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GLOBAL EXISTENCE OF SOLUTIONS TO NAVIER–STOKES EQUATIONS IN CYLINDRICAL DOMAINS

Abstract. We prove the existence of global and regular solutions to the Navier–Stokes equations in cylindrical type domains under boundary slip conditions, where coordinates are chosen so that the x_3-axis is parallel to the axis of the cylinder. Regular solutions have already been obtained on the interval $[0,T]$, where $T>0$ is large, on the assumption that the L_2-norms of the third component of the force field, of derivatives of the force field, and of the velocity field with respect to the direction of the axis of the cylinder are small. In this paper we continue the solution to all times.

1. Introduction. We consider the following initial-boundary value problem:

$$
\begin{aligned}
&v_{,t}+v\cdot\nabla v-\operatorname{div}\mathbb{T}(v,p)=f &&\text{in } \Omega\times(0,\infty),\\
&\operatorname{div} v=0 &&\text{in } \Omega\times(0,\infty),\\
&v\cdot n=0 &&\text{on } S\times(0,\infty),\\
&n\cdot\mathbb{T}(v,p)\cdot\tau_\alpha=0,\quad \alpha=1,2, &&\text{on } S\times(0,\infty),\\
&v|_{t=0}=v(0) &&\text{in } \Omega.
\end{aligned}
\tag{1.1}
$$

The domain Ω is an open and bounded subset of $\mathbb{R}^3$ of cylindrical type, not axially symmetric but parallel to the x_3-axis in the Cartesian coordinate system $x=(x_1,x_2,x_3)$. The velocity field is denoted by $v=v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))\in\mathbb{R}^3$, the external force field is denoted by $f=f(x,t)=(f_1(x,t),f_2(x,t),f_3(x,t))\in\mathbb{R}^3$, and $p=p(x,t)\in\mathbb{R}^1$ is the pressure. We denote by n the unit outward normal vector and by τ_α, $\alpha=1,2$, the tangent vectors to the boundary S. Moreover, $\mathbb{T}(v,p)$ is the stress tensor,

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which is equal to $\nu\mathbb{D}(v) - p\mathbb{I}$, where $\mathbb{D}(v) = \nabla v + (\nabla v)^T$ is the symmetric dilatation tensor and $\nu > 0$ is the constant viscosity coefficient.

The aim of this paper is to show the global in time existence of regular solutions to (1.1). We base on [4], where the existence of a regular solution for large time has been proved by the Leray–Schauder fixed point theorem. Using the existence of the solution on $[0,T]$ we continue it on $\mathbb{R}_+$ by a recursive procedure employing some cut-off functions. A similar technique has been used in [2]. The main results of the paper are stated in the following theorems.

THEOREM 1 (local existence). *Let*

$$\delta_k(T) := \|f_{,x_3}\|_{L_2(\Omega\times(kT,(k+1)T))} + \|f_3\|_{L_2(S_2\times(kT,(k+1)T))} + \|v_{,x_3}(kT)\|_{L_2(\Omega)},$$

where $k \in \mathbb{N}$. *Assume that*

$$\begin{aligned}
&f \in L_\infty(kT,(k+1)T;L_{6/5}(\Omega)) \cap L_2(\Omega\times(kT,(k+1)T)),\\
&f_3 \in L_2(S_2\times(kT,(k+1)T)),\\
&(\operatorname{rot} f)_3 \in L_2(kT,(k+1)T;L_{6/5}(\Omega)),\\
&f_{,x_3} \in L_2(\Omega\times(kT,(k+1)T)) \cap L_\sigma(\Omega\times(kT,(k+1)T)),\\
&v_{,x_3} \in L_\sigma(\Omega\times((k-1)T,kT))
\end{aligned}$$

and $v(kT) \in H^1(\Omega)$. *Then*, *if* $\delta_k(T)$ *is small enough*, *then there exists a solution to* (1.1) *such that*

$$\|v_{,x_3}\|_{W^{2,1}_\sigma(\Omega\times(kT,(k+1)T))} + \|\nabla p_{,x_3}\|_{L_\sigma(\Omega\times(kT,(k+1)T)} < A$$

and

$$\|v\|_{W^{2,1}_2(\Omega\times(kT,(k+1)T))} + \|\nabla p\|_{L_2(\Omega\times(kT,(k+1)T)} < c(A^2+1)$$

for any $\sigma \in (25/8, 10/3)$. *The constant* A *is chosen for a given* T *and it satisfies the inequalities*

$$\varphi(3A + D_k)\delta_k(T) + cE_k \le A, \qquad cE_k < A,$$

where φ *is some nonlinear*, *positive and increasing function*, *the constant* c *comes from an imbedding theorem for Sobolev spaces*, *and the constants* D_k *and* E_k *are given by*

$$\begin{aligned}
D_k :=\ & \|f\|_{L_\infty(kT,(k+1)T;L_{6/5}(\Omega))} + \|f_3\|_{L_2(S_2\times(kT,(k+1)T))} + \|f\|_{L_2(\Omega\times(kT,(k+1)T))}\\
&+ \|(\operatorname{rot} f)_3\|_{L_2(kT,(k+1)T;L_{6/5}(\Omega))} + \|f_{,x_3}\|_{L_2(\Omega\times(kT,(k+1)T))} + d_1 + d_2,\\
E_k :=\ & \|f_{,x_3}\|_{L_\sigma(\Omega\times(kT,(k+1)T))},
\end{aligned}$$

where d_1 *and* d_2 *come from the energy estimates of weak solutions to the problem* (1.1) (*see Lemma* 2.2).

THEOREM 2 (global existence). *Under the assumptions of Theorem* 1 *on external data there exists a global and regular solution to the problem* (1.1) *such that*

$$\|v_{,x_3}\|_{W^{2,1}_\sigma(\Omega\times(kT,(k+1)T))} + \|\nabla p_{,x_3}\|_{L_\sigma(\Omega\times(kT,(k+1)T))} < A$$

and

$$\|v\|_{W^{2,1}_2(\Omega\times(kT,(k+1)T))} + \|\nabla p\|_{L_2(\Omega\times(kT,(k+1)T))} < c(A^2+1),$$

where σ and A are as in Theorem 1 *and the constant A does not depend on k.*

THEOREM 3 (uniqueness). *Any solution to the problem* (1.1) *which belongs to the space $L_\infty(kT,(k+1)T;W^1_3(\Omega))$ is unique.*

The proof of Theorem 1 in the case $k=0$ is presented in [4]. In this paper we will show how to obtain the constant A independent of k.

2. Notation and auxiliary results. Let Ω^{kT} denote $\Omega\times(kT,(k+1)T)$. We introduce the spaces

$$V^m_2(\Omega^{kT}) = \Big\{u\colon \|u\|_{V^m_2(\Omega^{kT})} = \operatorname*{ess\,sup}_{t\in(kT,(k+1)T)} \|v\|_{H^m(\Omega)} + \Big(\int_{kT}^{(k+1)T} \|\nabla u\|^2_{H^m(\Omega)}\,dt\Big)^{1/2} < \infty\Big\},$$

$$W^{2,1}_\sigma(\Omega^{kT}) = \Big\{u\colon \|v\|_{W^{2,1}_\sigma(\Omega^{kT})} = \Big(\int_{\Omega^{kT}} (|u|^\sigma + |Du|^\sigma + |D^2u|^\sigma + |\partial_t u|^\sigma)\,dx\,dt\Big)^{1/\sigma} < \infty\Big\}$$

and

$$W^l_\sigma(\Omega) = \Big\{u\colon \|u\|^\sigma_{W^l_\sigma(\Omega)} = \|u\|_{L_\sigma(\Omega)} + \sum_{0\le l'\le[l]} \int_\Omega |D^{l'}_x u(x)|^\sigma\,dx + \int_\Omega\int_\Omega \frac{|D^{[l]}_x u(x) - D^{[l]}_{x'} u(x')|^\sigma}{|x-x'|^{3+\sigma(l-[l])}}\,dx\,dx' < \infty\Big\},$$

where $k,m\in\mathbb{N}\cup\{0\}$, l is any positive real number and L_σ is the Lebesgue space.

Our approach requires the energy estimates of weak solutions to the problem (1.1). They are obtained by application of the following

LEMMA 2.1 (Korn inequality). *Assume that $v\in H^1(\Omega)$ satisfies*

$$\|\mathbb{D}(v)\|^2_{L_2(\Omega)} < \infty,$$
$$v\cdot n|_S = 0,$$
$$\operatorname{div} v = 0.$$

If Ω is not axially symmetric, then there exists a constant c_1 such that

$$\|v\|^2_{H^1(\Omega)} \le c_1 \|\mathbb{D}(v)\|^2_{L_2(\Omega)}.$$

The proof can be found in [6, Sec. 1, Lemma 1.2].

Our considerations involve three global quantities:

$$a_1 = \sup_t \|f(t)\|_{L_{6/5}(\Omega)},$$
$$d_1^2 = \frac{c}{\nu_1}\, a_1^2 + \|v(0)\|^2_{L_2(\Omega)},$$
$$d_2^2 = (\min(1,\nu_2))^{-1} e^{\nu_1 T}\left(\frac{c}{\nu_1}\, a_1^2 + d_1^2\right),$$

which do not depend on $k \in \mathbb{N}$ and $\nu/c_1 = \nu_1 + \nu_2$, where c_1 is the constant from the Korn inequality (Lemma 2.1).

Finally we can present the energy estimates of weak solutions to the problem (1.1).

Lemma 2.2. *Assume that $a_1 < \infty$, $v(0) \in L_2(\Omega)$ and $T > 0$ are given. Then*

$$\|v(t)\|_{L_2(\Omega)} \le d_1 \qquad \text{for any } t \ge 0,$$
$$\|v\|_{V_2^0(\Omega\times(kT,t))} \le d_2 \qquad \text{for } t \in \big(kT,(k+1)T\big),\ k \in \mathbb{N}.$$

For convenience of notation we will use the following quantities that have been introduced in [4]:

$$h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3}, \quad \chi = (\operatorname{rot} v)_3 = v_{2,x_1} - v_{1,x_2},$$
$$w = v_3, \quad F_3 = (\operatorname{rot} f)_3.$$

These functions solve equations and satisfy estimates which we recall in the following lemmas.

Lemma 2.3. *The pair of functions (h,q) is a solution to the problem*

$$\begin{aligned}
&h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + g && \text{in } \Omega^T \equiv \Omega\times(0,T),\\
&\operatorname{div} h = 0 && \text{in } \Omega^T,\\
&h\cdot n = 0, \quad n\cdot\mathbb{T}(h,q)\cdot\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T \equiv S_1\times(0,T),\\
&h_i = 0, \quad i = 1,2, \quad h_{3,x_3} = 0 && \text{on } S_2^T \equiv S_2\times(0,T),\\
&h|_{t=0} = h(0) && \text{in } \Omega.
\end{aligned} \tag{2.1}$$

Lemma 2.4. *Let v be a weak solution to the problem (1.1). Assume that $h \in L_\infty(0,T;L_3(\Omega))$, $g \in L_2(\Omega^T)$, $f_3 \in L_2(S_2^T)$ and $h(0) \in L_2(\Omega)$. Then*

$$\|h\|^2_{V_2^0(\Omega^t)} \le c d_2^2 \|h\|^2_{L_\infty(0,t;L_3(\Omega))} + c(\|f_3\|^2_{L_2(S_2^t)} + \|g\|^2_{L_2(\Omega^t)} + \|h(0)\|^2_{L_2(\Omega)})$$

for all $t \le T$.

Lemma 2.5. *Let v be a weak solution to the problem* (1.1). *Assume that* $\nabla v \in L_2(0,T;L_3(\Omega))$, $h \in L_\infty(0,T;L_3(\Omega))$, $g \in L_2(\Omega^T)$, $f_3 \in L_2(S_2^T)$ *and* $h(0) \in L_2(\Omega)$. *Then*

$$\|h\|_{L_2(\Omega^t)} \le c(\|\nabla v\|_{L_2(0,t;L_3(\Omega))} \exp(c\|\nabla v\|^2_{L_2(0,t;L_3(\Omega))}) + 1) \cdot (\|f_3\|_{L_2(S_2^t)} + \|g\|_{L_2(\Omega^t)} + \|h(0)\|_{L_2(\Omega)})$$

for all $t \le T$.

Lemma 2.6. *Let q and f_3 be given. Then w is a solution to the problem*

$$\begin{aligned} &w_{,t} + v\cdot\nabla w - \nu\Delta w = q + f_3 && \text{in } \Omega^T,\\ &w_{,n} = 0 && \text{on } S_1^T,\\ &w = 0 && \text{on } S_2^T,\\ &w|_{t=0} = w(0) && \text{in } \Omega. \end{aligned}$$

Lemma 2.7. *Let F_3, h and v be given. Then χ is a solution to the problem*

$$\begin{aligned} &\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi = F_3 && \text{in } \Omega^T,\\ &\chi = v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + v\cdot\tau_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\ &\chi_{,x_3} = 0 && \text{on } S_2^T,\\ &\chi|_{t=0} = \chi(0) && \text{in } \Omega. \end{aligned} \tag{2.2}$$

For the detailed proofs of Lemmas 2.3–2.7 we refer the reader to [6].

3. Estimates. In this section we will present the estimates for v and h in the norms of $W_2^{2,1}(\Omega^t)$ and $W_\sigma^{2,1}(\Omega^t)$ respectively (σ will be defined later, see Lemma 3.4) in terms of the initial and the external data and of the quantity $\|h\|_{L_\infty(0,t;L_3(\Omega))}$. These estimates are obtained on any time interval of the form $(kT,(k+1)T)$ by application of cut-off functions defined by

$$\zeta^{(k_n)}(t) = \begin{cases} 1 & \text{for } t \in ((k-n)T,(k+1)T),\\ 0 & \text{for } t \le (k-n-1)T, \end{cases}$$

where $\zeta^{(k_n)} \in \mathcal{C}_0^\infty(0,\infty)$ and $\dot\zeta^{(k_n)} \le 1/T$. It is easy to see that for fixed k and increasing n we have the inclusions $\operatorname{supp}\zeta^{(k_0)} \subset \operatorname{supp}\zeta^{(k_1)} \subset \cdots \subset \operatorname{supp}\zeta^{(k_n)}$.

From now on we will use the notation $u^{(k_n)} = u\cdot\zeta^{k_n}$, where $0 \le t \le (k+1)T$.

The first step is to estimate the third component of the vorticity field, which we denote by χ. Since we integrate by parts, we expect the boundary integrals to vanish. Therefore we consider a function $\bar\chi$ defined as a solution to the problem

$$
(3.1)\qquad
\begin{aligned}
&\bar\chi_{,t}-\nu\Delta\bar\chi=0 && \text{in } \Omega^T,\\
&\overline{\chi}=\chi_* && \text{on } S_1^T,\\
&\overline{\chi}_{,x_3}=0 && \text{on } S_2^T,\\
&\overline{\chi}|_{t=0}=0 && \text{in } \Omega
\end{aligned}
$$

and subtract it from the function χ,

$$
(3.2)\qquad \chi'=\chi-\overline{\chi}.
$$

Then χ' is a solution to the problem

$$
(3.3)\qquad
\begin{aligned}
&\chi'_{,t}+v\cdot\nabla\chi'-h_3\chi'+h_2w_{,x_1}-h_1w_{,x_2}-\nu\Delta\chi'\\
&\quad=F_3-v\nabla\overline{\chi}+h_3\overline{\chi} && \text{in } \Omega^T,\\
&\chi'=0 && \text{on } S_1^T,\\
&\chi'_{,x_3}=0 && \text{on } S_2^T,\\
&\chi'|_{t=0}=\chi(0) && \text{in } \Omega.
\end{aligned}
$$

LEMMA 3.1. *Let* $h^{(k_n)}\in L_\infty(0,t;L_3(\Omega))$, $v'^{(k_n)}\in L_\infty(0,t;H^{5/6}(\Omega))$, $F_3^{(k_n)}\in L_2(0,t;L_{6/5}(\Omega))$. *Then a solution to the problem* (2.2) *satisfies*

$$
(3.4)\qquad \|\chi^{(k_n)}\|^2_{V_2^0(\Omega^t)}\le cd_2^2\|h^{(k_n)}\|^2_{L_\infty(0,t;L_3(\Omega))}+c\|F_3^{(k_n)}\|^2_{L_2(0,t;L_{6/5}(\Omega))}
$$
$$
+c(d_2^2+1)\|v'^{(k_n)}\|_{L_\infty(0,t;H^{5/6}(\Omega))}+c\|v'^{(k_n)}\|_{W_2^{1,1/2}(\Omega^t)}+cd_1^2+2(n+2)d_2^2.
$$

Proof. Multiplying $(3.3)_1$ by $\zeta^{(k_n)}$, then by $\chi'^{(k_n)}$ and integrating over Ω and using the boundary conditions $(3.3)_2$, $(3.3)_3$ and $(1.1)_2$ yields

$$
\begin{aligned}
\frac12\frac{d}{dt}\|\chi'^{(k_n)}\|^2_{L_2(\Omega)}+\nu\|\nabla\chi'^{(k_n)}\|^2_{L_2(\Omega)}&=\int_\Omega h_3(\chi'^{(k_n)})^2\,dx\\
&\quad-\int_\Omega(h_2w_{,x_1}-h_1w_{,x_2})^{(k_n)}\chi'^{(k_n)}\,dx+\int_\Omega F_3^{(k_n)}\chi'^{(k_n)}\,dx\\
&\quad-\int_\Omega(v\cdot\nabla\overline{\chi})^{(k_n)}\chi'^{(k_n)}\,dx+\int_\Omega h_3\overline{\chi}^{(k_n)}\chi'^{(k_n)}\,dx+\dot\zeta^{(k_n)}\int_\Omega\chi'\chi'^{(k_n)}\,dx.
\end{aligned}
$$

Now we estimate the terms on the right-hand side above. We have

$$
\begin{aligned}
&\frac{d}{dt}\|\chi'^{(k_n)}\|^2_{L_2(\Omega)}+2\overline{\nu}\|\chi'^{(k_n)}\|^2_{H^1(\Omega)}\\
&\quad\le\varepsilon_1\|\chi'^{(k_n)}\|^2_{L_6(\Omega)}+\frac{1}{\varepsilon_1}\|h^{(k_n)}\|^2_{L_3(\Omega)}\|\chi'\|^2_{L_2(\Omega)}\\
&\qquad+\varepsilon_2\|\chi'^{(k_n)}\|^2_{L_6(\Omega)}+\frac{1}{\varepsilon_2}\|h^{(k_n)}\|^2_{L_3(\Omega)}\|\nabla w\|^2_{L_2(\Omega)}
\end{aligned}
$$

$$+\varepsilon_3\|\chi'^{(k_n)}\|^2_{L_6(\Omega)}+\frac{1}{\varepsilon_3}\|F_3^{(k_n)}\|^2_{L_{6/5}(\Omega)}$$
$$+\varepsilon_4\|\nabla\chi'^{(k_n)}\|^2_{L_2(\Omega)}+\frac{1}{\varepsilon_4}\|\overline{\chi}^{(k_n)}\|^2_{L_3(\Omega)}\|v\|^2_{L_6(\Omega)}$$
$$+\varepsilon_5\|\chi'^{(k_n)}\|^2_{L_6(\Omega)}+\frac{1}{\varepsilon_5}\|h\|^2_{L_2(\Omega)}\|\overline{\chi}^{(k_n)}\|^2_{L_3(\Omega)}$$
$$+\varepsilon_6\|\chi'\dot{\zeta}^{(k_n)}\|^2_{L_2(\Omega)}+\frac{1}{\varepsilon_6}\|\chi'^{(k_n)}\|^2_{L_2(\Omega)},$$

where $\overline{\nu}=\nu/c$ and c comes from the Poincaré inequality. Now we take $\varepsilon_1,\dots,\varepsilon_6$ sufficiently small and use equality (3.2) and the Minkowski inequality for $\|\chi'\dot{\zeta}^{(k_n)}\|^2_{L_2(\Omega)}$ to obtain

$$\begin{aligned}\frac{d}{dt}\|\chi'^{(k_n)}\|^2_{L_2(\Omega)}+\overline{\nu}\|\chi'^{(k_n)}\|^2_{H^1(\Omega)}&\le c\|h^{(k_n)}\|^2_{L_3(\Omega)}(\|\chi\|^2_{L_2(\Omega)}+\|\overline{\chi}\|^2_{L_2(\Omega)})\\&+c\|h^{(k_n)}\|^2_{L_3(\Omega)}\|\nabla w\|^2_{L_2(\Omega)}+\|F_3^{(k_n)}\|^2_{L_{6/5}(\Omega)}\\&+c\|\overline{\chi}^{(k_n)}\|^2_{L_3(\Omega)}\|v\|^2_{L_6(\Omega)}+c\|h\|^2_{L_2(\Omega)}\|\overline{\chi}^{(k_n)}\|^2_{L_3(\Omega)}\\&+c(\|\chi\dot{\zeta}^{(k_n)}\|^2_{L_2(\Omega)}+\|\overline{\chi}\dot{\zeta}^{(k_n)}\|^2_{L_2(\Omega)}).\end{aligned}$$

Integrating with respect to $t\in((k-n-1)T,(k+1)T)$ and using inequality (3.6) yields

$$\begin{aligned}\|\chi'^{(k_n)}(t)\|^2_{L_2(\Omega)}+\overline{\nu}\int_{(k-n-1)T}^{t}\|\chi'^{(k_n)}(s)\|^2_{H^1(\Omega)}\,ds&\le 2cd_2^2\|h^{(k_n)}\|^2_{L_\infty(0,t;L_3(\Omega))}\\&+c\|F_3^{(k_n)}\|^2_{L_2(0,t;L_{6/5}(\Omega))}+2cd_2^2\|\overline{\chi}^{(k_n)}\|^2_{L_\infty(0,t;L_3(\Omega))}+2(n+2)d_2^2,\end{aligned}$$

because from the definition of χ and χ' it follows that

$$\int_{(k-n-1)T}^{(k+1)T}(\|\chi\dot{\zeta}^{(k_n)}\|^2_{L_2(\Omega)}+\|\overline{\chi}\dot{\zeta}^{(k_n)}\|^2_{L_2(\Omega)})\,dt<2(n+2)d_2^2,$$

where d_2 comes from Lemma 2.2. For a solution to (3.1) we have

$$\|\overline{\chi}^{(k_n)}\|_{L_\infty(0,t;L_3(\Omega))}\le c\|v'^{(k_n)}\|_{L_\infty(0,t;H^{5/6}(\Omega))},$$

hence

$$\begin{aligned}\|\chi'^{(k_n)}\|^2_{V_2^0(\Omega^t)}&\le cd_2^2\|h^{(k_n)}\|^2_{L_\infty(0,t;L_3(\Omega))}+c\|F_3^{(k_n)}\|^2_{L_2(0,t;L_{6/5}(\Omega))}\\&+cd_2^2\|v'^{(k_n)}\|^2_{L_\infty(0,t;H^{5/6}(\Omega))}+2(n+2)d_2^2.\end{aligned}$$

The trace theorem implies that

$$\|\overline{\chi}^{(k_n)}\|^2_{L_\infty(0,t;L_2(\Omega))} \le c\|v'^{(k_n)}\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))},$$
$$\|\overline{\chi}^{(k_n)}\|^2_{L_2(0,t;H^1(\Omega))} \le c\|v'^{(k_n)}\|^2_{W_2^{1,1/2}(\Omega^t)}.$$

Since

$$\|v'^{(k_n)}\|^2_{L_\infty(0,t;H^{1/2+\varepsilon}(\Omega))} \le c\|v'^{(k_n)}\|^2_{L_\infty(0,t;H^{5/6}(\Omega))} + cd_1^2,$$

the proof is finished. ∎

We can finally find the estimate for v in the $W_2^{2,1}(\Omega \times (kT,(k+1)T))$-norm (see Lemma 3.3). However, first we need an auxiliary inequality for $v' = (v_1, v_2)$ in the $V_2^1(\Omega^t)$-norm (see Lemma 3.2). Consider the problem

$$\begin{aligned} &v_{1,x_2} - v_{2,x_1} = \chi && \text{in } \Omega',\\ &v_{1,x_1} + v_{2,x_2} = -h_3 && \text{in } \Omega',\\ &v'\cdot\overline{n}' = 0 && \text{on } S_1', \end{aligned} \tag{3.5}$$

where $\Omega' = \Omega \cap \{x_3 = \text{const} \in (-a,a)\}$ and S_1' is defined analogously.

Lemma 3.2. *Let* $h^{(k_n)} \in L_\infty(0,t;L_3(\Omega))$, $v'^{(k_n)} \in H^{1/2}(0,t;L_2(\Omega))$, $g^{(k_n)} \in L_2(\Omega^t)$, $f_3^{(k_n)} \in L_2(S_2^t)$, $F_3^{(k_n)} \in L_2(0,t;L_{6/5}(\Omega))$. *Then*

$$\begin{aligned} \|v'^{(k_n)}\|^2_{V_2^1(\Omega^t)} &\le cd_2^2\|h^{(k_n)}\|^2_{L_\infty(0,t;L_3(\Omega))} + \|v'^{(k_n)}\|^2_{H^{1/2}(0,t;L_2(\Omega))}\\ &\quad + c\|F_3^{(k_n)}\|^2_{L_2(0,t;L_{6/5}(\Omega))} + c\|f_3^{(k_n)}\|^2_{L_2(S_2^t)}\\ &\quad + c\|g^{(k_n)}\|^2_{L_2(\Omega^t)} + cd_1^2 + (c+3(n+2))d_2^2. \end{aligned}$$

Proof. First we observe that applying the cut-off function $\zeta^{(k_n)}$ in the proof of Lemma 2.4 (see [6, Sec. 4, Lemma 4.2]) gives

$$\begin{aligned} \|h^{(k_n)}\|^2_{V_2^0(\Omega^T)} &\le cd_2^2\|h^{(k_n)}\|^2_{L_\infty(0,t;L_3(\Omega))} + c\|f_3^{(k_n)}\|^2_{L_2(S_2^t)}\\ &\quad + c\|g^{(k_n)}\|^2_{L_2(\Omega^t)} + (n+2)d_2^2. \end{aligned} \tag{3.6}$$

For a solution to the problem (3.5) we have the estimate

$$\|v'^{(k_n)}\|^2_{V_2^1(\Omega^t)} \le c\|h^{(k_n)}\|^2_{V_2^0(\Omega^t)} + c\|\chi^{(k_n)}\|^2_{V_2^0(\Omega^t)}. \tag{3.7}$$

It follows by interpolation that

$$\|v'^{(k_n)}\|^2_{L_\infty(0,t;H^{5/6}(\Omega))} \le \varepsilon\|v'^{(k_n)}\|^2_{L_\infty(0,t;H^1(\Omega))} + c(1/\varepsilon)d_1^2$$

and from Lemma 2.2 that

$$\|v'^{(k_n)}\|^2_{L_2(0,t;H^1(\Omega))} \le cd_2^2.$$

Applying Lemma 3.1, the inequalities (3.6) and (3.7) and using the above interpolation result ends the proof. ∎

LEMMA 3.3. *Let $k \in \mathbb{N} \setminus \{0,1\}$ be fixed and $k-n-1>0$. Assume that*

$$(D^{(k_n)})^2 = \|F_3^{(k_n)}\|^2_{L_2(0,t;L_{6/5}(\Omega))} + \|f_3^{(k_n)}\|^2_{L_2(S_2^t)} + \|g^{(k_n)}\|^2_{L_2(\Omega^t)} + cd_1^2 + (c+3(n+2))d_2^2 < \infty$$

and $\|h^{(k_{n+1})}\|_{L_\infty(0,t;L_3(\Omega))}, \|h^{(k_{n+1})}\|_{L_{10/3}(\Omega^t)}, \|f^{(k_{n+1})}\|_{L_2(\Omega^t)} < \infty$. *Then for any solution to the problem* (1.1) *we have the estimate*

$$\|v^{(k_n)}\|_{W_2^{2,1}(\Omega^t)} + \|\nabla p^{(k_n)}\|_{L_2(\Omega^t)} \le cd_2^2(\|h^{(k_{n+1})}\|^2_{L_\infty(0,t;L_3(\Omega))} + \|h^{(k_{n+1})}\|^2_{L_{10/3}(\Omega^t)} + \|f^{(k_{n+1})}\|^2_{L_{5/3}(\Omega^t)}) + c(D^{(k_{n+1})})^2 + c\|f^{(k_{n+1})}\|_{L_2(\Omega^t)}. \tag{3.8}$$

Proof. Let us consider the problem (1.1) in the form

$$\begin{aligned} & v_{,t}^{(k_n)} - \operatorname{div}\mathbb{T}(v^{(k_n)}, p^{(k_n)}) \\ & \qquad = -v'^{(k_n)}\cdot\nabla' v - wh^{(k_n)} + f^{(k_n)} + \dot\zeta^{(k_n)} v && \text{in } \Omega^t, \\ & \operatorname{div} v^{(k_n)} = 0 && \text{in } \Omega^t, \\ & v^{(k_n)}\cdot n = 0 && \text{on } S^t, \\ & n\cdot\mathbb{T}(v^{(k_n)}, p^{(k_n)})\cdot\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } S^t, \\ & v^{(k_n)}|_{t=(k-n-1)T} = 0 && \text{in } \Omega, \end{aligned} \tag{3.9}$$

where $v' = (v_1, v_2)$, $\nabla' = (\partial_{x_1}, \partial_{x_2})$. In view of [7, Lemma 3.7] the inequality $\|v'\|_{L_{10}(\Omega^t)} \le c\|v'\|_{V_2^1(\Omega^t)}$ holds. Hence

$$\|v'^{(k_n)}\cdot\nabla' v\|_{L_{5/3}(\Omega^t)} \le \|v'^{(k_n)}\|_{L_{10}(\Omega^t)}\|\nabla' v\|_{L_2(\Omega^t)} \le cd_2\|v'^{(k_n)}\|_{V_2^1(\Omega^t)},$$

$$\|wh^{(k_n)}\|_{L_{5/3}(\Omega^t)} \le \|w\|_{L_{10/3}(\Omega^t)}\|h^{(k_n)}\|_{L_{10/3}(\Omega^t)} \le cd_2\|h^{(k_n)}\|_{L_{10/3}(\Omega^t)}.$$

In view of Lemma 3.2 we obtain, for any solution to the problem (3.9),

$$\|v^{(k_n)}\|_{W_{5/3}^{2,1}(\Omega^t)} \le cd_2(\|v'^{(k_n)}\|_{V_2^1(\Omega^t)} + \|h^{(k_n)}\|_{L_{10/3}(\Omega^t)}) + c\|f^{(k_n)}\|_{L_{5/3}(\Omega^t)}.$$

Applying now the interpolation result

$$\|v'^{(k_n)}\|_{H^{1/2}(0,t;L_2(\Omega))} \le \varepsilon\|v'^{(k_n)}\|_{W_{5/3}^{2,1}(\Omega^t)} + c(1/\varepsilon)d_2$$

we get

$$\|v^{(k_n)}\|_{W_{5/3}^{2,1}(\Omega^t)} \le cd_2(\|h^{(k_n)}\|_{L_\infty(0,t;L_3(\Omega))} + \|h^{(k_n)}\|_{L_{10/3}(\Omega^t)} + D^{(k_n)}) + c\|f^{(k_n)}\|_{L_{5/3}(\Omega^t)}. \tag{3.10}$$

Let us now rewrite problem (1.1) in the form

$$
(3.11)\qquad \begin{aligned}
&v_{,t}^{(k_n)} - \operatorname{div} \mathbb{T}(v^{(k_n)}, p^{(k_n)}) \\
&\qquad = -v' \cdot \nabla' v^{(k_n)} - w^{(k_n)} h + f^{(k_n)} + \dot{\zeta}^{(k_n)} v && \text{in } \Omega^t, \\
&\operatorname{div} v^{(k_n)} = 0 && \text{in } \Omega^t, \\
&v^{(k_n)} \cdot n = 0 && \text{on } S^t, \\
&n \cdot \mathbb{T}(v^{(k_n)}, p^{(k_n)}) \cdot \tau_\alpha = 0, \quad \alpha = 1, 2, && \text{on } S^t, \\
&v^{(k_n)}|_{t=(k-n-1)T} = 0 && \text{in } \Omega.
\end{aligned}
$$

Then using (3.10) we get

$$
\begin{aligned}
\|v' \cdot \nabla' v^{(k_n)}\|_{L_2(\Omega^t)} &\le \|v'^{(k_{n+1})}\|_{V_2^1(\Omega^t)} \|v^{(k_n)}\|_{W_{5/3}^{2,1}(\Omega^t)} \\
&\le c d_2^2 (\|h^{(k_{n+1})}\|^2_{L_\infty(0,t;L_3(\Omega))} + \|h^{(k_{n+1})}\|^2_{L_{10/3}(\Omega^t)}) \\
&\quad + c(D^{(k_{n+1})})^2 + c\|f^{(k_n)}\|^2_{L_{5/3}(\Omega^t)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|w^{(k_n)} h\|_{L_2(\Omega^t)} &\le \|w^{(k_n)}\|_{W_{5/3}^{2,1}(\Omega^t)} \|h^{(k_{n+1})}\|_{L_{10/3}(\Omega^t)} \\
&\le c d_2^2 (\|h^{(k_{n+1})}\|^2_{L_\infty(0,t;L_3(\Omega))} + \|h^{(k_{n+1})}\|^2_{L_{10/3}(\Omega^t)}) \\
&\quad + c(D^{(k_{n+1})})^2 + c\|f^{(k_n)}\|^2_{L_{5/3}(\Omega^t)},
\end{aligned}
$$

which concludes the proof. ∎

Lemma 3.4. *Let $\sigma \in (1, 10)$ and assume that the norms $\|f^{(k_{n+1})}\|_{L_2(\Omega^t)}$ and $\|g^{(k_n)}\|_{L_\sigma(\Omega^t)}$ are finite for any $k \in \mathbb{N} \setminus \{0, 1\}$. Let $k - n - 1 > 0$. Then for any solution to the problem (2.1) we have*

$$
(3.12)\qquad \begin{aligned}
&\|h^{(k_n)}\|_{W_\sigma^{2,1}(\Omega^t)} \\
&\quad \le \varphi(\|h\|_{W_\sigma^{2,1}(\Omega \times ((k-n-4)T, kT))} + \|h^{(k_0)}\|_{W_\sigma^{2,1}(\Omega^t)} + D^{(k_{n+1})} \\
&\qquad + \|f^{(k_{n+1})}\|_{L_2(\Omega^t)}) \delta_k(T) + c\|g^{(k_n)}\|_{L_\sigma(\Omega^t)} \\
&\qquad + \frac{1}{T} \|h\|_{L_\sigma(\Omega \times ((k-n-1)T, (k-n)T))},
\end{aligned}
$$

where φ is some nonlinear, positive and increasing function.

Proof. Let us consider problem (2.1) in the form

$$
(3.13)\qquad
\begin{aligned}
& h^{(k_n)}_{,t} - \operatorname{div} \mathbb{T}(h^{(k_n)}, q^{(k_n)}) \\
&\qquad = -v\cdot\nabla h^{(k_n)} - h^{(k_n)}\cdot\nabla v + g^{(k_n)} + \dot\zeta^{(k_n)} h && \text{in } \Omega^t,\\
& \operatorname{div} h^{(k_n)} = 0 && \text{in } \Omega^t,\\
& h^{(k_n)}\cdot n = 0,\quad n\cdot\mathbb{T}(h^{(k_n)}, q^{(k_n)})\cdot\tau_\alpha = 0,\quad \alpha=1,2, && \text{on } S_1^t,\\
& h^{(k_n)}_i = 0,\quad i=1,2,\quad h^{(k_n)}_{3,x_3} = 0 && \text{on } S_2^t,\\
& h^{(k_n)}|_{t=(k-n-1)T} = 0 && \text{in } \Omega.
\end{aligned}
$$

Repeating the proof of [4, Lemma 3.4] we obtain the inequality

$$
\begin{aligned}
\|h^{(k_n)}\|_{W^{2,1}_\sigma(\Omega^t)} + \|\nabla q^{(k_n)}\|_{L_\sigma(\Omega^t)} &\le \varphi(\|v^{(k_{n+1})}\|_{W^{2,1}_2(\Omega^t)})\|h^{(k_n)}\|_{L_2(\Omega^t)}\\
&\quad + c\|g^{(k_n)}\|_{L_\sigma(\Omega^t)} + \frac{1}{T}\|h\|_{L_\sigma(\Omega\times((k-n-1)T,(k-n)T))},
\end{aligned}
$$

for any $\sigma\in(1,10)$ and φ some nonlinear, positive and increasing function. Next we estimate $\|h^{(k_n)}\|_{L_2(\Omega^t)}$. Therefore we multiply $(3.13)_1$ by $h^{(k_n)}$, integrate by parts over Ω and repeat the proof of Lemma 2.5 (for details, see [6, Sec. 4, Lemma 4.2]). Finally, we get

$$
\begin{aligned}
\|h^{(k_n)}\|_{L_2(\Omega^t)} &\le c\sqrt{n+2}(\|\nabla v^{(k_{n+2})}\|_{L_2(0,t;L_3(\Omega))}\\
&\qquad\cdot \exp(c\|\nabla v^{(k_{n+2})}\|^2_{L_2(0,t;L_3(\Omega))}) + 1)\delta_k(T).
\end{aligned}
$$

Next we estimate the right-hand side using Lemma 3.3. Observing that

$$
\|h^{(k_{n+2})}\|_{W^{2,1}_\sigma(\Omega^t)} \le \|h\|_{W^{2,1}_\sigma(\Omega\times((k-n-4)T,kT))} + \|h^{(k_0)}\|_{W^{2,1}_\sigma(\Omega^t)}
$$

we conclude the proof. ∎

Proof of Theorem 1. Let $T_0 = 4T$, so $k=4$ and $n=0$ (when $n=0$ we write k instead of k_0). In view of [4, Lemma 3.5],

$$
(3.14)\qquad \|h\|_{L_\sigma(\Omega\times(3T,4T))} \le \|h^{(3)}\|_{W^{2,1}_\sigma(\Omega^t)} \le \|h\|_{W^{2,1}_\sigma(\Omega\times(0,T_0))} \le A,
$$

where the constant A is such that

$$
(3.15)\qquad c(\|g\|_{L_\sigma(\Omega\times(0,T_0))} + \|h(0)\|_{W^{2-2/\sigma}_\sigma(\Omega)}) < A
$$

and the constant c depends on n, p and T (for details see [4, Lemma 3.5]). Let us observe that without loss of generality we can assume that

$$
(3.16)\qquad c\|g\|_{L_\sigma(\Omega\times(kT,(k+1)T))} < A
$$

for any $k\in\mathbb{N}$. Then (3.12) implies that

$$
\begin{aligned}
\|h^{(4)}\|_{W^{2,1}_\sigma(\Omega^t)} &\le \varphi(A + \|h^{(4)}\|_{W^{2,1}_\sigma(\Omega^t)} + D^{(4_1)} + \|f^{(4_1)}\|_{L_2(\Omega^t)})\delta_4(T)\\
&\quad + c\|g^{(4)}\|_{L_\sigma(\Omega^t)} + \frac{1}{T}\|h\|_{L_\sigma(\Omega\times(3T,4T))}.
\end{aligned}
$$

In view of (3.14) and for $\delta_4(T)$ sufficiently small,

$$\varphi(A+\|h^{(4)}\|_{W^{2,1}_\sigma(\Omega^t)}+D^{(4_1)}+\|f^{(4_1)}\|_{L_2(\Omega^t)})\delta_4(T) + c\|g^{(4)}\|_{L_\sigma(\Omega^t)}+\frac{A}{T}\le A$$

where

$$c\|g^{(4)}\|_{L_\sigma(\Omega^t)}+\frac{A}{T}<A,$$

if only T is large enough. Hence

$$\|h^{(4)}\|_{W^{2,1}_\sigma(\Omega^t)}\le A.$$

Assume now that for $n=0$ and $4\le m\le s\in\mathbb{N}$ we have

$$\|h^{(m)}\|_{W^{2,1}_\sigma(\Omega^t)}\le A. \tag{3.17}$$

We will show that

$$\|h^{(s+1)}\|_{W^{2,1}_\sigma(\Omega^t)}\le A. \tag{3.18}$$

From (3.12) it follows that

$$\begin{aligned}
&\|h^{(s+1)}\|_{W^{2,1}_\sigma(\Omega^t)}\\
&\quad\le \varphi(\|h^{(s-2)}\|_{W^{2,1}_\sigma(\Omega^t)}+\|h^{(s-1)}\|_{W^{2,1}_\sigma(\Omega^t)}+\|h^{(s)}\|_{W^{2,1}_\sigma(\Omega^t)}\\
&\qquad+\|h^{(s+1)}\|_{W^{2,1}_\sigma(\Omega^t)}+D^{(s+1)_1}+\|f^{(s+1)_1}\|_{L_2(\Omega^t)})\delta_{s+1}(T)\\
&\qquad+c\|g^{(s+1)}\|_{L_\sigma(\Omega^t)}+\frac{1}{T}\|h\|_{L_\sigma(\Omega\times(sT,(s+1)T))}\\
&\quad\le \varphi(3A+\|h^{(s+1)}\|_{W^{2,1}_\sigma(\Omega^t)}+D^{(s+1)_1}+\|f^{(s+1)_1}\|_{L_2(\Omega^t)})\delta_{s+1}(T)\\
&\qquad+c\|g^{(s+1)}\|_{L_\sigma(\Omega^t)}+\frac{1}{T}\|h\|_{L_\sigma(\Omega\times(sT,(s+1)T))}.
\end{aligned}$$

If $\delta_{s+1}(T)$ is small enough, then using (3.17) to estimate the last term on the right-hand side we can see that (3.18) holds for T sufficiently large. The existence of functions v and h can be proved as in [4, Sec. 4]. This concludes the proof. ∎

Proof of Theorem 2. It remains to show that $\|h(kT)\|_{L_2(\Omega)}$ is equally small for any $k\in\mathbb{N}$. We first differentiate (1.1) with respect to x_3, then multiply by h, integrate over Ω, use the boundary conditions and apply the Stokes theorem and the Korn, Hölder and Young inequalities. Then we get

$$\frac{d}{dt}\|h\|^2_{L_2(\Omega)}+\nu\|h\|^2_{H^1(\Omega)}\le c\|h\|^2_{L_2(\Omega)}\|\nabla v\|^2_{L_3(\Omega)}+c\|g\|^2_{L_2(\Omega)}+c\|f_3\|^2_{L_2(S_2)}.$$

Using the Gronwall inequality on the time interval $(kT,(k+1)T)$ yields

$$\|h((k+1)T)\|^2_{L_2(\Omega)} \le ce^{-\overline{\nu}T+c\|\nabla v\|^2_{L_2(kT,(k+1)T;L_3(\Omega))}} \cdot (\|h(kT)\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\times(kT,(k+1)T))} + \|f_3\|^2_{L_2(S_2\times(kT,(k+1)T))}).$$

From Theorem 1 it follows that $\|\nabla v\|^2_{L_2(kT,(k+1)T;L_3(\Omega))} < c(A^4+1)$. Since the constant A is chosen in such a way that it satisfies (3.15) and (3.16) we can take T large enough so that $-\overline{\nu}T + c(A^4+1) < 0$ and

$$ce^{-\overline{\nu}T+c\|\nabla v\|^2_{L_2(kT,(k+1)T;L_3(\Omega))}} \cdot (\|h(kT)\|^2_{L_2(\Omega)} + \|g\|^2_{L_2(\Omega\times(kT,(k+1)T))} + \|f_3\|^2_{L_2(S_2\times(kT,(k+1)T))}) \le ce^{-\overline{\nu}T+c(A^4+1)} \cdot 2\varepsilon < \varepsilon.$$

Applying Theorem 1 on any time interval $(kT,(k+1)T)$ ends the proof. ■

Proof of Theorem 3. Let (v_i,p_i) for $i=1,2$ be two solutions to the problem (1.1). Let $V = v_1 - v_2$ and $P = p_1 - p_2$. Then the pair $(V^{(k_n)}, P^{(k_n)})$ solves the problem

$$\begin{aligned}
&V^{(k_n)}_{,t} - \operatorname{div}\mathbb{T}(V^{(k_n)},P^{(k_n)}) = -V^{(k_n)}\cdot\nabla v_1 - v_2\cdot\nabla V^{(k_n)} + \dot\zeta V && \text{in } \Omega^t,\\
&\operatorname{div} V^{(k_n)} = 0 && \text{in } \Omega^T,\\
&V^{(k_n)}\cdot n = 0 && \text{on } S^T,\\
&n\cdot\mathbb{T}(V^{(k_n)},P^{(k_n)})\cdot\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } S^T,\\
&V^{(k_n)}|_{t=k-n-1} = 0 && \text{in } \Omega.
\end{aligned}$$

Multiplying the first equation by $V^{(k_n)}$ and integrating over Ω gives

$$\frac{d}{dt}\|V^{(k_n)}\|^2_{L_2(\Omega)} + \nu\|V^{(k_n)}\|^2_{H^1(\Omega)} \le c\|\nabla v_1^{(k_{n+1})}\|^2_{L_3(\Omega)}\|V^{(k_n)}\|^2_{L_2(\Omega)}.$$

Since $v_1^{(k_{n+1})} \in L_\infty((k-n-2)T,(k+1)T;W^1_3(\Omega))$, the Gronwall inequality implies that $\|V^{(k_n)}(t)\|_{L_2(\Omega)} = 0$. This ends the proof. ■

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