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WEAK CONVERGENCE OF MUTUALLY INDEPENDENT X_n^B AND X_n^A UNDER WEAK CONVERGENCE OF $X_n \equiv X_n^B - X_n^A$

Abstract. For each $n \geq 1$, let $\{v_{n,k}, k \geq 1\}$ and $\{u_{n,k}, k \geq 1\}$ be mutually independent sequences of nonnegative random variables and let each of them consist of mutually independent and identically distributed random variables with means $\overline{v}_n$ and $\overline{u}_n$, respectively. Let $X_n^B(t) = (1/c_n)\sum_{j=1}^{[nt]}(v_{n,j} - \overline{v}_n)$, $X_n^A(t) = (1/c_n)\sum_{j=1}^{[nt]}(u_{n,j} - \overline{u}_n)$, $t \geq 0$, and $X_n = X_n^B - X_n^A$. The main result gives conditions under which the weak convergence $X_n \xrightarrow{\mathcal{D}} X$, where X is a Lévy process, implies $X_n^B \xrightarrow{\mathcal{D}} X^B$ and $X_n^A \xrightarrow{\mathcal{D}} X^A$, where X^B and X^A are mutually independent Lévy processes and $X = X^B - X^A$.

1. Introduction. Let $X = \{X(t), t \geq 0\}$ be a Lévy process (see [3]) without Gaussian component and with sample paths in the space $D[0,\infty)$. Then the characteristic function of $X(t)$ has the form $E\exp(iuX(t)) = \exp(t\psi_{b,\nu}(u))$, where

$$(1)\quad \psi_{b,v}(u) = iub + \int\limits_{|x|\geq r} (e^{iux} - 1)\,\nu(dx) + \int\limits_{0<|x|<r} (e^{iux} - 1 - iux)\,\nu(dx),$$

the *drift* b is a real number, the *spectral measure* ν is a positive measure on $(-\infty,\infty)$ such that $\nu(\{0\}) = 0$ and it integrates the function $\min(1, x^2)$ on $(-\infty,\infty)$, while r is a positive number such that the points $-r$ and r are continuity points of the spectral measure ν. The function $\psi_{b,\nu}(u)$ is called the characteristic exponent of the process X. It is well known (see Theorem 6.1 in [3]) that $E|X(1)| < \infty$, if and only if $\int_{|x|>1} |x|\,\nu(dx) < \infty$. In such a

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situation the characteristic exponent $\psi_{b,v}(u)$ can be written in the form

$$\psi_{b,v}(u) = iub(r) + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux)\,\nu(dx) \tag{2}$$

where $b(r) = b + \int_{|x|>r} x\,\nu(dx)$ and $b(r) = EX(1)$. Hence, if $EX(t) = 0$, then $b = -\int_{|x|>r} x\,\nu(dx)$.

A Lévy process can be considered as the limiting process of the processes $X_n(t) = c_n^{-1} \sum_{j=1}^{[nt]} \zeta_{n,j}$, $t \geq 0$, $n \geq 1$, where $\zeta_{n,k}$ are r.v.'s. Below we recall some special case of Prokhorov's classical result providing conditions for such a convergence in the case when for each $n \geq 1$, $\{\zeta_{n,k}, k \geq 1\}$ is a sequence of independent and identically distributed (briefly iid) r.v.'s with distribution function F_n. First we introduce the definition of Prokhorov's condition for $\{F_n\}$ with a spectral measure ν defined by means of real nondecreasing and right continuous functions M and N on $(-\infty, 0)$ and $(0, \infty)$, respectively, such that $M(x) \geq 0$, $-N(x) \geq 0$ and $\lim_{x\to-\infty} M(x) = \lim_{x\to\infty} N(x) = 0$. Namely, the spectral measure ν on $(-\infty, \infty)$ is defined by its values on the intervals $(a, b]$ in the following way: $\nu(a, b] = M(b) - M(a)$ for $-\infty < a \leq b < 0$, $\nu(a, b] = N(b) - N(a)$ for $0 < a < b < \infty$ and $\nu(\{0\}) = 0$.

Definition 1. A sequence $\{F_n\}$ of distribution functions satisfies the *Prokhorov condition* (briefly, *condition* P) with drift b_r and spectral measure ν if the following conditions hold:

P1 $nF_n(yc_n) \to M(y)$ and $n(1 - F_n(xc_n)) \to -N(x)$ as $n \to \infty$, for all continuity points $y < 0$ and $x > 0$ of the functions M and N, respectively,

P2 $\lim_{x\to\infty} \sup_n n(1 - F_n(xc_n) + F_n(-xc_n)) = 0$,

P3 $b_r := \lim_{n\to\infty} \frac{n}{c_n} \int_{|x|\leq rc_n} x\,dF_n(x)$ and $|b_r| < \infty$,

P4 $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{c_n^2} \int_{|x|<\varepsilon c_n} x^2\,dF_n(x) = 0$.

Proposition 1 ([2]). *For each $n \geq 1$, let $\{\zeta_{n,k}, k \geq 1\}$ be a sequence of iid random variables with distribution function F_n and let X be a Lévy process with the characteristic exponent given by* (1) *with the pair (b_r, ν). Then $X_n \xrightarrow{\mathcal{D}} X$ in $D[0, \infty)$ equipped with the J_1 Skorokhod topology if and only if $\{F_n\}$ satisfies condition* P *with drift b_r and spectral measure ν.*

The main result of the paper deals with a special case of a Lévy process X from Proposition 1 and special r.v.'s $\zeta_{n,k}$. Namely, we assume that

$EX(1) = 0$, i.e. (b_r, ν) satisfies

P5 $$\int_{|x|>1} |x|\,\nu(dx) < \infty \quad \text{and} \quad b_r = -\int_{|x|>r} x\,\nu(dx).$$

Then

(3) $$\begin{aligned} b_r &= -\int_{-\infty}^{-r} x\,dM(x) - \int_{r}^{\infty} x\,dN(x) \\ &= rM(-r) + rN(r) + \int_{-\infty}^{-r} M(x)\,dx + \int_{r}^{\infty} N(x)\,dx. \end{aligned}$$

For $\zeta_{n,k}$ we assume that for each $n \geq 1$,

$$\zeta_{n,k} := (v_{n,k} - Ev_{n,k}) - (u_{n,k} - Eu_{n,k}),$$

where $\{v_{n,k}, k \geq 1\}$ and $\{u_{n,k}, k \geq 1\}$ are mutually independent sequences of nonnegative r.v.'s with finite expectations $\overline{v}_n := Ev_{n,k}$, $\overline{u}_n := Eu_{n,k}$ and each of them is a sequence of iid r.v.'s with distribution functions $F_n^B(x) := P(v_{n,k} - \overline{v}_n \leq x)$ and $F_n^A(x) := P(u_{n,k} - \overline{u}_n \leq x)$, respectively. Below we refer to the following conditions:

P6 $$\overline{v}_n/c_n \to 0 \quad \text{and} \quad \overline{u}_n/c_n \to 0 \quad \text{as } n \to \infty,$$

P7 $$\lim_{n\to\infty} \int_{-\infty}^{-r} nF_n(xc_n)\,dz = \int_{-\infty}^{-r} M(x)\,dx.$$

The main result of the paper, Theorem 1, says the following. If $\{F_n\}$ satisfies conditions P1–P4 with (b_r, ν) satisfying condition P5 and additionally conditions P6–P7 hold, then the sequences $\{F_n^B\}$ and $\{F_n^A\}$ satisfy conditions P1–P4 with the pairs (b_r^B, ν^B) and (b_r^A, ν^A), respectively, where $b_r = b_r^B - b_r^A$ and the spectral measures ν^B and ν^A are defined by their values on the intervals (a, b) in the following way: $\nu^B(a, b) = \nu(a, b)$, $\nu^A(a, b) = \nu(-b, -a)$ for $0 < a < b$ and $\nu^B(a, b) = \nu^A(a, b) = 0$ for $a < b < 0$.

With the notation

$$X_n^B(t) = \frac{1}{c_n}\sum_{j=1}^{[nt]}(v_{n,j} - \overline{v}_n), \qquad X_n^A(t) = \frac{1}{c_n}\sum_{j=1}^{[nt]}(u_{n,j} - \overline{u}_n), \qquad t \geq 0,$$

and $X_n = X_n^B - X_n^A$ the main result can be expressed in the following way. If $X_n \xrightarrow{\mathcal{D}} X$, where X is a Lévy process with pair (b_r, ν) and conditions P5–P7 hold, then $X_n^B \xrightarrow{\mathcal{D}} X^B$, $X_n^A \xrightarrow{\mathcal{D}} X^A$ and X^B and X^A are mutually independent Lévy processes with appropriate pairs (b_r^B, ν^B), (b_r^A, ν^A) and $X = X^B - X^A$.

It is obvious that if $\{F_n^B\}$ and $\{F_n^A\}$ satisfy conditions P1–P4 with pairs (b_r^B, ν^B) and (b_r^A, ν^A), respectively, then $X_n \equiv X_n^B - X_n^A \xrightarrow{\mathcal{D}} X^B - X^A$.

2. Main result

THEOREM 1 (Main result). *Let $\{F_n\}$ satisfy condition* P *with drift b_r and spectral measure ν satisfying condition* P5 *and let conditions* P6 *and* P7 *hold. Then $\{F_n^B\}$ and $\{F_n^A\}$ satisfy condition* P *with drifts b_r^B, b_r^A, respectively, such that $b_r = b_r^B - b_r^A$ and with spectral measures ν^B and ν^A, respectively, defined as $\nu^B(a,b) = \nu(a,b)$, $\nu^A(a,b) = \nu(-b,-a)$ for $0 < a < b$ and $\nu^B(-b,-a) = \nu^A(-b,-a) = 0$ for $0 < a < b$.*

Set $v_n = v_{n,1},\ u_n = u_{n,1}$ and $\widetilde{v}_n = v_n - \overline{v}_n$, $\widetilde{u}_n = u_n - \overline{u}_n$.

LEMMA 1. *If $\{F_n\}$ satisfies condition* P1 *with functions N and M, and* P6 *holds, then $\{F_n^B\}$ and $\{F_n^A\}$ satisfy condition* P1 *with functions N^B, M^B and N^A, M^A, respectively, such that $N^B = N$, $M^B \equiv 0$ and $N^A(x) = -M(-x)$ for $x > 0$, $M^A \equiv 0$.*

Proof. Let $\varepsilon > 0$ and $n(\varepsilon)$ be such that $\overline{v}_n/c_n < \varepsilon$ and $\overline{u}_n/c_n < \varepsilon$ for $n \ge n(\varepsilon)$. Then for all $x > 0$ such that $x - \varepsilon > 0$ we have

$$P(\widetilde{v}_n > xc_n) \ge P(\widetilde{v}_n - \widetilde{u}_n > xc_n + \overline{u}_n) \ge P(\widetilde{v}_n - \widetilde{u}_n > c_n(x+\varepsilon))$$

and

$$\begin{aligned} P(\widetilde{v}_n > xc_n)P(\widetilde{u}_n \le \varepsilon c_n) &= P(\widetilde{v}_n > xc_n,\, \widetilde{u}_n \le \varepsilon c_n) \\ &\le P(\widetilde{v}_n > xc_n + \widetilde{u}_n - \varepsilon c_n,\, \widetilde{u}_n \le \varepsilon c_n) \le P(\widetilde{v}_n - \widetilde{u}_n > c_n(x-\varepsilon)). \end{aligned}$$

Hence for $n \ge n(\varepsilon)$ and all $x > 0$ such that $x - \varepsilon > 0$ we get

$$1 - F_n(c_n(x+\varepsilon)) \le 1 - F_n^B(xc_n) \le (1 - F_n(c_n(x-\varepsilon)))(F_n^A(\varepsilon c_n))^{-1}. \tag{4}$$

This and condition P1 for $\{F_n\}$ and convergence in probability $\widetilde{u}_n/c_n \xrightarrow{p} 0$ give the inequalities

$$-N(x+\varepsilon) \le \liminf_n n(1 - F_n^B(xc_n)) \le \limsup_n n(1 - F_n^B(xc_n)) \le -N(x-\varepsilon)$$

if $x+\varepsilon$ and $x-\varepsilon > 0$ are continuity points of N. Hence if $x > 0$ is a continuity point of N then

$$\lim_n n(1 - F_n^B(xc_n)) = -N(x) \equiv -N^B(x). \tag{5}$$

Now, if $y < 0$ and $y + \varepsilon < 0$, then for $n \ge n(\varepsilon)$ we have

$$P(\widetilde{v}_n \le yc_n) = P(v_n \le c_n(y + \overline{v}_n/c_n)) \le P(v_n \le c_n(y+\varepsilon)) = 0.$$

Hence we get

$$\lim_n nF_n^B(yc_n) = \lim_n nP(\widetilde{v}_n \le yc_n) = 0 \equiv M^B(y) \quad \text{for all } y < 0. \tag{6}$$

This means that $\{F_n^B\}$ satisfies condition P1 with functions N^B and M^B equal to $N^B \equiv N$ and $M^B \equiv 0$, respectively.

In a similar way we get the inequalities

$$P(\widetilde{u}_n > c_n x) \geq P(\widetilde{u}_n - \widetilde{v}_n > c_n(x+\varepsilon)) = P(\widetilde{v}_n - \widetilde{u}_n < -c_n(x+\varepsilon)) \\ = F_n(-c_n(x+\varepsilon)-)$$

and

$$P(\widetilde{u}_n > xc_n)P(\widetilde{v}_n \leq \varepsilon c_n) \leq P(\widetilde{v}_n - \widetilde{u}_n < -c_n(x-\varepsilon)) = F_n(-c_n(x-\varepsilon)-),$$

for $n \geq n(\varepsilon)$, $x-\varepsilon > 0$, where $F_n(x-)$ is the left hand limit of F_n at x. Hence for $n \geq n(\varepsilon)$ and $x - \varepsilon > 0$ we get

$$(7) \qquad F_n(-c_n(x+\varepsilon)-) \leq 1 - F_n^A(xc_n) \leq F_n(-c_n(x-\varepsilon)-)(F_n^B(\varepsilon c_n))^{-1}.$$

But by condition P1 for $\{F_n\}$ we have $nF_n(xc_n-) \to M(x)$ whenever $x < 0$ is a continuity point of M. This together with (7) and condition P1 for $\{F_n\}$ gives

$$M(-x-\varepsilon) \leq \liminf_n n(1-F_n^A(xc_n)) \leq \limsup_n n(1-F_n^A(xc_n)) \leq M(-x+\varepsilon)$$

provided $-x-\varepsilon$ and $-x+\varepsilon$ are continuity points of M. Hence if $x > 0$ is a continuity point of M then

$$\lim_n n(1 - F_n^A(xc_n)) = M(-x) \equiv -N^A(x).$$

Now, reasoning in a similar way as for the sequence $\{F_n^B\}$ in (6) we get

$$\lim_{n\to\infty} nF_n^A(yc_n) = 0 \equiv M^A(y) \quad \text{for } y < 0.$$

All this implies that $\{F_n^A\}$ satisfies condition P1 with functions $N^A(x) \equiv -M(-x)$ for $x > 0$ and $M^A \equiv 0$. ∎

LEMMA 2. *If* $\{F_n\}$ *satisfies conditions* P1–P2 *and* P6, *then* $\{F_n^B\}$ *and* $\{F_n^A\}$ *satisfy condition* P2.

Proof. Condition P2 for $\{F_n^B\}$ and $\{F_n^A\}$ follows from condition P2 for $\{F_n\}$ and from inequalities (4) and (7), respectively. ∎

LEMMA 3. *Let* ξ *be a r.v. with distribution function* F *and* $E\xi = 0$. *Then for any* $c > 0$,

$$(8) \qquad \frac{1}{c} E\xi I(|\xi| \leq rc) = -r(1-F(rc)) + rF(-rc) \\ + \int_{-\infty}^{-r} F(cx)\,dx - \int_r^{\infty} (1-F(cx))\,dx.$$

Proof. Notice that

$$E\xi I(|\xi|\le r) = -E\xi I(|\xi|>r) = -\int_r^{\infty} x\,dF(x) - \int_{-\infty}^{-r} x\,dF(x)$$
$$= -r(1-F(r)) - \int_r^{\infty}(1-F(x))\,dx + rF(-r) + \int_{-\infty}^{-r} F(x)\,dx.$$

Replacing ξ by ξ/c we get the assertion of the lemma. ∎

LEMMA 4. *If $\{F_n\}$ satisfies condition* P1 *then for each $m \ge r$,*

$$\lim_n \int_{-m}^{-r} nF_n(xc_n)\,dx = \int_{-m}^{-r} M(x)\,dx \tag{9}$$

and

$$\lim_n \int_r^m n(1-F_n(xc_n))\,dx = -\int_r^m N(x)\,dx. \tag{10}$$

Proof. Let $f_n(x) := nP(\widetilde{v}_n - \widetilde{u}_n \le xc_n)$ for $x<0$. Then each f_n is nondecreasing on $(-\infty,0)$ and by condition P1 we have $f_n(x)\to M(x)$ for all $x<0$ that are continuity points of M. Hence for all $x<-r$ and some $\delta>0$ we have

$$0 \le f_n(x) \le f_n(-r) \le M(-r)+\delta \equiv f_0(x) \quad \text{and} \quad \int_{-m}^{-r} f_0(x)\,dx < \infty.$$

Therefore by Lebesgue's dominated convergence theorem we get

$$\lim_n \int_{-m}^{-r} f_n(x)\,dx = \int_{-m}^{-r} \lim_n f_n(x)\,dx = \int_{-m}^{-r} M(x)\,dx$$

for all $m\ge r$, which gives the first assertion.

To prove the second assertion note that the functions $g_n(x) := nP(\widetilde{v}_n - \widetilde{u}_n > xc_n)$ for $x>0$ are nonincreasing and $g_n(x)\to -N(x)$ for all $x>0$ that are continuity points of N. Hence for $x\in(r,m)$ and some $\delta>0$ we have

$$0 \le g_n(x) \le g_n(r) \le -N(r)+\delta \equiv g_0(x) \quad \text{and} \quad \int_r^m g_0(x)\,dx < \infty.$$

Therefore by Lebesgue's dominated convergence theorem we get

$$\lim_n \int_r^m g_n(x)\,dx = \int_r^m \lim_n g_n(x)\,dx = -\int_r^m N(x)\,dx$$

for all $m\ge r$, which gives the second assertion and finishes the proof of the Lemma. ∎

LEMMA 5. *If $\{F_n\}$ satisfies conditions* P1–P3 *and* P5, *then for* $r_1, r_2 \geq r$,

$$\lim_n \Big(\int_{-\infty}^{-r_1} nF_n(xc_n)\,dx - \int_{r_2}^{\infty} n(1-F_n(xc_n))\,dx \Big) = \int_{-\infty}^{-r_1} M(x)\,dx + \int_{r_2}^{\infty} N(x)\,dx. \tag{11}$$

Moreover if condition P7 *holds, then*

$$\lim_n \int_{r_2}^{\infty} n(1-F_n(xc_n))\,dx = -\int_{r_2}^{\infty} N(x)\,dx.$$

Proof. Putting $\xi = \widetilde{v}_n - \widetilde{u}_n$ and $c = c_n$ in Lemma 3 we get

$$\begin{aligned} b_{n,r} &:= \frac{n}{c_n} E(\widetilde{v}_n - \widetilde{u}_n) I(|\widetilde{v}_n - \widetilde{u}_n| \leq rc_n) \\ &= -rn(1-F_n(rc_n) + rnF_n(-rc_n) \\ &\quad + \int_{-\infty}^{-r} nF_n(xc_n)\,dx - \int_{r}^{\infty} n(1-F_n(xc_n))\,dx. \end{aligned} \tag{12}$$

Using P3 and P1 we get

$$b_r = rN(r) + rM(-r) + \lim_n \Big(\int_{-\infty}^{-r} nF_n(xc_n)\,dx - \int_{r}^{\infty} n(1-F_n(xc_n))\,dx \Big),$$

and this in view of Lemma 4 gives, for $r_1, r_2 \geq r$,

$$\int_{-\infty}^{-r_1} M(x)\,dx + \int_{r_2}^{\infty} N(x)\,dx = \lim_n \Big(\int_{-\infty}^{-r_1} nF_n(xc_n)\,dx - \int_{r_2}^{\infty} n(1-F_n(xc_n))\,dx \Big),$$

which finishes the proof of the first assertion of the lemma. The second assertion follows immediately from the first and from assumption P7. ∎

LEMMA 6. *Let $\{F_n\}$ satisfy conditions* P1–P3 *and* P5–P7. *Then $\{F_n^B\}$ and $\{F_n^A\}$ satisfy condition* P3 *with $b_r^B = rN^B(r) + \int_r^\infty N^B(x)\,dx$ and $b_r^A = rN^A(r) + \int_r^\infty N^A(x)\,dx$, respectively.*

Proof. Using Lemma 3 for $\xi = \widetilde{v}_n$ and $c = c_n$ we get

$$\begin{aligned} b_{n,r}^B &:= \frac{n}{c_n} E\widetilde{v}_n I(|\widetilde{v}_n| \leq rc_n) \\ &= -rn(1-F_n^B(rc_n)) + rnF_n^B(-rc_n) \\ &\quad - \int_{r}^{\infty} n(1-F_n^B(xc_n))\,dx + \int_{-\infty}^{-r} nF_n^B(xc_n)\,dx. \end{aligned} \tag{13}$$

Let $\varepsilon > 0$ be such that $r - \varepsilon > 0$ and $n(\varepsilon)$ be such that $\overline{v}_n/c_n < \varepsilon$ and $\overline{u}_n/c_n < \varepsilon$ for $n \geq n(\varepsilon)$, which is guaranteed by P6. Then for $x \leq -r$ we have

$$F_n^B(xc_n) = P(\widetilde{v}_n \le xc_n) = P(v_n \le c_n(x + \overline{v}_n/c_n)) = 0.$$

Hence for $n \ge n(\varepsilon)$ we get

$$(14)\qquad b_{n,r}^B = -rn(1 - F_n^B(rc_n)) - \int_r^\infty n(1 - F_n^B(xc_n))\,dx.$$

By inequality (4) we get

$$n(1 - F_n^B(xc_n)) \le n(1 - F_n(c_n(x - \varepsilon)))(F_n^A(\varepsilon c_n))^{-1}.$$

Applying this together with the convergences $F_n^A(\varepsilon c_n) \to 1$ and

$$\lim_n \int_r^\infty n(1 - F_n(xc_n))\,dx = \int_r^\infty \lim_n n(1 - F_n(xc_n))\,dx = -\int_r^\infty N(x)\,dx$$

and the Lebesgue dominated convergence theorem to the sequence $\{F_n^B\}$, which satisfies condition P1, we obtain the convergence

$$b_r^B := \lim_n b_{n,r}^B = rN^B(r) + \int_r^\infty N^B(x)\,dx.$$

In a similar way we get the convergence

$$b_r^A := \lim_n b_{n,r}^A = rN^A(r) + \int_r^\infty N^A(x)\,dx. \ \blacksquare$$

LEMMA 7. *If* $\{F_n\}$ *satisfies conditions* P1–P7 *then* $\{F_n^B\}$ *and* $\{F_n^A\}$ *satisfy condition* P4.

Proof. Notice that for any mutually independent random variables v and u and any number $\delta > 0$ we have

$$\big(vI(|v| \le \delta)I(|u| \le \delta) - uI(|u| \le \delta)I(|v| \le \delta)\big)^2 \le (v - u)^2 I(|v - u| \le 2\delta)$$

and

$$(15)\qquad E\big(vI(|v| \le \delta)I(|u| \le \delta) - uI(|u| \le \delta)I(|v| \le \delta)\big)^2 \le E(v - u)^2 I(|v - u| \le 2\delta).$$

But the left hand side of (15) equals

$$(16)\qquad Ev^2 I(|v| \le \delta)P(|u| \le \delta) + Eu^2 I(|u| \le \delta)P(|v| \le \delta) - 2EvI(|v| \le \delta)EuI(|u| \le \delta).$$

Putting $v = \widetilde{v}_n$, $u = \widetilde{u}_n$ and $\delta = c_n\varepsilon$ for $\varepsilon > 0$ in (15) and (16) we get

$$(17)\qquad E\widetilde{v}_n^2 I(|\widetilde{v}_n| \le c_n\varepsilon)P(|\widetilde{v}_n| \le c_n\varepsilon) + E\widetilde{u}_n^2 I(|\widetilde{u}_n| \le c_n\varepsilon)P(|\widetilde{u}_n| \le c_n\varepsilon) - 2E\widetilde{v}_n I(|\widetilde{v}_n| \le c_n\varepsilon)E\widetilde{u}_n I(|\widetilde{u}_n| \le c_n\varepsilon) \le E(\widetilde{v}_n - \widetilde{u}_n)^2 I(|\widetilde{v}_n - \widetilde{u}_n| \le 2\varepsilon c_n).$$

By the assumptions and Lemma 6 we know that $\{F_n^B\}$ and $\{F_n^A\}$ satisfy condition P3. Hence

$$\frac{n}{c_n}\, E\widetilde{v}_n I(|\widetilde{v}_n| \le \varepsilon c_n) = O(1) \quad \text{and} \quad \frac{n}{c_n}\, E\widetilde{u}_n I(|\widetilde{u}_n| \le \varepsilon c_n) = O(1),$$

which implies

$$\lim_n \frac{1}{c_n}\, E\widetilde{v}_n I(|\widetilde{v}_n| \le \varepsilon c_n) = \lim_n \frac{1}{c_n}\, E\widetilde{u}_n I(|\widetilde{u}_n| \le \varepsilon c_n) = 0.$$

All this implies

$$\lim_{\varepsilon\to 0} \limsup_n nE\frac{\widetilde{v}_n}{c_n}\, I(|\widetilde{v}_n| \le \varepsilon c_n) E\frac{\widetilde{u}_n}{c_n}\, I(|\widetilde{u}_n| \le \varepsilon c_n) = 0.$$

Together with (17) and P4 for $\{F_n\}$, this gives

$$\lim_{\varepsilon\to 0} \limsup_n \frac{n}{c_n^2} E\widetilde{v}_n^2 I(|\widetilde{v}_n| \le \varepsilon c_n) = 0$$

and

$$\lim_{\varepsilon\to 0} \limsup_n \frac{n}{c_n^2}\, E\widetilde{u}_n^2 I(|\widetilde{u}_n| \le \varepsilon c_n) = 0,$$

which means that $\{F_n^B\}$ and $\{F_n^A\}$ satisfy condition P4. ∎

Proof of Theorem 1. The assertion follows from Lemmas 1–7. ∎

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