
&\qquad + \mathbb{E}_\theta(f(Y)) - \exp(-t^2/2)\\
&\quad=: I_{1,\theta} + I_{2,\theta} + I_{3,\theta} + I_{4,\theta}.
\end{aligned}$$
Using Taylor expansion we obtain
$$\begin{aligned}
|I_{1,\theta}| &\le \|f'\|_\infty \mathbb{E}_\theta|S - Z| \le |t| \mathbb{E}_\theta^{1/2}(S - Z)^2\\
&= \frac{|t|}{\sqrt{n}} \mathbb{E}_\theta^{1/2}\Big(\sum_{j=1}^{k} \sum_{s\in B_j'} \tilde{X}_s + \sum_{s\in R_n} \tilde{X}_s\Big)^2\\
&\le \frac{|t|}{\sqrt{n}} \Big(\mathbb{E}_\theta^{1/2}\Big(\sum_{j=1}^{k} \sum_{s\in B_j'} \tilde{X}_s\Big)^2 + \mathbb{E}_\theta^{1/2}\Big(\sum_{s\in R_n} \tilde{X}_s\Big)^2\Big)
\end{aligned}$$

where $R_n := \{1,\dots,n\} \setminus \bigcap_{j=1}^{k}(B_j \cup B'_j)$. From stationarity of the sequence $(\tilde{X}_s)$, we obtain

$$\mathbb{E}_\theta\Big(\sum_{j=1}^{k}\sum_{s\in B'_j}\tilde{X}_s\Big)^2 \le 2kq\sum_{j=0}^{\infty}|\mathrm{Cov}_\theta(\tilde{X}_1,\tilde{X}_{1+j})|.$$

From (A1b) and (A4), we have

$$\begin{aligned}\sup_{\theta\in\Theta}\mathbb{E}_\theta\Big(\sum_{j=1}^{k}\sum_{s\in B'_j}\tilde{X}_s\Big)^2 &\le 2kq\sup_{\theta\in\Theta}\sum_{j=0}^{\infty}|\mathrm{Cov}_\theta(\tilde{X}_1,\tilde{X}_{1+j})|\\ &\le 2kq\sup_{\theta\in\Theta}\frac{1}{\sigma^2_{\mathrm{as}}(\theta)}\sum_{j=0}^{\infty}|\mathrm{Cov}_\theta(X_1,X_{1+j})| \le C_1kq\end{aligned}$$

for some constant C_1. Similarly,

$$\sup_{\theta\in\Theta}\mathbb{E}_\theta\Big(\sum_{s\in R_n}\tilde{X}_s\Big)^2 \le C_2(n-k(p+q)) \le C_2(p+q)$$

for some constant C_2. Hence,

$$\sup_{\theta\in\Theta}|I_{1,\theta}| \le C'_1|t|\left(\sqrt{\frac{q}{p}}+\sqrt{\frac{p+q}{n}}\right) \le 2C'_1|t|\left(\sqrt{\frac{q}{p}}+\sqrt{\frac{p}{n}}\right) \tag{6}$$

for some constant C'_1.

Observe that

$$|\mathbb{E}_\theta(f(Z)-f(Z^*))| \le \sum_{j=2}^{k}\bigg|\mathrm{Cov}_\theta\bigg\{\exp\Big(\frac{it}{\sqrt{n}}\sum_{s=1}^{j-1}U_s\Big),\exp\Big(\frac{it}{\sqrt{n}}U_j\Big)\bigg\}\bigg|.$$

Then using (A3), we get

$$\sup_{\theta\in\Theta}|I_{2,\theta}| \le C'_2|t|b_n \tag{7}$$

for some constant C'_2.

Clearly,

$$\begin{aligned}|I_{3,\theta}| &= \bigg|\prod_{j=1}^{k}\mathbb{E}_\theta\exp\Big(\frac{it}{\sqrt{n}}\,U_j\Big) - \prod_{j=1}^{k}\mathbb{E}_\theta\exp\Big(\frac{it}{\sqrt{n}}\,N_j\Big)\bigg|\\ &\le k\bigg|\mathbb{E}_\theta\exp\Big(\frac{it}{\sqrt{n}}\,U_1\Big) - \mathbb{E}_\theta\exp\Big(\frac{it}{\sqrt{n}}\,N_1\Big)\bigg|.\end{aligned}$$

From the Taylor formula we have

$$\mathbb{E}_\theta\exp\Big(\frac{it}{\sqrt{n}}\,U_1\Big) = 1-\frac{\mathbb{E}_\theta U_1^2}{2n}\,t^2-\frac{i}{6n^{3/2}}\,\mathbb{E}_\theta(U_1^3)t^3\eta_1$$

and

$$\mathbb{E}_\theta \exp\left(\frac{it}{\sqrt{n}} N_1\right) = 1 - \frac{\mathbb{E}_\theta N_1^2}{2n}\, t^2 - \frac{i}{6n^{3/2}}\, \mathbb{E}_\theta(N_1^3) t^3 \eta_2$$
$$= 1 - \frac{\mathbb{E}_\theta U_1^2}{2n}\, t^2 - \frac{i}{6n^{3/2}}\, \mathbb{E}_\theta(U_1^3) t^3 \eta_2$$

for some $|\eta_1|, |\eta_2| \in (0,1)$. This and (A5) yield

$$\sup_{\theta\in\Theta} |I_{3,\theta}| \le k\, \frac{|t|^3}{6n^{3/2}} \sup_{\theta\in\Theta} \mathbb{E}_\theta |U_1|^3 \le C_3' |t|^3\, \frac{kp^{3/2}}{n^{3/2}} \le C_3' |t|^3 \sqrt{\frac{p}{n}} \tag{8}$$

for some constant C_3'.

Observe that

$$\begin{aligned} |I_{4,\theta}| &= \left| \left(\mathbb{E}_\theta \exp\left(\frac{it}{\sqrt{n}}\, N_1\right)\right)^k - \exp(-t^2/2)\right| \\ &= \left| \exp\left(-\frac{t^2 k \operatorname{Var}_\theta(N_1)}{2n}\right) - \exp(-t^2/2)\right| \\ &\le \frac{t^2}{2} \left| 1 - \frac{k}{n} \operatorname{Var}_\theta(N_1)\right| = \frac{t^2}{2} \left| 1 - \frac{k}{n} \operatorname{Var}_\theta(U_1)\right|. \end{aligned} \tag{9}$$

Let $D_p := \frac{1}{p} \operatorname{Var}_\theta(\sum_{i\in B_1} X_i)$. Then

$$1 - \frac{k}{n} \operatorname{Var}_\theta(U_1) = 1 - \frac{k}{n} \operatorname{Var}_\theta\Big(\sum_{i\in B_1} \tilde{X}_i\Big) = 1 - \frac{kp}{n}\, \frac{D_p}{\sigma_{\mathrm{as}}^2(\theta)}. \tag{10}$$

Now,

$$1 - \frac{kp}{n}\, \frac{D_p}{\sigma_{\mathrm{as}}^2(\theta)} = 1 - \frac{kp}{n} - \frac{kp}{n}\, \frac{D_p - \sigma_{\mathrm{as}}^2(\theta)}{\sigma_{\mathrm{as}}^2(\theta)}. \tag{11}$$

Therefore and from (A1a)–(A1b), we obtain

$$\left| 1 - \frac{kp}{n}\, \frac{D_p}{\sigma_{\mathrm{as}}^2(\theta)}\right| \le \left| 1 - \frac{kp}{n}\right| + \frac{kp}{n}\, \frac{|D_p - \sigma_{\mathrm{as}}^2(\theta)|}{\sigma_{\mathrm{as}}^2(\theta)} \le \frac{p+q}{n} + \frac{a_p}{M}. \tag{12}$$

From (9)–(12), we have, for some constant C_4,

$$\sup_{\theta\in\Theta} |I_{4,\theta}| \le C_4 t^2 \left(\frac{p}{n} + a_p\right). \tag{13}$$

From (6), (7), (8), (13) we obtain (2) for

$$c_n = \mathcal{O}\left(\max\left(a_p, b_n, \sqrt{\frac{q}{p}}, \sqrt{\frac{p}{n}}\right)\right), \tag{14}$$

where a_p is the pth term in the sequence defined by (A1a). ∎

3. Linear processes. We consider the following linear process (LP):

$$X_n = \sum_{r=0}^{\infty} a_r(\theta) Z_{n-r},$$

where the innovations (Z_n) are i.i.d. r.v.'s with mean zero and unit variance, and $a_r(\theta)$ is a nonrandom sequence depending on the parameter θ. We will consider the following assumptions:

$$(\mathrm{a}_0) \qquad \sup_{\theta\in\Theta} \sum_{r=0}^{\infty} |a_r(\theta)| < \infty,$$

$$(\mathrm{a}_1) \qquad \sup_{\theta\in\Theta} \sum_{r=j}^{\infty} a_r^2(\theta) = \mathcal{O}(j^{-t}) \quad \text{for some } t > 1 \text{ (as } j \to \infty),$$

$$(\mathrm{b}_1) \qquad \mathbb{E}|Z_1|^3 < \infty.$$

PROPOSITION 3. *Under assumptions* (a_0), (a_1), (b_1) *we obtain* (A3)–(A5).

Proof. First we will show (A3). Let

$$h_1(U) := \exp\biggl(\frac{it}{\sqrt{n}} \sum_{s=1}^{j-1} U_s\biggr), \quad h_2(U) := \exp\biggl(\frac{it}{\sqrt{n}} U_j\biggr),$$

$$h_1(\hat{U}) := \exp\biggl(\frac{it}{\sqrt{n}} \sum_{s=1}^{j-1} \hat{U}_s\biggr), \quad h_2(\hat{U}) := \exp\biggl(\frac{it}{\sqrt{n}} \hat{U}_j\biggr),$$

$$\hat{U}_s := \sum_{k\in B_s} \frac{\hat{X}_k - \mu(\theta)}{\sigma_{\mathrm{as}}(\theta)}, \qquad \hat{X}_k := \sum_{r=0}^{q-1} a_r(\theta) Z_{k-r}.$$

Then

$$\begin{aligned}\mathrm{Cov}_\theta(h_1(U), h_2(U)) &= \mathrm{Cov}_\theta(h_1(U) - h_1(\hat{U}), h_2(U)) \\ &\quad + \mathrm{Cov}_\theta(h_1(\hat{U}), h_2(U) - h_2(\hat{U})) \\ &\quad + \mathrm{Cov}_\theta(h_1(\hat{U}), h_2(\hat{U})).\end{aligned}$$

Hence

$$\sigma(h_1(\hat{U})) \subset \sigma(\ldots, Z_{(j-1)p+(j-2)q})$$

and

$$\sigma(h_2(\hat{U})) \subset \sigma(Z_{(j-1)(p+q)+2-q}, \ldots, Z_{jp+(j-1)q}),$$

so the r.v.'s $h_1(\hat{U})$ and $h_2(\hat{U})$ are independent, which implies

$$(15) \qquad \mathrm{Cov}_\theta(h_1(\hat{U}), h_2(\hat{U})) = 0.$$

From the elementary inequality

$$|\exp(ia) - \exp(ib)| \le |a - b|, \qquad a, b \in \mathbb{R},$$

bounding h_2 we obtain

$$(16)\qquad |\mathrm{Cov}_\theta(h_1(U)-h_1(\hat U),h_2(U))| \le 2\mathbb{E}_\theta|h_1(U)-h_1(\hat U)|$$
$$\le 2\,\frac{|t|}{\sigma_{\mathrm{as}}(\theta)\sqrt{n}}\sum_{s=1}^{j-1}\sum_{l\in B_s} M_{l,q}(\theta) = 2|t|n^{-1/2}(j-1)pM_{1,q}(\theta),$$

where

$$M_{l,q}(\theta) := \mathbb{E}_\theta\Big|\sum_{r=q}^{\infty} a_r(\theta)Z_{l-r}\Big|.$$

Similarly we obtain

$$(17)\qquad |\mathrm{Cov}_\theta(h_1(\hat U),h_2(U)-h_2(\hat U))| \le 2\mathbb{E}_\theta|h_2(U)-h_2(\hat U)|$$
$$\le 2\,\frac{|t|}{\sigma_{\mathrm{as}}(\theta)\sqrt{n}}\sum_{l\in B_j} M_{l,q}(\theta).$$

From (15)–(17),

$$|\mathrm{Cov}_\theta(h_1(U),h_2(U))| \le C|t|n^{-1/2}jpM_{1,q}(\theta),$$

and from (A1b) we have

$$(18)\qquad \sum_{j=2}^{k}\Big|\mathrm{Cov}_\theta\Big(\exp\Big\{\frac{it}{\sqrt{n}}\sum_{s=1}^{j-1}U_s\Big\},\exp\Big\{\frac{it}{\sqrt{n}}U_j\Big\}\Big)\Big|$$
$$\le C|t|\,\frac{p}{\sigma_{\mathrm{as}}(\theta)\sqrt{n}}\sum_{j=2}^{k} jM_{1,q}(\theta) \le C|t|\frac{p}{\sigma_{\mathrm{as}}(\theta)\sqrt{n}}\,M_{1,q}(\theta)\sum_{j=2}^{k} j$$
$$\le C|t|\,\frac{pk^2}{\sigma_{\mathrm{as}}(\theta)\sqrt{n}}\,M_{1,q}(\theta) \le C'|t|p^{-1}n^{3/2}M_{1,q}(\theta)$$

for some constants C, C'. By (a_2) we get

$$(19)\qquad \sup_{\theta\in\Theta} M_{1,q}(\theta) \le \sup_{\theta\in\Theta}\mathbb{E}_\theta^{1/2}\Big(\sum_{r=q}^{\infty} a_r(\theta)Z_{l-r}\Big)^2$$
$$\le \sup_{\theta\in\Theta}\Big(\sum_{r=q}^{\infty} a_r^2(\theta)\Big)^{1/2} \le Cq^{-(1/2+\gamma)}$$

for some $\gamma>0$. Hence choosing

$$p(n)\sim n^{1-\varepsilon/2},\qquad q(n)\sim n^{1-\varepsilon}\qquad \text{for some } \varepsilon>0,$$

we find that the r.h.s. of (18) is less than $C|t|b_n$, where $b_n\to 0$. This proves (A3).

From (a_0), we easily obtain (A4). From Theorem 2.1 of Furmańczyk (2008) for $Q=3$ we get

$$\sup_{\theta\in\Theta}\mathbb{E}_\theta\Big|\sum_{i=1}^{n}\tilde{X}_i\Big|^3 \le C\sup_{\theta\in\Theta}\frac{n^{3/2}}{\sigma_{\mathrm{as}}^{3/2}(\theta)}\Big(\sum_{r=0}^{\infty}|a_r(\theta)|\Big)^3 \le C'' n^{3/2},$$

which implies (A5). ∎

Corollary 4. *Under assumptions* (a_0), (a_1) *for some* $t>7/2$, (b_1) *and* (A1a) *we obtain* (3) *for*

$$c_n = \mathcal{O}(n^{-1/8}). \tag{20}$$

Moreover, the constant in (A1a)–(A1b) *has the form*

$$\sigma_{\mathrm{as}}^2(\theta) = \sum_{s=0}^{\infty} a_s^2(\theta) + 2\sum_{j=1}^{\infty}\sum_{s=0}^{\infty} a_s(\theta)a_{s+j}(\theta). \tag{21}$$

Proof. By (a_0),

$$\sup_{\theta\in\Theta}\sum_{j=1}^{\infty}|\mathrm{Cov}_\theta(X_1,X_{1+j})| < \infty,$$

and from stationarity, we easily obtain (21).

Observe that

$$\Big|\frac{1}{n}\mathrm{Var}_\theta\Big(\sum_{i=1}^{n}X_i\Big) - \sigma_{\mathrm{as}}^2(\theta)\Big| = 2\Big|\sum_{j=n+1}^{\infty}\mathrm{Cov}_\theta(X_1,X_{1+j})\Big| \le 2\sum_{j=n+1}^{\infty}\sum_{s=0}^{\infty}|a_s(\theta)|\,|a_{s+j}(\theta)|.$$

From (a_0) and the Schwarz inequality, we have (A1a) for

$$a_n = \mathcal{O}\bigg(\sum_{j=n+1}^{\infty}\sqrt{\sup_{\theta\in\Theta}\sum_{s=0}^{\infty}a_{s+j}^2(\theta)}\bigg) = \mathcal{O}\Big(\sum_{j=n+1}^{\infty}\sqrt{j^{-t}}\Big) = \mathcal{O}(n^{-t/2+1}),$$

therefore putting $p(n)\sim n^{3/4}$ and $q(n)\sim n^{1/2}$, we obtain $a_p = \mathcal{O}(p^{-t/2+1}) = \mathcal{O}(n^{-1/8})$. From (19) and condition (a_1) for some $t>7/2$ we obtain

$$b_n = \mathcal{O}\Big(\frac{n^{3/2}}{pq^{t/2}}\Big) = \mathcal{O}(n^{-1/8}).$$

Observe that $\sqrt{q/p} = \mathcal{O}(n^{-1/8})$ and $\sqrt{p/n} = \mathcal{O}(n^{-1/8})$. Hence from (14) we obtain (20). Therefore from Lemma 1, Theorem 2 and Proposition 3 we obtain (3). ∎

We now consider an AR(1) process with parameter $\theta \in (-1;1)$ of the form

$$X_n = \sum_{r=0}^{\infty} \theta^r Z_{n-r}. \tag{22}$$

PROPOSITION 5. *If* $\theta \in (-1+\delta; 1-\delta)$ *for some* $\delta > 0$ *and* (b_1) *holds, then conditions* (a_0), (a_1), (A1a)–(A1b) *are satisfied and the process* X_n *satisfies uniform CLT* (3) *for* c_n *of the form* (20).

Proof. Since the AR(1) process is a linear process with $a_r(\theta) = \theta^r$ and $\Theta = (-1+\delta; 1-\delta)$, we have

$$\sup_{\theta\in\Theta} \sum_{r=0}^{\infty} |a_r(\theta)| = \sup_{\theta\in\Theta} \sum_{r=0}^{\infty} |\theta|^r = \sup_{\theta\in\Theta} \frac{|\theta|}{1-|\theta|} \le \frac{1-\delta}{\delta},$$

and condition (a_0) is satisfied. Similarly

$$\sup_{\theta\in\Theta} \sum_{r=j}^{\infty} a_r^2(\theta) = \sup_{\theta\in\Theta} \frac{\theta^{2j}}{1-\theta^2} \le \frac{(1-\delta)^{2j}}{\delta(2-\delta)} = \mathcal{O}(A^j) \quad \text{for some } A < 1,$$

and condition (a_1) holds. From (21) we have

$$\begin{aligned}\left|\frac{1}{n} \mathrm{Var}_\theta \Big(\sum_{i=1}^{n} X_i\Big) - \sigma_{\mathrm{as}}^2(\theta)\right| &= 2\Big|\sum_{j=n+1}^{\infty} \mathrm{Cov}_\theta(X_1, X_{1+j})\Big| \\ &= 2\Big|\sum_{j=n+1}^{\infty}\sum_{s=0}^{\infty} \theta^{2s+j}\Big|\end{aligned}$$

and

$$\begin{aligned}\sup_{\theta\in\Theta}\left|\frac{1}{n} \mathrm{Var}_\theta \Big(\sum_{i=1}^{n} X_i\Big) - \sigma_{\mathrm{as}}^2(\theta)\right| &= 2\sup_{\theta\in\Theta} \frac{|\theta|^{n+1}}{(1-\theta^2)(1-\theta)} \\ &\le \frac{(1-\delta)^{n+1}}{\delta^2(2-\delta)} = \mathcal{O}(A_1^n)\end{aligned}$$

for some $0 < A_1 < 1$. Therefore, we obtain condition (A1a). From (21) we have

$$\sigma_{\mathrm{as}}^2(\theta) = \sum_{s=0}^{\infty} a_s^2(\theta) + 2\sum_{j=1}^{\infty}\sum_{s=0}^{\infty} a_s(\theta)a_{s+j}(\theta) = \frac{1}{(1-\theta)^2}$$

because $a_s(\theta) = \theta^s$. Then

$$\inf_{\theta\in\Theta} \frac{1}{(1-\theta)^2} > \frac{1}{(2-\delta)^2},$$

and we obtain condition (A1b). Consequently, from Corollary 4 and Lemma 1 we get uniform CLT (3) for X_n. ∎

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