APPLICATIONES MATHEMATICAE
36,2 (2009), pp. 157–168

I. Malinowska and D. Szynal (Lublin)

# RELATIONS FOR CHARACTERISTIC FUNCTIONS OF $k$-TH RECORD VALUES FROM GENERALIZED PARETO AND INVERSE GENERALIZED PARETO DISTRIBUTION

*Abstract.* Relations for the marginal, joint, conditional characteristic functions of $k$-th upper and lower record values from generalized Pareto distribution and inverse generalized Pareto distribution are given.

**1. Introduction.** Let $\{X_n,\, n \geq 1\}$ be a sequence of independent identically distributed i.i.d. random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The $j$-th order statistic of a sample $(X_1, \ldots, X_n)$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$ we define the sequence $\{L_k(n),\, n \geq 1\}$ of *$k$-th lower record times* of $\{X_n,\, n \geq 1\}$ as follows:

$$L_k(1) = 1,$$
$$L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}, \quad n \geq 1.$$

Then $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L_k(n)+k-1},\ n \geq 1$, is called the sequence of *$k$-th lower record values* of $\{X_n, n \geq 1\}$. Note that $Z_1^{(k)} = \max\{X_1, \ldots, X_k\}$ and $Z_n^{(1)} = X_{L(n)},\ n \geq 1$, are lower record values (cf. [4]). It is known that

$$f_{Z_1^{(k)},\ldots,Z_n^{(k)}}(z_1,\ldots,z_n) = \begin{cases} k^n (F(z_n))^{k-1} f(z_n) \displaystyle\prod_{i=1}^{n-1} \frac{f(z_i)}{F(z_i)}, & z_1 > \cdots > z_n, \\ 0, & \text{elsewhere.} \end{cases}$$

Hence the pdfs of $Z_n^{(k)}$ and $(Z_m^{(k)}, Z_n^{(k)})$ are

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!} (-\ln F(x))^{n-1} (F(x))^{k-1} f(x), \quad n \geq 1, \tag{1.1}$$

---

2000 *Mathematics Subject Classification*: 62E99, 60E10.
*Key words and phrases*: generalized Pareto distribution, inverse generalized Pareto distribution, characteristic functions, $k$-th record values, moments.

DOI: 10.4064/am36-2-4  

$$(1.2)\qquad f_{Z_m^{(k)},Z_n^{(k)}}(x,y)=\frac{k^n}{(m-1)!\,(n-m-1)!}[\ln F(x)-\ln F(y)]^{n-m-1}$$
$$\times(-\ln F(x))^{m-1}\,\frac{f(x)}{F(x)}\,(F(y))^{k-1}f(y),\qquad x>y,\ n\geq 2,\ m<n,$$

respectively.

Now we recall the definition of $k$-th upper record values. With the above notation, for a fixed $k\geq 1$ we define the sequence $\{U_k(n),\,n\geq 1\}$ of $k$-th *upper record times* of $\{X_n,\,n\geq 1\}$ as follows:

$$U_k(1)=1,$$
$$U_k(n+1)=\min\{j>U_k(n):X_{j:j+k-1}>X_{U_k(n):U_k(n)+k-1}\},\quad n\geq 1.$$

Then $\{Y_n^{(k)},n\geq 1\}$ with $Y_n^{(k)}=X_{U_k(n):U_k(n)+k-1}$, $n\geq 1$, is called the sequence of *$k$-th upper record values* of $\{X_n,\,n\geq 1\}$ (cf. [2]). Note that $Y_1^{(k)}=\min\{X_1,\ldots,X_k\}$, and $Y_n^{(1)}=X_{U(n)}$, $n\geq 1$, with $U(n)=\min\{j>U(n-1):X_j>X_{U(n-1)}\}$ are upper record values. It is known that the joint pdf of $(Y_1^{(k)},\ldots,Y_n^{(k)})$ is given by

$$f_{Y_1^{(k)},\ldots,Y_n^{(k)}}(x_1,\ldots,x_n)=\begin{cases}k^n\displaystyle\prod_{i=1}^{n-1}\frac{f(x_i)}{\overline{F}(x_i)}\,(\overline{F}(x_n))^{k-1}f(x_n), & x_1<\cdots<x_n,\\ 0, & \text{elsewhere.}\end{cases}$$

and the pdfs of $Y_n^{(k)}$ and $(Y_m^{(k)},Y_n^{(k)})$ are

$$(1.3)\qquad f_{Y_n^{(k)}}(x)=\frac{k^n}{(n-1)!}\,(-\ln\overline{F}(x))^{n-1}(\overline{F}(x))^{k-1}f(x),\quad n\geq 1,$$

$$(1.4)\qquad f_{Y_m^{(k)},Y_n^{(k)}}(x,y)=\frac{k^n}{(m-1)!(n-m-1)!}\,[\ln\overline{F}(x)-\ln\overline{F}(y)]^{n-m-1}$$
$$\times(-\ln\overline{F}(x))^{m-1}\,\frac{f(x)}{\overline{F}(x)}\,(\overline{F}(y))^{k-1}f(y),\qquad x<y,\ n\geq 2,\ m<n,$$

respectively, where $\overline{F}(\cdot)=1-F(\cdot)$.

The properties of record values have been extensively studied in the literature. Recurrence relations for single and product moments of $k$-th record values for generalized Pareto, inverse generalized Pareto, Burr, inverse Burr, Weibull, exponential, generalized extreme values distributions are given in [4]–[7]. In [1], recurrence relations are given for moment generating functions of record values for Pareto and Gumbel distributions.

This paper contains relations for the marginal, joint and conditional characteristic functions of $k$-th upper and lower record values from generalized Pareto distribution and inverse generalized Pareto distribution. These relations yield as special cases the relations for moments of generalized Pareto distribution and inverse generalized Pareto distribution given in [4]–[6].

The pdf of the generalized Pareto distribution (GPar($\alpha$)) is

$$f(x) = \begin{cases} (1+\alpha x)^{-(1+\alpha^{-1})}, & x \ge 0,\, \alpha > 0, \\ (1+\alpha x)^{-(1+\alpha^{-1})}, & 0 < x < -\alpha^{-1},\, \alpha < 0 \\ e^{-x}, & x \ge 0,\ \alpha = 0, \\ 0, & \text{else.} \end{cases}$$

Note that

$$f(x)(1+\alpha x) = 1 - F(x) \quad \text{(cf. [5])}. \tag{1.5}$$

The pdf of the inverse generalized Pareto distribution (IGPar($\theta$,$\tau$)) is

$$f(x) = \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}, \qquad x > 0,\, \theta > 0,\, \tau > 0,$$

and one can observe that

$$x(x+\theta)f(x) = \theta\tau F(x) \quad \text{(cf. [6])}. \tag{1.6}$$

**2. Relation for the marginal characteristic function.** We start our study with the generalized Pareto distribution. Using (1.3) and (1.5) we obtain the following relation for the marginal characteristic function of $k$-th record values. Let ${}_k\varphi_n(t)$ denote the characteristic function of the $k$-th upper record value $Y_n^{(k)}$.

THEOREM 2.1. *For $k \ge 1$, $n \ge 2$, and $\alpha \ne 0$,*

$$(k - it)\,{}_k\varphi_n(t) - \alpha t \frac{d}{dt}\,{}_k\varphi_n(t) = k\,{}_k\varphi_{n-1}(t), \qquad \textit{where } i = \sqrt{-1}. \tag{2.1}$$

*Proof.* For GPar($\alpha$) we have

$${}_k\varphi_n(t) = \mathrm{E}\, e^{itY_n^{(k)}} = \int_0^\infty e^{itx} f_{Y_n^{(k)}}(x)\,dx.$$

Thus by (1.3) and (1.5) we get

$${}_k\varphi_n(t) + \frac{\alpha}{i}\frac{d}{dt}\,{}_k\varphi_n(t) = \frac{k^n}{(n-1)!}\int_0^\infty e^{itx}(-\ln \overline{F}(x))^{n-1}(\overline{F}(x))^k dx. \tag{2.2}$$

Integrating the right side of (2.2) we obtain

$$\begin{aligned} {}_k\varphi_n(t) + \frac{\alpha}{i}\frac{d}{dt}\,{}_k\varphi_n(t) &= \frac{k^{n+1}}{it(n-1)!}\int_0^\infty e^{itx}(-\ln \overline{F}(x))^{n-1}(\overline{F}(x))^{k-1} f(x)\,dx \\ &\quad - \frac{k^n}{it(n-2)!}\int_0^\infty e^{itx}(-\ln \overline{F}(x))^{n-2}(\overline{F}(x))^{k-1} f(x)\,dx \\ &= \frac{k}{it}\,{}_k\varphi_n(t) - \frac{k}{it}\,{}_k\varphi_{n-1}(t), \end{aligned}$$

which gives (2.1).

Corollary 2.1.1. *For $n \geq 2$ and $j = 0, 1, 2, \ldots,$*

$$(2.3)\qquad (k - it - j\alpha)({}_k\varphi_n(t))^{(j)} = k({}_k\varphi_{n-1}(t))^{(j)} + ij({}_k\varphi_n(t))^{(j-1)} + \alpha t({}_k\varphi_n(t))^{(j+1)},$$

*where $({}_k\varphi_n(t))^{(j)}$ is the $j$-th derivative of ${}_k\varphi_n(t)$ with $({}_k\varphi_n(t))^{(-1)} = 0$.*

Corollary 2.1.2. *For the exponential distribution, i.e. when $\alpha = 0$, we have*

$$(2.4)\qquad (k - it)({}_k\varphi_n(t))^{(j)} = k({}_k\varphi_{n-1}(t))^{(j)} + ij({}_k\varphi_n(t))^{(j-1)},$$

*with $({}_k\varphi_n(t))^{(-1)} = 0$.*

In [3], "a new characterization of exponential distribution" (Theorem 1.1) is given together with an application to constructing goodness-of-fit tests for exponentiality. Namely, it is proved that $X$ has exponential distribution iff

$$(2.5)\qquad \mathrm{E}(\sin(tX)) - t\,\mathrm{E}(\cos(tX)) = 0, \qquad t \in \mathbb{R}.$$

We see that this condition can be derived from (2.4) after letting $j = 0$, $k = 1$ and $n = 1$. Moreover, we note that for $j = 1$, $k = 1$ and $n = 1$ in (2.4) on can get

$$(1 - it)(\varphi(t))^{(1)} = i\varphi(t).$$

From this equation we conclude that $X$ has exponential distribution iff

$$\begin{cases} \mathrm{E}(1 - X)\sin tX + t\,\mathrm{E}\,X\cos X = 0, \\ \mathrm{E}(X - 1)\cos tX + t\,\mathrm{E}\,X\sin X = 0, \end{cases}$$

Using these conditions one can try to construct a goodness-of-fit test for exponentiality as was done in [3].

Remark 2.1.1. Setting $t = 0$ in (2.3) and replacing $j$ by $j + 1$ we get recurrence relations for single moments of $k$-th record values from generalized Pareto distribution:

$$\mathrm{E}(Y_n^{(k)})^{j+1} = \frac{k}{k - (j+1)\alpha}\,\mathrm{E}(Y_{n-1}^{(k)})^{j+1} + \frac{j+1}{k - (j+1)\alpha}\,\mathrm{E}(Y_n^{(k)})^j \quad \text{(cf. [4])}.$$

Remark 2.1.2. Setting $t = 0$ in (2.4) and replacing $j$ by $j + 1$ we get recurrence relations for single moments of $k$-th record values from exponential distribution:

$$\mathrm{E}(Y_n^{(k)})^{j+1} = \mathrm{E}(Y_{n-1}^{(k)})^{j+1} + \frac{j+1}{k}\,\mathrm{E}(Y_n^{(k)})^j \qquad \text{(cf. [4])}.$$

Now let ${}_k\psi_n(t)$ denote the characteristic function of the $k$-th lower record value $Z_n^{(k)}$ from IGPar$(\theta, \tau)$, i.e.

$${}_k\psi_n(t) = \mathrm{E}\,e^{itZ_n^{(k)}} = \int_0^\infty e^{itx} f_{Z_n^{(k)}}(x)\,dx.$$

THEOREM 2.2. *For $k \geq 1$ and $n \geq 2$,*

$$\frac{t}{\tau k}\frac{d}{dt}\,{}_k\psi_n(t) - \frac{it}{\tau\theta k}\frac{d^2}{dt^2}\,{}_k\psi_n(t) = {}_k\psi_{n-1}(t) -{}_k \psi_n(t). \tag{2.6}$$

*Proof.* Using (1.1) and (1.6) we have

$$\frac{\theta}{i}\frac{d}{dt}\,{}_k\psi_n(t) - \frac{d^2}{dt^2}\,{}_k\psi_n(t) = \frac{\theta\tau k^n}{(n-1)!}\int_0^\infty e^{itx}(-\ln F(x))^{n-1}(F(x))^k\,dx. \tag{2.7}$$

Integrating the right side of (2.7) by parts we obtain

$$\begin{aligned}\frac{\theta}{i}\frac{d}{dt}\,{}_k\psi_n(t) &- \frac{d^2}{dt^2}\,{}_k\psi_n(t)\\ &= \frac{\theta\tau k^n}{it(n-2)!}\int_0^\infty e^{itx}(-\ln F(x))^{n-2}(F(x))^{k-1}f(x)\,dx\\ &\quad - \frac{\theta\tau k^{n+1}}{it(n-1)!}\int_0^\infty e^{itx}(-\ln F(x))^{n-1}(F(x))^{k-1}f(x)\,dx\\ &= \frac{\theta\tau k}{it}[\,{}_k\psi_{n-1}(t) - {}_k\psi_n(t)],\end{aligned}$$

which gives (2.6).

COROLLARY 2.2.1. *For $n \geq 2$ and $j = 0, 1, 2, \ldots,$*

$$\begin{aligned}({}_k\psi_n(t))^{(j)} = \frac{k\tau}{j+k\tau}\,({}_k\psi_{n-1}(t))^{(j)} &- \frac{t\theta - ij}{\theta(j+k\tau)}\,({}_k\psi_n(t))^{(j+1)}\\ &+ \frac{it}{\theta(j+k\tau)}\,({}_k\psi_n(t))^{(j+2)},\end{aligned} \tag{2.8}$$

*where $({}_k\psi_n(t))^{(j)}$ is the $j$-th derivative of ${}_k\psi_n(t)$.*

REMARK 2.2.1. Setting $t = 0$ in (2.8) we get relations for single moments of $k$-th record values from IGPar$(\theta, \tau)$:

$$\mathrm{E}(Z_n^{(k)})^j = \frac{k\tau}{j+k\tau}\,\mathrm{E}(Z_{n-1}^{(k)})^j - \frac{j}{\theta(j+k\tau)}\,\mathrm{E}(Z_n^{(k)})^{j+1} \qquad \text{(cf. [6]).}$$

**3. Relations for the joint characteristic function.** Let ${}_k\varphi_{m,n}(t_1, t_2)$ denote the characteristic function of the $m$-th and $n$-th upper $k$-th record values $Y_m^{(k)}$ and $Y_n^{(k)}$ from GPar$(\alpha)$, i.e

$${}_k\varphi_{m,n}(t_1, t_2) = \mathrm{E}\,e^{it_1 Y_m^{(k)}}e^{it_2 Y_n^{(k)}} = \int_0^\infty\int_0^\infty e^{it_1x}e^{it_2y}f_{Y_m^{(k)},Y_n^{(k)}}(x,y)\,dx\,dy.$$

THEOREM 3.1. *For $k \geq 1$, $2 \leq m < n-1$, and $\alpha \neq 0$,*

$$(3.1)\qquad it_1\, {}_k\varphi_{m,n}(t_1,t_2) + t_1\alpha \frac{\partial}{\partial t_1}\, {}_k\varphi_{m,n}(t_1,t_2) = k({}_k\varphi_{m,n-1}(t_1,t_2) - {}_k\varphi_{m-1,n-1}(t_1,t_2)).$$

*Proof.* Using (1.4) and (1.5) we get

$$(3.2)\qquad {}_k\varphi_{m,n}(t_1,t_2) + \frac{\alpha}{i}\frac{\partial}{\partial t_1}\, {}_k\varphi_{m,n}(t_1,t_2)$$
$$= \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_0^y e^{it_1x}e^{it_2y}(-\ln \overline{F}(x))^{m-1}$$
$$\times (\overline{F}(y))^{k-1}[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1} f(y)\,dx\,dy$$
$$= \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty e^{it_2y}(\overline{F}(y))^{k-1} f(y) I(y)\,dy$$

where

$$(3.3)\qquad I(y) = \int_0^y e^{it_1x}(-\ln \overline{F}(x))^{m-1}[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}dx.$$

Integrating the right side of (3.3), we obtain

$$(3.4)\qquad I(y) = \frac{n-m-1}{it_1}\int_0^y e^{it_1x}(-\ln \overline{F}(x))^{m-1}$$
$$\times [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-2}\frac{f(x)}{\overline{F}(x)}\,dx - \frac{m-1}{it_1}$$
$$\times \int_0^y e^{it_1x}(-\ln \overline{F}(x))^{m-2}[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}\frac{f(x)}{\overline{F}(x)}\,dx.$$

Making use of (3.4) in (3.2), after some simplifications we are led to (3.1).

COROLLARY 3.1.1. *For $2 \leq m \leq n-2$ and $j, s = 0, 1, 2, \ldots,$*

$$(3.5)\qquad ({}_k\varphi_{m-1,n-1}(t_1,t_2))^{(j,s)} = ({}_k\varphi_{m,n-1}(t_1,t_2))^{(j,s)}$$
$$- \frac{it_1 + j\alpha}{k}({}_k\varphi_{m,n}(t_1,t_2))^{(j,s)} - \frac{ji}{k}({}_k\varphi_{m,n}(t_1,t_2))^{(j,s)}$$
$$- \frac{\alpha t_1}{k}({}_k\varphi_{m,n}(t_1,t_2))^{(j+1,s)},$$

*where $({}_k\varphi_{m,n}(t_1,t_2))^{(j,s)} = \frac{\partial^{j+s}}{\partial t_1^j \partial t_2^s}\, {}_k\varphi_{m,n}(t_1,t_2)$ with $({}_k\varphi_{m,n}(t_1,t_2))^{(-1,s)} = 0$.*

REMARK 3.1.1. Letting $\alpha = 0$, one can get recurrence relations for the characteristic functions of joint $k$-th record values from exponential distribution.

REMARK 3.1.2. Setting $t_1 = 0$ and $t_2 = 0$ in (3.5) we get a relation for product moments of $k$-th upper record values from generalized Pareto distribution:

$$\mathrm{E}(Y_{m-1}^{(k)})^j(Y_{n-1}^{(k)})^s = \mathrm{E}(Y_m^{(k)})^j(Y_{n-1}^{(k)}) - \frac{j\alpha}{k}\,\mathrm{E}(Y_m^{(k)})^j(Y_n^{(k)})^s - \frac{j}{k}\,\mathrm{E}(Y_m^{(k)})^{j-1}(Y_n^{(k)})^s.$$

REMARK 3.1.3. Letting $t_1 = 0$ and $t_2 = 0$ in (3.5) with $\alpha = 0$ and replacing $j$ by $j+1$ one can get relations for product moments of $k$-th record values from exponential distribution (cf. [7]).

Now let ${}_k\psi_{m,n}(t_1,t_2)$ denote the characteristic function of the $m$-th and $n$-th lower $k$-th record values $Z_m^{(k)}$ and $Z_n^{(k)}$. Making use of (1.2) and (1.6) we get relations for $\mathrm{IGPar}(\theta,\tau)$:

THEOREM 3.2. *For $k \geq 1$ and $2 \leq m < n-1$,*

$$\frac{t_1}{\tau}\frac{\partial}{\partial t_1}\,{}_k\psi_{m,n}(t_1,t_2) - \frac{it_1}{\theta\tau}\frac{\partial^2}{\partial t_1^2}\,{}_k\psi_{m,n}(t_1,t_2) = k({}_k\psi_{m-1,n-1}(t_1,t_2) - {}_k\psi_{m,n-1}(t_1,t_2)). \tag{3.6}$$

*Proof.* Using (1.2) and (1.6) we have

$$\frac{\theta}{i}\frac{\partial}{\partial t_1}\,{}_k\psi_{m,n}(t_1,t_2) - \frac{\partial^2}{\partial t_1^2}\,{}_k\psi_{m,n}(t_1,t_2) = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty e^{it_2y}(F(y))^{k-1}f(y)D(y)\,dy \tag{3.7}$$

where

$$D(y) = \tau\theta\int_y^\infty e^{it_1x}(-\ln F(x))^{m-1}[\ln F(x) - \ln F(y)]^{n-m-1}dx. \tag{3.8}$$

Integrating the right side of (3.8), we get

$$D(y) = -\tau\theta\,\frac{n-m-1}{it_1}\int_y^\infty e^{it_1x}(-\ln F(x))^{m-1} \times [\ln F(x) - \ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}\,dx + \tau\theta\,\frac{m-1}{it_1} \times \int_y^\infty e^{it_1x}(-\ln F(x))^{m-2}[\ln F(x) - \ln F(y)]^{n-m-1}\frac{f(x)}{F(x)}\,dx. \tag{3.9}$$

Making use of (3.9) in (3.7), after some simplifications we get (3.6).

Corollary 3.2.1. *For* $2 \le m \le n-2$ *and* $j, s = 0, 1, 2, \ldots$,

$$(3.10)\qquad ({}_k\psi_{m-1,n-1}(t_1,t_2))^{(j,s)} = ({}_k\psi_{m,n-1}(t_1,t_2))^{(j,s)} + \frac{j}{k\tau}({}_k\psi_{m,n}(t_1,t_2))^{(j,s)} + \frac{t_1\theta - ji}{k\theta\tau}({}_k\psi_{m,n}(t_1,t_2))^{(j+1,s)} - \frac{it_1}{k\theta\tau}({}_k\psi_{m,n}(t_1,t_2))^{(j+2,s)},$$

*where*

$$({}_k\psi_{m,n}(t_1,t_2))^{(j,s)} = \frac{\partial^{j+s}}{\partial t_1^j \partial t_2^s}({}_k\psi_{m,n}(t_1,t_2)).$$

Remark 3.2.1. Setting $t_1 = 0$ and $t_2 = 0$ in (3.10), we get relations between product moments of $k$-th lower record values from IGPar$(\theta, \tau)$:

$$\mathrm{E}(Z_{m-1}^{(k)})^j (Z_{n-1}^{(k)})^s = \mathrm{E}(Z_m^{(k)})^j (Z_{n-1}^{(k)})^s + \frac{j}{k\tau}\mathrm{E}(Z_m^{(k)})^j (Z_{n-1}^{(k)})^s + \frac{j}{k\theta\tau}\mathrm{E}(Z_m^{(k)})^{j+1}(Z_n^{(k)})^s.$$

**4. Relations for the conditional characteristic function.** Let $Y_n^{(k)}$, $n \ge 1$, be the $k$-th upper values from GPar$(\alpha)$ and let ${}_k\varphi_{n|m}(t)$ be the conditional characteristic function of $Y_n^{(k)}$ given $Y_m^{(k)} = x$. We have

Theorem 4.1. *For* $k \ge 1$, $1 \le m < n-1$, *and* $\alpha \ne 0$,

$$(4.1)\qquad (k - it)\,{}_k\varphi_{n|m}(t) - t\alpha\frac{d}{dt}\,{}_k\varphi_{n|m}(t) = k({}_k\varphi_{n-1|m}(t)).$$

*Proof.* From (1.3) and (1.4) we have

$$(4.2)\qquad f_{Y_n^{(k)}|Y_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!(\overline{F}(x))^k} \times [\ln\overline{F}(x) - \ln\overline{F}(y)]^{n-m-1}(\overline{F}(y))^{k-1}f(y), \quad x < y,\ 1 \le m < n.$$

Using (4.2) and (1.5) we get

$$(4.3)\qquad {}_k\varphi_{n|m}(t) + \frac{\alpha}{i}\frac{d}{dt}\,{}_k\varphi_{n|m}(t) = \frac{k^{n-m}}{(n-m-1)!(\overline{F}(x))^k} \times \int_x^\infty e^{ity}[\ln\overline{F}(x) - \ln\overline{F}(y)]^{n-m-1}(\overline{F}(y))^k dy.$$

Integrating the right side of (4.3) leads to (4.1).

Corollary 4.1.1. *For* $1 \le m < n-1$ *and* $j = 0, 1, 2, \ldots$,

$$(4.4)\qquad (k - it - j\alpha)({}_k\varphi_{n|m}(t))^{(j)} = k({}_k\varphi_{n-1|m}(t))^{(j)} + ij({}_k\varphi_{n|m}(t))^{(j-1)} + t\alpha({}_k\varphi_{n|m}(t))^{(j+1)},$$

*where* $({}_k\varphi_{n|m}(t))^{(j)} = \frac{d^j}{dt^j}({}_k\varphi_{n|m}(t))$ *with* $({}_k\varphi_{n|m}(t))^{(-1)} = 0$.

Let ${}_k\varphi_{m|n}(t)$ be the conditional characteristic function of $Y_m^{(k)}$ given $Y_n^{(k)} = y$. Then we have

THEOREM 4.2. *For* $k \geq 1$, $2 \leq m < n-1$, *and* $\alpha \neq 0$,

$$it\, {}_k\varphi_{m|n}(t) + t\alpha \frac{d}{dt}\, {}_k\varphi_{m|n}(t) = \frac{n-1}{\ln \overline{F}(y)}\, {}_k\varphi_{m-1|n-1}(t) - \frac{n-1}{\ln \overline{F}(y)}\, {}_k\varphi_{m|n-1}(t). \tag{4.5}$$

*Proof.* From (1.3) and (1.4) we have

$$f_{Y_m^{(k)}|Y_n^{(k)}}(x|y) = \frac{(n-1)!}{(m-1)!(n-m-1)!(-\ln \overline{F}(y))^{n-1}} \times [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}(-\ln \overline{F}(x))^{m-1} \frac{f(x)}{\overline{F}(x)}, \quad x < y,\ 1 \leq m < n. \tag{4.6}$$

Using (4.6) and (1.5) we get

$${}_k\varphi_{m|n}(t) + \frac{\alpha}{i} \frac{d}{dt}\, {}_k\varphi_{m|n}(t) = \frac{(n-1)!}{(m-1)!(n-m-1)!(-\ln \overline{F}(y))^{n-1}} \times \int_0^y e^{itx} [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}(-\ln \overline{F}(x))^{m-1}\, dx. \tag{4.7}$$

Integrating the right side in (4.7), we get (4.5).

COROLLARY 4.2.1. *For* $2 \leq m \leq n-2$ *and* $j = 1, 2, \ldots,$

$$({}_k\varphi_{m-1|n-1}(t))^{(j)} = ({}_k\varphi_{m|n-1}(t))^{(j)} + \frac{ji \ln \overline{F}(y)}{n-1} ({}_k\varphi_{m|n}(t))^{(j-1)} + \frac{(ti + j\alpha) \ln \overline{F}(y)}{n-1} ({}_k\varphi_{m|n}(t))^{(j)} + \frac{t\alpha \ln \overline{F}(y)}{n-1} ({}_k\varphi_{m|n}(t))^{(j+1)}, \tag{4.8}$$

*where* $({}_k\varphi_{m|n}(t))^{(j)} = \frac{d^j}{dt^j}({}_k\varphi_{m|n}(t))$.

REMARK 4.2.1. Setting $t = 0$ in (4.4) and (4.8), we get relations between conditional moments of $k$-th upper record values for GPar($\alpha$):

$$\mathrm{E}(Y_n^{(k)} \mid Y_m^{(k)})^{j+1} = \frac{k}{k-(j+1)\alpha} \mathrm{E}(Y_{n-1}^{(k)} \mid Y_m^{(k)})^{j+1} + \frac{j+1}{k-(j+1)\alpha} \mathrm{E}(Y_n^{(k)} \mid Y_m^{(k)})^{j},$$

$$\mathrm{E}(Y_{m-1}^{(k)} \mid Y_{n-1}^{(k)})^j = \mathrm{E}(Y_m^{(k)} \mid Y_{n-1}^{(k)})^j + \frac{j \ln \overline{F}(y)}{n-1} \mathrm{E}(Y_m^{(k)} \mid Y_n^{(k)})^{j-1} + \frac{j\alpha \ln \overline{F}(y)}{n-1} \mathrm{E}(Y_m^{(k)} \mid Y_n^{(k)})^j.$$

REMARK 4.2.2. Setting $t = 0$ in (4.4) and (4.8) with $\alpha = 0$ we get relations for conditional moments of $k$-th record values from exponential distribution:

$$\mathrm{E}(Y_n^{(k)} \mid Y_m^{(k)})^{j+1} = \mathrm{E}(Y_{n-1}^{(k)} \mid Y_m^{(k)})^{j+1} + \frac{j+1}{k} \mathrm{E}(Y_n^{(k)} \mid Y_m^{(k)})^j,$$

$$\mathrm{E}(Y_{m-1}^{(k)} \mid Y_{n-1}^{(k)})^j = \mathrm{E}(Y_m^{(k)} \mid Y_{n-1}^{(k)})^j + \frac{j \ln \overline{F}(y)}{n-1} \mathrm{E}(Y_m^{(k)} \mid Y_n^{(k)})^{j-1}.$$

Now let $Z_n^{(k)}$, $n \geq 1$, be the $k$-th lower values from IGPar$(\theta, \tau)$ and let ${}_k\psi_{n|m}(t)$ be the conditional characteristic function of $Z_n^{(k)}$ given $Z_m^{(k)} = x$. Then we have

THEOREM 4.3. *For $k \geq 1$ and $1 \leq m < n-1$,*

$$t\theta \frac{d}{dt}\, {}_k\psi_{n|m}(t) - it \frac{d^2}{dt^2}\, {}_k\psi_{n|m}(t) = k\theta\tau({}_k\psi_{n-1|m}(t) - {}_k\psi_{n|m}(t)). \tag{4.9}$$

*Proof.* From (1.1) and (1.2) we have

$$f_{Z_n^{(k)}|Z_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!(F(x))^k} [\ln F(x) - \ln F(y)]^{n-m-1} (F(y))^{k-1} f(y), \quad y < x, \leq m < n. \tag{4.10}$$

Hence by (4.10) and (1.6) we see that

$$\frac{\theta}{i} \frac{d}{dt}\, {}_k\psi_{n|m}(t) - \frac{d^2}{dt^2}\, {}_k\psi_{n|m}(t) = \frac{\theta\tau k^{n-m}}{(n-m-1)!(F(x))^k} \times \int_0^x e^{ity} [\ln F(x) - \ln F(y)]^{n-m-1} (F(y))^k \, dy. \tag{4.11}$$

Integrating the right side of (4.11), we get (4.9).

COROLLARY 4.3.1. *For $1 \leq m < n-1$ and $j = 0, 1, 2, \ldots,$*

$$({}_k\psi_{n|m}(t))^{(j)} = \frac{ji - t\theta}{\theta(j+\tau k)} ({}_k\psi_{n|m}^k(t))^{(j+1)} + \frac{it}{\theta(j+\tau k)} ({}_k\psi_{n|m}^k(t))^{(j+2)} + \frac{\tau k}{(j+\tau k)} ({}_k\psi_{n-1|m}^k(t))^{(j)}, \tag{4.12}$$

*where $({}_k\psi_{n|m}(t))^{(j)} = \frac{d^j}{dt^j}({}_k\psi_{n|m}(t))$.*

Let ${}_k\psi_{m|n}(t)$ be the conditional characteristic function of $Z_m^{(k)}$ given $Z_n^{(k)} = y$. Then we have

THEOREM 4.4. *For $k \geq 1$ and $2 \leq m < n-1$,*

$$t\theta \frac{d}{dt}\, {}_k\psi_{m|n}(t) - it \frac{d^2}{dt^2}\, {}_k\psi_{m|n}(t) = \frac{(n-1)\theta\tau}{-\ln F(y)} \left({}_k\psi_{m-1|n-1}(t) - {}_k\psi_{m|n-1}(t)\right). \tag{4.13}$$

*Proof.* From (1.1) and (1.2) we have

$$f_{Z_m^{(k)}|Z_n^{(k)}}(x|y) = \frac{(n-1)!}{(m-1)!(n-m-1)!(-\ln F(y))^{n-1}} \times [\ln F(x) - \ln F(y)]^{n-m-1}(-\ln F(x))^{m-1} \frac{f(x)}{F(x)}, \quad y < x,\ 1 \leq m < n. \tag{4.14}$$

Hence by (4.14) and (1.6) we see that

$$\frac{\theta}{i}\frac{d}{dt}\, {}_k\psi_{m|n}(t) - \frac{d^2}{dt^2}\, {}_k\psi_{m|n}(t) = \frac{\theta\tau(n-1)!}{(m-1)!(n-m-1)!(-\ln F(y))^{n-1}} \times \int_y^\infty e^{itx}[\ln F(x) - \ln F(y)]^{n-m-1}(-\ln F(x))^{m-1}\,dx. \tag{4.15}$$

Integrating the right side of (4.15), we get (4.13).

COROLLARY 4.4.1. *For $2 \leq m \leq n-2$ and $j = 0, 1, 2, \ldots,$*

$$({}_k\psi_{m-1|n-1}(t))^{(j)} = ({}_k\psi_{m|n-1}(t))^{(j)} - \frac{j \ln F(y)}{\tau(n-1)} ({}_k\psi_{m|n}(t))^{(j)} - \frac{(t\theta - ji)\ln F(y)}{\theta\tau(n-1)} ({}_k\psi_{m|n}(t))^{(j+1)} + \frac{it \ln F(y)}{\theta\tau(n-1)} ({}_k\psi_{m|n}(t))^{(j+2)} \tag{4.16}$$

*where* $({}_k\psi^k_{m|n}(t))^{(j)} = \frac{d^j}{dt^j}({}_k\psi^k_{m|n}(t))$.

REMARK 4.4.1. Setting $t = 0$ in (4.12) and (4.16), we get relations between conditional moments of $k$-th lower record values for IGPar$(\theta, \tau)$:

$$\mathrm{E}(Z_n^{(k)} \mid Z_m^{(k)})^j = -\frac{j}{\theta(j+\tau k)} \mathrm{E}(Z_n^{(k)} \mid Z_m^{(k)})^{j+1} + \frac{\tau k}{(j+\tau k)} \mathrm{E}(Z_{n-1}^{(k)} \mid Z_m^{(k)})^j,$$

$$\mathrm{E}(Z_{m-1}^{(k)} \mid Z_{n-1}^{(k)})^j = \mathrm{E}(Z_m^{(k)} \mid Z_{n-1}^{(k)})^j - \frac{j \ln F(y)}{\tau(n-1)} \mathrm{E}(Z_m^{(k)} \mid Z_n^{(k)})^j - \frac{j \ln F(y)}{\theta\tau(n-1)} \mathrm{E}(Z_m^{(k)} \mid Z_n^{(k)})^{j+1}.$$

## References


[1] M. Ahsanullah and M. Z. Raqab, *Relations for marginal and joint moment generating of record values from power function distribution*, J. Appl. Statist. Sci. 10 (2000), 27–36.

[2] W. Dziubdziela and B. Kopociński, *Limiting properties of the k-th record values*, Zastos. Mat. 15 (1976), 187–190.

[3] N. Henze and S. G. Meitanis, *Goodness-of-fit based on a new characterization of the exponential distribution*, Comm. Statist. 31 (2002), 1479–1497.

[4] P. Pawlas and D. Szynal, *Relations for single and product moments of k-th record values from exponential and Gumbel distributions*, J. Appl. Statist. Sci. 7 (1998), 53–62.

[5] —, —, *Recurrence relations for single and product moments of k-th record values from Pareto, generalized Pareto and Burr distributions*, Comm. Statist. 28 (1999), 1699–1709.

[6] —, —, *Relations for single and product moments of k-th lower record values from the inverse distributions of Pareto's type and characterizations*, Discussiones Math. 20 (2000), pp. 223–231.

[7] —, —, *Relations for single and product moments of k-th lower record values from Weibull distribution and a characterization*, J. Appl. Statist. Sci. 10 (2000), 17–26.



Iwona Malinowska
Department of Mathematics
Lublin University of Technology
Nadbystrzycka 38a
20-618 Lublin, Poland
E-mail: i.malinowska@pollub.pl

Dominik Szynal
Institute of Mathematics
Maria Curie-Skłodowska University
Pl. M. Curie-Skłodowskiej 1
20-031 Lublin, Poland
E-mail: szynal@poczta.umcs.lublin.pl



*Received on 30.1.2008;*
*revised version on 24.3.2009* (1911)