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JACOBIANS OF CERTAIN TRANSFORMATIONS OF SINGULAR MATRICES

Abstract. In this study various Jacobians of transformations of singular random matrices are found. An alternative proof of Uhlig's first conjecture (Uhlig (1994)) is proposed. Furthermore, we propose various extensions of this conjecture under different singularities. Finally, an application of the theory of singular distributions is discussed.

1. Introduction. The Jacobian computations of matrix transformations play a fundamental role in several areas of multivariate statistics, especially in the theory of distributions and its applications. This topic has been the objective of many papers in statistical literature. For example, the literature concerning the Lebesgue measure, i.e. when the random and constant matrices in the transformations have complete rank, includes Deemer and Olkin (1951), Olkin (1953), James (1954), Olkin and Sampson (1972), Muirhead (1982), Mathai (1997), Olkin (1998) and Olkin (2002); they have proposed several Jacobians for a lot of linear and non-linear transformations of random matrices (random vectors). Recently, some works in the context of singular random matrices have been published; specifically, they treat the densities and the measures with respect to these densities (Hausdorff measure) and the computation of the Jacobians of the transformations; see Khatri (1968), Uhlig (1994), Díaz-García et al. (1997), Díaz-García and Gutiérrez (1997), Srivastava (2003), Díaz-García and Gutiérrez-Jáimez (2006), Díaz-García and González-Farías (2005a), Díaz-García and González-Farías (2005b), Ip, Wong and Liu (2007) and Díaz-García (2007).

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Now observe that, for a given singular random matrix $Y \in \mathbb{R}^{N\times m}$, if the measure (dY) with respect to Y has a density, then the explicit form of (dY) can be given as a function of a certain decomposition of the matrix Y; that function will depend on the choice of the base and the coordinate system of the subspace where the measure is defined. In particular, when the QR decomposition of the matrix X is considered (see Eaton (1983, p. 160)), we get

$$(1)\qquad (dY) = \prod_{i=1}^{q} t_{ii}^{N-i}(H_1' dH_1)(dT)$$

where H_1 is a semi-orthogonal matrix, i.e. $H_1' H_1 = I_q$, and T is a quasi-triangular matrix (see Díaz-García and González-Farías (1999) and/or Díaz-García and González-Farías (2005a)).

In general, it is possible to propose alternative definitions to (1) for the measure (dY), as a function of other decompositions. Under alternative decompositions, the explicit form of the measure (dY) has been studied in Díaz-García and González-Farías (1999) and Díaz-García and González-Farías (2005a) and some applications to the theory of distributions were given in Díaz-García and González-Farías (2005b) and Díaz-García (2007).

In the present article, we propose an alternative proof to that given by Díaz-García and Gutiérrez (1997) of Uhlig's first conjecture (Uhlig (1994); see Theorem 2.1). An extension of this latter result is studied for other singularities (see Theorem 2.3). Two Jacobians that involve the Moore–Penrose inverse are proposed in Theorems 2.2 and 2.4. The study concludes by applying some of the Jacobians studied in Section 3 to the theory of singular distributions.

2. Jacobians.

In this section we propose an alternative proof for Uhlig's first conjecture, and an extension to more general cases. First, however, some notation should be established.

Let $\mathcal{L}_{m,N}(q)$ be the linear space of all $N \times m$ real matrices of rank $q \le \min(N,m)$ and $\mathcal{L}_{m,N}^{+}(q)$ be the linear space of all $N \times m$ real matrices of rank $q \le \min(N,m)$ with q distinct singular values. The set of matrices $H_1 \in \mathcal{L}_{m,N}$ such that $H_1' H_1 = I_m$ is a manifold, denoted $\mathcal{V}_{m,N}$ and called the *Stiefel manifold*. In particular, $\mathcal{V}_{m,m}$ is the group $\mathcal{O}(m)$ of orthogonal matrices. Denote by $\mathcal{S}_m$ the homogeneous space of $m \times m$ positive definite symmetric matrices, and by $\mathcal{S}_m^{+}(q)$ the $(mq - q(q-1)/2)$-dimensional manifold of rank q positive semidefinite $m \times m$ symmetric matrices with q distinct positive eigenvalues.

Now, consider the following preliminary results.

LEMMA 2.1. *Let $Z \in \mathcal{L}_{m,N}^{+}(N)$ be such that $Z = VDW_1'$ with $W_1 \in \mathcal{V}_{N,m}$, $V \in \mathcal{O}(N)$ and $D = \operatorname{diag}(d_1,\dots,d_N)$, $d_1 > \dots > d_N > 0$. Then*

$$(dZ) = 2^{-N}|D|^{m-N}\prod_{i<j}^{N}(d_i^2 - d_j^2)(dD)(V'dV)(W_1'dW_1) \tag{2}$$

where $(dD) \equiv \bigwedge_{i=1}^{N} dD_{ii}$, *and*

$$(W_1'dW_1) \equiv \bigwedge_{i=1}^{N}\bigwedge_{j=i+1}^{m} w_j'dw_i \quad \text{and} \quad (V'dV) \equiv \bigwedge_{i=1}^{N}\bigwedge_{j=i+1}^{N} v_j'dv_i$$

define an invariant measure on $\mathcal{V}_{N,m}$ *and on* $\mathcal{O}(m)$, *respectively* (*see Uhlig* (1994) *and/or James* (1954)).

Lemma 2.2. *Under the assumptions of Lemma* 2.1, *define* $S = Z'Z = W_1'LW_1 \in \mathcal{S}_m^+(N)$, *where* $L = \operatorname{diag}(l_1, \dots, l_N)$, $l_1 > \cdots > l_N > 0$. *Then*

$$(dS) = 2^{-N}|L|^{m-N}\prod_{i<j}^{N}(l_i - l_j)(dL)(W_1'dW_1),$$

$$(dZ) = 2^{-N}|L|^{(N-m-1)/2}(dS)(V'dV).$$

Proof. This follows from Lemma 2.1 and from Theorem 2 in Uhlig (1994) (see also Díaz-García et al. (1997)). ∎

Lemma 2.3. *Let* $X \in \mathcal{L}_{m,N}^+(q)$, *let* $A \in \mathcal{L}_{N,p}^+(r_A)$ *and* $B \in \mathcal{L}_{n,m}^+(r_B)$ *be constant, and let* $Y \in \mathcal{L}_{n,p}^+(q)$, *with* $r_A \geq q$ *and* $r_B \geq q$. *Let* r_C, r_E *satisfy* $q = \min(r_C, r_E)$. *Let* $C \in \mathcal{L}_{r_C,N}^+(r_C)$ *and* $E \in \mathcal{L}_{m,r_E}^+(r_E)$ *be such that* $X = CZE$ *with* $Z \in \mathcal{L}_{r_E,r_C}^+(q)$. *If* $Y = AXB$, *then*

$$(dY) = \frac{\prod_{i=1}^{r_C}\operatorname{ch}_i(ACC'A')^{r_E/2}\prod_{j=1}^{r_E}\operatorname{ch}_j(B'E'EB)^{r_C/2}}{\prod_{i=1}^{r_C}\operatorname{ch}_i(CC')^{r_E/2}\prod_{j=1}^{r_E}\operatorname{ch}_j(E'E)^{r_C/2}}\,(dX), \tag{3}$$

where $\operatorname{ch}_i(M)$ *is the* ith *non-null eigenvalue of* M.

Proof. The proof is given in Díaz-García (2007). ∎

When the matrices A and B are non-singular, and when X and Y have full rank, the result follows from Lemma 2.3: just take $N = p = r_A$ and $m = n = r_B$, thus $C = I_N$ and $E = I_m$ (see e.g. Deemer and Olkin (1951, Theorem 3.6) and Muirhead (1982, Theorem 2.1.5, p. 58)).

The Jacobian studied in Theorem 2.1 below was proposed as a conjecture in Uhlig (1994). An indirect proof was provided in Díaz-García and Gutiérrez (1997) based on the following idea:

Let X and Y be random matrices with density functions $f_X(X)$ and $g_Y(Y)$, respectively. Let $X = h(Y)$ be a transformation such that $Y = h^{-1}(X)$. Then by the change of variables theorem,

$$f_X(X) = g_Y(h^{-1}(X))|J(Y \to X)|.$$

Finally, it is assumed that both density functions, $f_X(X)$ and $g_Y(Y)$, are known explicitly. Then

$$|J(Y \to X)| = \frac{f_X(X)}{g_Y(h^{-1}(X))}.$$

This approach can be used in all circumstances in which both density functions, $f_X(X)$ and $g_Y(Y)$, are known. Unfortunately, this is not always so. In the next result we propose an alternative proof of this Jacobian, which can be applied under more general conditions, as shown below.

THEOREM 2.1 (First Uhlig's conjecture). *Let $X, Y \in \mathcal{S}_m^+(N)$ be such that $X = B'YB$, with $B \in \mathcal{L}_{m,m}^+(m)$ fixed. Additionally, let $X = G_1KG_1'$ and $Y = H_1LH_1'$ with $G_1, H_1 \in \mathcal{V}_{N,m}$ and $K = \operatorname{diag}(k_1, \ldots, k_N)$, $k_1 > \cdots > k_N > 0$, $L = \operatorname{diag}(l_1, \ldots, l_N)$, $l_1 > \cdots > l_N > 0$. Then*

$$\begin{aligned}
(dX) &= |G_1'BH_1|^{m+1-N}|B|^N(dY) && (4)\\
&= |H_1'B'G_1|^{m+1-N}|B|^N(dY) &&\\
&= |K|^{(m+1-N)/2}|L|^{-(m+1-N)/2}|B|^N(dY) && (5)
\end{aligned}$$

with

$$(dY) = 2^{-N}|L|^{m-N}\prod_{i<j}^{N}(l_i - l_j)(dL)(H_1'dH_1).$$

Proof. Let $Z \in \mathcal{L}_{m,N}^+(N)$ be such that $Y = Z'Z$. Then

$$X = B'YB = B'Z'ZB = \Lambda'\Lambda \quad \text{with} \quad \Lambda = ZB. \tag{6}$$

From Lemma 2.2,

$$(d\Lambda) = 2^{-N}|K|^{(N-m-1)/2}(dX)(V'dV). \tag{7}$$

Here $\Lambda = VDG_1'$, with $V \in \mathcal{O}(N)$, $G_1 \in \mathcal{V}_{N,m}$ and $D^2 = K$. Note that $d\Lambda = dZB$, and so $(d\Lambda) = |B|^N(dZ)$, from which, substituting in (7), we obtain

$$|B|^N(dZ) = 2^{-N}|K|^{(N-m-1)/2}(dX)(V'dV). \tag{8}$$

Now $Y = Z'Z$, and from Lemma 2.2,

$$(dZ) = 2^{-N}|L|^{(N-m-1)/2}(dY)(V_z'dV_z) \tag{9}$$

where $Z = V_zD_zH_1'$, $D_z^2 = L$ and $V_z \in \mathcal{O}(N)$. Moreover, note that, due to the uniqueness of Haar measure, $(V'dV) = (V_z'dV_z)$ (see James (1954)). Thus, substituting (9) in (8), we obtain

$$\begin{aligned}
(dX) &= |L|^{(N-m-1)/2}|K|^{-(N-m-1)/2}|B|^N(dY)\\
&= |K|^{(m-N+1)/2}|L|^{-(m-N+1)/2}|B|^N(dY).
\end{aligned}$$

Finally, note that

$$|K| = |G_1'XG_1| = |G_1'B'YBG_1| = |G_1'B'H_1LH_1'BG_1| \\ = |G_1'B'H_1|\,|L|\,|H_1'BG_1|,$$

from which, taking into account that $|A| = |A'|$, we obtain the other expressions for (dX). ∎

The non-singular case follows from the expression (4) when $m = N$; this result was studied by other authors (see Deemer and Olkin (1951, Theorem 3.7), Olkin (2002) or Mathai (1997, Theorem 1.20, p. 32)).

REMARK 2.1. Alternative expressions, with respect to different explicit forms of the measure (dY), can be obtained for (5), with alternative decompositions, like QR decomposition, polar decomposition and LDM, etc.

Now, let us assume that A and B have a non-singular Wishart distribution; the matrix $R = A^{-1/2}BA'^{-1/2}$ then has a multivariate F (or beta type II) distribution, in which $A^{1/2}$ is a root of the matrix A, such that $A = A^{1/2}A'^{1/2}$ (see Srivastava and Khatri (1979, p. 92) and Gupta and Nagar (2000, p. 190)). An alternative definition, proposed by various authors, is given by the expression $R_1 = B^{1/2}A^{-1}B'^{1/2}$ (see James (1964), Muirhead (1982, p. 449) and Gupta and Nagar (2000, p. 192)). A similar situation occurs in the case of the beta type I distribution (Srivastava (1968) and Díaz-García and Gutiérrez (2001)). We now present various results that enable us to extend the densities of R and R_1 to the case in which both B and A are singular random matrices. First, however, consider the following lemma, the proof of which is given in Díaz-García and Gutiérrez-Jáimez (2006).

LEMMA 2.4. *Assume that $X \in \mathcal{S}_m^+(N)$ and let $Y = X^+$ (the Moore–Penrose inverse of X, see Campbell and Meyer* (1979)). *Then*

$$(dY) = |K|^{-2m+N-1}(dX)$$

with $X = G_1KG_1'$, $K = \operatorname{diag}(k_1, \dots, k_N)$, $k_1 > \dots > k_N > 0$.

THEOREM 2.2. *Let $X, Y \in \mathcal{S}_m^+(N)$ be such that $X = B'Y^+B$ with $B \in \mathcal{L}_{m,m}^+(m)$ fixed. Moreover, let $X = G_1KG_1'$ and $Y = H_1LH_1'$ with $G_1, H_1 \in \mathcal{V}_{N,m}$ and $K = \operatorname{diag}(k_1, \dots, k_N)$, $k_1 > \dots > k_N > 0$, $L = \operatorname{diag}(l_1, \dots, l_N)$, $l_1 > \dots > l_N > 0$. Then*

$$(dX) = |K|^{(m+1-N)/2}|L|^{-(5m+3-3N)/2}|B|^N(dY). \tag{10}$$

Proof. Define $Z = Y^+$. Then from Theorem 2.1,

$$(dX) = |K|^{(m-N+1)/2}|L|^{-(m-N+1)/2}|B|^N(dZ).$$

The result follows from Lemma 2.4, upon observing that $(dZ) = |L|^{-2m+N-1}(dY)$. ∎

We now generalize the results from Theorems 2.1 and 2.2 to the case in which B is a fixed singular matrix such that $r(B) \geq N$.

THEOREM 2.3. *Let $X, Y \in \mathcal{S}_m^+(n)$ be such that $X = B'YB$ with $B \in \mathcal{L}_{m,m}^+(r)$ fixed and $r \geq n$. Moreover, let $X = G_1 K G_1'$ and $Y = H_1 L H_1'$ with $G_1, H_1 \in \mathcal{V}_{n,m}$ and $K = \operatorname{diag}(k_1, \dots, k_n)$, $k_1 > \cdots > k_n > 0$, $L = \operatorname{diag}(l_1, \dots, l_n)$, $l_1 > \cdots > l_n > 0$. Then*

$$\begin{aligned}
(dX) &= |G_1' B H_1|^{m+1-n} \left(\frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} \right) (dY) \\
&= |H_1' B' G_1|^{m+1-n} \left(\frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} \right) (dY) \\
&= |K|^{(m+1-n)/2} |L|^{-(m+1-n)/2} \left(\frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} \right) (dY)
\end{aligned}$$

where $Y = Q'U'UQ$ with $U \in \mathcal{L}_{n,n}^+(n)$ and $Q \in \mathcal{L}_{m,n}^+(n)$.

Proof. Let $Z \in \mathcal{L}_{m,n}^+(n)$ be such that $Y = Z'Z$. Then

$$X = B'YB = B'Z'ZB = \mathit{\Lambda}'\mathit{\Lambda} \quad \text{with} \quad \mathit{\Lambda} = ZB. \tag{11}$$

From Lemma 2.2,

$$(d\mathit{\Lambda}) = 2^{-n} |K|^{(n-m-1)/2} (dX)(V'dV). \tag{12}$$

Here $\mathit{\Lambda} = VDG_1'$ with $V \in \mathcal{O}(n)$, $G_1 \in \mathcal{V}_{n,m}$ and $D^2 = K$. Note that $d\mathit{\Lambda} = dZB$, and then by Lemma 2.3,

$$(d\mathit{\Lambda}) = \frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} (dZ), \tag{13}$$

where $Z = UQ$ with $U \in \mathcal{L}_{n,n}^+(n)$ and $Q \in \mathcal{L}_{m,n}^+(n)$. By substituting (13) in (12), we have

$$\frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} (dZ) = 2^{-n} |K|^{(n-m-1)/2} (dX)(V'dV). \tag{14}$$

Now from (11), $Y = Z'Z$. Thus, applying Lemma 2.2 we obtain

$$(dZ) = 2^{-n} |L|^{(n-m-1)/2} (dY)(V_z' dV_z) \tag{15}$$

where $Z = V_z D_z H_1'$, $D_z^2 = L$ and $V_z \in \mathcal{O}(n)$. Moreover, note that, due to the uniqueness of Haar measure, $(V'dV) = (V_z' dV_z)$ (see James (1954)). Thus, by substituting (15) in (14), we obtain

$$\begin{aligned}
(dX) &= |L|^{(N-m-1)/2} |K|^{-(N-m-1)/2} \frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} (dY) \\
&= |K|^{(m-N+1)/2} |L|^{-(m-N+1)/2} \frac{\prod_{i=1}^n \operatorname{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^n \operatorname{ch}_i(Q'Q)^{n/2}} (dY).
\end{aligned}$$

Finally, observe that

$$|K| = |G_1' X G_1| = |G_1' B' Y B G_1| = |G_1' B' H_1 L H_1' B G_1| \\ = |G_1' B' H_1|\,|L|\,|H_1' B G_1|,$$

from which we obtain the other expressions for (dX) recalling that $|A| = |A'|$. ∎

THEOREM 2.4. *In Theorem* 2.3, *assume that* $X = B'Y^{+}B$. *Then*

$$(dX) = |K|^{(m+1-n)/2} |L|^{(3n-5m-3)/2} \left(\frac{\prod_{i=1}^{n} \mathrm{ch}_i(B'Q'QB)^{n/2}}{\prod_{i=1}^{n} \mathrm{ch}_i(Q'Q)^{n/2}} \right) (dY).$$

Proof. The proof is immediate from Theorem 2.3 and Lemma 2.4. ∎

3. Some applications. Finally, we present some applications of the results obtained in Section 3.

Let us assume that $S \in \mathcal{S}_m^{+}(r_S)$ has a singular Wishart or pseudo-Wishart distribution, that is, $S \sim \mathcal{W}_m(n, \Sigma)$ with $\Sigma \in \mathcal{S}_m^{+}(r_\Sigma)$. Additionally, assume that $A \in \mathcal{S}_m^{+}(r_A)$ is fixed and $r(A) \geq r(\Sigma)$ and $r(A'\Sigma A) = r(\Sigma)$. Then $V = A'SA \sim \mathcal{W}_m(n, A'\Sigma A)$ with $r(V) = r_V = r_S = r(S)$. This result is well known and can be obtained from the characteristic function technique. Now the proof is obtained by applying the Jacobians found in Section 3.

From Díaz-García et al. (1997), the function of S is given by

$$dG_S(S) = \frac{\pi^{n(r_S - r_\Sigma)/2} |L|^{(n-m-1)/2}}{2^{nr_\Sigma/2} \Gamma_{r_S}\left[\frac{1}{2}n\right] \prod_{i=1}^{r_\Sigma} \mathrm{ch}_i(\Sigma)^{n/2}} \,\mathrm{etr}\big(-\tfrac{1}{2}\Sigma^{-} S\big)(dS)$$

where $S = G_1 L G_1'$ is the non-singular part of the spectral decomposition of S, with $G_1 \in \mathcal{V}_{r_S,m}$, $L = \mathrm{diag}(l_1, \ldots, l_{r_S})$, $l_1 > \cdots > l_{r_S} > 0$, Σ^{-} is a symmetric generalised inverse of Σ and $\mathrm{etr}(\cdot) \equiv \exp(\mathrm{tr}(\cdot))$. Let $\Sigma = Q'Q$. Then from Theorem 2.3,

$$(dS) = |K|^{-(m+1-n)/2} |L|^{(m+1-n)/2} \left(\frac{\prod_{i=1}^{r_\Sigma} \mathrm{ch}_i(\Sigma)^{n/2}}{\prod_{i=1}^{r_\Sigma} \mathrm{ch}_i(A'\Sigma A)^{n/2}} \right) (dV)$$

with $K = \mathrm{diag}(k_1, \ldots, k_{r_V})$, $k_1 > \cdots > k_{r_V} > 0$, such that $V = W_1 K W_1'$ is the non-singular part of the spectral decomposition of V, and $W_1 \in \mathcal{V}_{r_V,m}$. Then

$$dG_V(V) = g_S(A'^{+} V A^{+}) |J(S \to V)| (dV) \\ = \frac{\pi^{n(r_V - r_\Sigma)/2} |K|^{(n-m-1)/2}}{2^{nr_\Sigma/2} \Gamma_{r_V}\left[\frac{1}{2}n\right] \prod_{i=1}^{r_{y\Sigma}} \mathrm{ch}_i(A'\Sigma A)^{n/2}} \,\mathrm{etr}\big(-\tfrac{1}{2}(A'\Sigma A)^{-} V\big)(dV)$$

where $\mathrm{tr}\,\Sigma^{-} S = \mathrm{tr}\,\Sigma^{-} A'^{+} V A^{+} = \mathrm{tr}\,A^{+} \Sigma^{-} A'^{+} V = \mathrm{tr}(A'\Sigma A)^{-} V$ (see Campbell and Meyer (1979)).

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