APPLICATIONES MATHEMATICAE
33,1 (2006), pp. 71–84

Wojciech Połowczuk (Wrocław)

ON TWO-POINT NASH EQUILIBRIA IN BIMATRIX GAMES WITH CONVEXITY PROPERTIES

Abstract. This paper considers bimatrix games with matrices having concavity properties. The games described by such payoff matrices well approximate two-person non-zero-sum games on the unit square, with payoff functions $F_1(x, y)$ concave in x for each y, and/or $F_2(x, y)$ concave in y for each x. For these games it is shown that there are Nash equilibria in players' strategies with supports consisting of at most two points. Also a simple search procedure for such Nash equilibria is given.

1. Introduction. The assumption of concavity of payoff functions is very often used, both in theoretical considerations and practical applications of noncooperative games. A classical result in this field belongs to Glicksberg [1] and concerns n-person games on $\mathbb{R}^k$ with continuous quasi-concave payoffs (see Theorem 1).

Non-zero-sum n-person finite games were first studied by Nash [3], who proved that such games always have Nash equilibria. Shapley [8] gave some conditions for existence of saddle points in zero-sum matrix games. Radzik [5, 7] extended Shapley's results to games with matrices having some concavity-convexity properties. Next, a generalization of his result from [5] to bimatrix games was given in [4]. In these two papers [4, 5], pure Nash equilibria were considered. In [7], Radzik discussed two-point optimal strategies, i.e. strategies having supports with at most two elements. In the present paper we use the same concept of solution, generalizing the results from [7] to bimatrix games.

The organization of the paper is as follows. In Section 2 we present basic definitions and some background results. Section 3 contains our new results for bimatrix games. Finally, Sections 4 and 5 are devoted to their proofs.

2000 *Mathematics Subject Classification*: Primary 91A05.
Key words and phrases: non-zero-sum game, concave/convex game, Nash equilibrium.

2. Preliminary results. In this section we recall four background theorems, essential for our further considerations. First we need to fix some notation. We will consider n-person non-zero-sum games in normal form $\mathbf{G} = \langle N, \{X_i\}_{i\in N}, \{F_i\}_{i\in N}\rangle$, where

1. $N = \{1, \ldots, n\}$ is a finite set of players;
2. for each $i \in N$, X_i is a space of pure strategies x_i of Player i.
3. for each $i \in N$ and $x = (x_1, \ldots, x_n) \in \prod_{i\in N} X_i$, $F_i(x)$ is the payoff function of Player i in the situation when the players use pure strategies $x_1, \ldots, x_n$, respectively.

A *mixed strategy* for Player i is any probability measure μ_i on X_i, $i = 1, \ldots, n$.

The first background theorem belongs to Glicksberg [1]. We recall that by definition, a real-valued function f on a convex set X is *quasi-concave* if for each real c, the set $\{x : f(x) \geq c\}$ is convex. Clearly, every concave function is quasi-concave.

THEOREM 1. *Let $X_i \subset \mathbb{R}^k$ be non-empty, convex and compact for all $i \in N$. If every function F_i is continuous on $\prod_{i\in N} X_i$ and quasi-concave in x_i, then the n-person non-zero-sum game $\mathbf{G}$ has a pure strategy Nash equilibrium.*

In the next two theorems, a two-person non-zero-sum game on the unit square $\mathbf{G}' = \langle\{1, 2\}, \{[0, 1], [0, 1]\}, \{F, G\}\rangle$ is considered. The payoff functions $F(x, y)$ and $G(x, y)$ for Players 1 and 2, respectively, are assumed to be bounded and bounded from above on $[0, 1] \times [0, 1]$, respectively. Both these results were proved by Radzik in [6]. (Here and throughout the paper, δ_t is the degenerate probability distribution concentrated at the point t.)

THEOREM 2. *Let $F(x, y)$ be concave in x for each y. Then for any $\varepsilon > 0$, the game $\mathbf{G}'$ has an ε-Nash equilibrium of the form $(\alpha\delta_a + (1-\alpha)\delta_b, \beta\delta_c + (1-\beta)\delta_d)$ for some $0 \leq \alpha, \beta, a, b, c, d \leq 1$ with $|a - b| < \varepsilon$.*

THEOREM 3. *Let $F(x, y)$ be convex in x for each y. Then for any $\varepsilon > 0$, the game $\mathbf{G}'$ has an ε-Nash equilibrium of the form $(\alpha\delta_0 + (1-\alpha)\delta_1, \beta\delta_c + (1-\beta)\delta_d)$ for some $0 \leq \alpha, \beta, c, d \leq 1$, where α is independent of ε.*

In many situations, the players' strategy spaces are finite and the above theorems cannot be applied. In this paper we try to answer the question if the theorems have any "discrete" counterparts. We will study this problem for the two-person non-zero-sum case.

For the rest of the paper we will consider two-person non-zero-sum finite games with strategy spaces of the form $X_1 = \{1, \ldots, m\}$ and $X_2 = \{1, \ldots, n\}$ for two natural numbers m and n, and with payoff functions F_1 and F_2 for Players 1 and 2, respectively. Let A and B denote the $(m \times n)$-matrices such that $a_{ij} = F_1(i,j)$ and $b_{ij} = F_2(i,j)$ for all i and j. We will denote this bimatrix game by $\Gamma(A,B)$. It will also be called an $(m \times n)$-*game* $\Gamma(A,B)$ or denoted by $\Gamma(A,B)_{m\times n}$, to emphasize the size of the payoff matrices A and B.

Now we give the definitions of concavity for finite games, which are basic for our paper.

DEFINITION 1. A bimatrix game $\Gamma(A,B)_{m\times n}$ is said to be *column-concave* [*row-concave*] if there exists a function $F_1(x,y)$ [$F_2(x,y)$] on the unit square, concave in x for each y [concave in y for each x] and if there are two strictly increasing sequences $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ in $[0,1]$ such that $F_1(x_i,y_j) = a_{ij}$ [$F_2(x_i,y_j) = b_{ij}$] for all i and j. A *column-convex* [*row-convex*] game is defined analogously.

DEFINITION 2. A game $\Gamma(A,B)$ is called *concave* [*convex*] when it is column-concave and row-concave [column-convex and row-convex]. For a zero-sum game ($B = -A$), the equality $F_2 = -F_1$ is also required.

For a given game $\Gamma(A,B)$ it is rather difficult to check directly if it is concave or not. It turns out, however, that there exists an alternative (equivalent) characterization of concavity of bimatrix games, which allows us to check this property without difficulty. The proof of the following result is identical to the one for zero-sum two-person games, given in Radzik [7].

THEOREM 4. *A game $\Gamma(A,B)_{m\times n}$ is column-concave [row-concave] if and only if there exist positive numbers $\theta_1,\dots,\theta_{m-1}$ [$\tau_1,\dots,\tau_{n-1}$] such that*

$$(1)\quad \theta_1(a_{2j}-a_{1j}) \ge \theta_2(a_{3j}-a_{2j}) \ge \dots \ge \theta_{m-1}(a_{mj}-a_{m-1,j}) \quad \textit{for all } j$$

$$(2)\quad [\tau_1(b_{i2}-b_{i1}) \ge \tau_2(b_{i3}-b_{i2}) \ge \dots \ge \tau_{n-1}(b_{in}-b_{i,n-1}) \quad \textit{for all } i].$$

When all the inequalities in (1) *and/or in* (2) *are reverse, the game $\Gamma(A,B)$ is column-convex and/or row-convex.*

REMARK 1. Note that (1) and/or (2) hold with positive $\theta_1,\dots,\theta_{m-1}$ and $\tau_1,\dots,\tau_{n-1}$ if and only if for any $1 \le k \le m-2$ and $1 \le l \le n-2$ there are $\alpha_k > 0$ and $\beta_l > 0$ such that

$$(3)\qquad \alpha_k(a_{k+1,j}-a_{kj}) \ge a_{k+2,j}-a_{k+1,j} \quad \text{for all } j$$

and/or

$$(4)\qquad \beta_l(b_{i,l+1}-b_{il}) \ge b_{i,l+2}-b_{i,l+1} \quad \text{for all } i.$$

These two conditions are easily verifiable, allowing one to check whether a game is column-concave and/or row-concave. An analogous algorithm can be used in the "convex" case.

To end this section, we quote Theorem 4.3 from Radzik [7], essential for our paper. It describes the structure of players' optimal strategies in two-person zero-sum concave matrix games. This result can be seen as a "discrete" counterpart of Theorem 1 for zero-sum games, with quasi-concavity replaced by concavity.

THEOREM 5. *Let $\Gamma(A,-A)_{m\times n}$ be a concave zero-sum matrix game. Then there exist $0 \le \lambda, \gamma \le 1$ and $1 \le s < m$ and $1 \le r < n$ such that $(\mu^*, \nu^*) = (\lambda\delta_s + (1-\lambda)\delta_{s+1}, \gamma\delta_r + (1-\gamma)\delta_{r+1})$ is a pair of optimal strategies in $\Gamma(A,-A)$.*

It is worth adding here that a simple procedure of searching for optimal strategies described in the above theorem is also given in [7].

REMARK 2. In view of Theorem 1, one could ask if Theorem 5 remains true when "concavity" of $\Gamma(A,-A)$ is replaced by "quasi-concavity" (defined analogously to concavity). Unfortunately, as shown in [7], the result of Theorem 5 does not hold under this new weaker assumption.

In the next sections, we study two problems. The first is to generalize Theorem 5 to concave bimatrix games. The second problem is to get discrete counterparts of Theorems 2 and 3.

3. Main theorems. In this section we formulate our four main results. The first of them (Theorem 6) generalizes Theorem 5 to non-zero-sum games. It may also be seen as a discrete counterpart of Theorem 1 for the two-person case. To formulate it, we need to introduce some notation.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be fixed matrices of the same size $m \times n$, $m, n \ge 2$.

The game $\Gamma_1(A_1, B_1)$ is said to be a *subgame* of $\Gamma(A,B)$ if the matrices A_1 and B_1 can be obtained by removing some rows and (or) columns from A and B (the same for A and B).

Now let $\Gamma^{ij}_{kl} = \Gamma(A^{ij}_{kl}, B^{ij}_{kl})$, $1 \le i \le k \le m$, $1 \le j \le l \le n$, where for any $(m \times n)$-matrix $W = [w_{sr}]$ we put

$$W^{ij}_{kl} = \begin{bmatrix} w_{ij} & w_{i,j+1} & \dots & w_{il} \\ w_{i+1,j} & w_{i+1,j+1} & \dots & w_{i+1,l} \\ \vdots & \vdots & & \vdots \\ w_{kj} & w_{k,j+1} & \dots & w_{kl} \end{bmatrix}.$$

Obviously, each game Γ^{ij}_{kl} is a subgame of $\Gamma(A,B)$.

Further, define

$$\underline{\lambda^{ij}_{kl}} = \min(b^{ij}_{kl}, b^{i,j+1}_{k,l+1}), \qquad \overline{\lambda^{ij}_{kl}} = \max(b^{ij}_{kl}, b^{i,j+1}_{k,l+1}), \tag{5}$$

$$\underline{\gamma^{ij}_{kl}} = \min(a^{ij}_{kl}, a^{i+1,j}_{k+1,l}), \qquad \overline{\gamma^{ij}_{kl}} = \max(a^{ij}_{kl}, a^{i+1,j}_{k+1,l}), \tag{6}$$

where

$$b_{kl}^{ij} = \frac{b_{kl} - b_{kj}}{b_{kl} - b_{kj} + b_{ij} - b_{il}}, \tag{7}$$

$$a_{kl}^{ij} = \frac{a_{il} - a_{kl}}{a_{il} - a_{kl} + a_{kj} - a_{ij}}. \tag{8}$$

REMARK 3. One can easily check that if a game of the form

$$\overline{\Gamma} = \Gamma\left(\begin{bmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{bmatrix}, \begin{bmatrix} b_{ij} & b_{il} \\ b_{kj} & b_{kl} \end{bmatrix}\right)$$

does not have a pure Nash equilibrium, then the pair (μ^*, ν^*) with $\mu^* = b_{kl}^{ij}\delta_i + (1 - b_{kl}^{ij})\delta_k$ and $\nu^* = a_{kl}^{ij}\delta_j + (1 - a_{kl}^{ij})\delta_l$ is a Nash equilibrium in $\overline{\Gamma}$.

Now we are ready to formulate our first main theorem.

THEOREM 6. *Let $\Gamma = \Gamma(A, B)_{m\times n}$ be a concave game. Then one of the following four cases must occur*:

CASE 1: *There exists a pure Nash equilibrium (s, r) in Γ.*

CASE 2: *There exists a (2×2)-subgame $\Gamma_{s+1,r+1}^{sr}$ of Γ without pure Nash equilibria.*

In this case there is a Nash equilibrium in $\Gamma_{s+1,r+1}^{sr}$ of the form $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ and $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$ with $\lambda = b_{s+1,r+1}^{sr}$ and $\gamma = a_{s+1,r+1}^{sr}$, which is also a Nash equilibrium in Γ.

CASE 3: *For some $k \geq 3$ there is a $(k \times 2)$-subgame $\Gamma_{s+k-1,r+1}^{sr}$ of Γ without pure Nash equilibrium, which satisfies*

$$b_{lr} = b_{l,r+1} \quad \text{for all } l \text{ with } s < l < s + k - 1. \tag{9}$$

In this case for every l with $s < l < s + k - 1$ and every γ with $\underline{\gamma_{l,r+1}^{l-1,r}} \leq \gamma \leq \overline{\gamma_{l,r+1}^{l-1,r}}$, there is a Nash equilibrium in $\Gamma_{s+k-1,r+1}^{sr}$ of the form $\mu^* = \delta_l$ and $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$, which is also a Nash equilibrium in Γ.

CASE 4: *For some $k \geq 3$ there is a $(2 \times k)$-subgame $\Gamma_{s+1,r+k-1}^{sr}$ of Γ without pure Nash equilibrium, for which*

$$a_{sl} = a_{s+1,l} \quad \text{for all } l \text{ with } r < l < r + k - 1. \tag{10}$$

In this case for every l with $r < l < r + k - 1$ and every λ with $\underline{\lambda_{s+1,l}^{s,l-1}} \leq \lambda \leq \overline{\lambda_{s+1,l}^{s,l-1}}$, there is a Nash equilibrium in $\Gamma_{s+1,r+k-1}^{sr}$ of the form $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ and $\nu^* = \delta_l$, which is also a Nash equilibrium in Γ.

REMARK 4. A zero-sum version of Theorem 6 was proved in [7, Theorem 4.3]. However, for zero-sum games it is enough to consider only (2×3)- and (3×2)-subgames in Cases 3 and 4 above.

Our second main theorem generalizes Theorem 6.1 of [7] to non-zero-sum finite games. It can also be seen as a discrete counterpart of Theorem 2 given in the previous section.

THEOREM 7. *Let* $\Gamma = \Gamma(A,B)_{m\times n}$ *be a column-concave game. Then one of the following three cases must occur*:

CASE 1: *There exists a pure Nash equilibrium* (s,r) *in* Γ.

CASE 2: *For some* $1 \le s < m$, *there exists a* $(2\times n)$*-subgame* $\Gamma^{s1}_{s+1,n}$ *of* Γ *without pure Nash equilibria.*

In this case there is a Nash equilibrium in $\Gamma^{s1}_{s+1,n}$ *of the form* $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ *and* $\nu^* = \gamma\delta_r + (1-\gamma)\delta_u$ *for some* $0 < \lambda < 1$, $0 \le \gamma \le 1$ *and* $1 \le r < u \le n$, *which is also a Nash equilibrium in* Γ.

CASE 3: *For some* $1 < l < m$ *and* $1 \le r < u \le n$ *there exists a* (3×2)*-subgame of* Γ *of the form*

$$\Gamma' = \Gamma\left(\begin{bmatrix} a_{l-1,r} & a_{l-1,u} \\ a_{lr} & a_{lu} \\ a_{l+1,r} & a_{l+1,u}\end{bmatrix}, \begin{bmatrix} b_{l-1,r} & b_{l-1,u} \\ b_{lr} & b_{lu} \\ b_{l+1,r} & b_{l+1,u}\end{bmatrix}\right)$$

satisfying

$$b_{lr} = b_{lu} \ge b_{lj} \quad \textit{for all } 1 \le j \le n \tag{11}$$

and

$$\text{(a)}\begin{cases} a_{l-1,r} < a_{lr} < a_{l+1,r} \\ a_{l-1,u} > a_{lu} > a_{l+1,u}\end{cases} \quad \textit{or} \quad \text{(b)}\begin{cases} a_{l-1,r} > a_{lr} > a_{l+1,r} \\ a_{l-1,u} < a_{lu} < a_{l+1,u}.\end{cases} \tag{12}$$

In this case for every γ *with* $\underline{\gamma^{l-1,r}_{lu}} \le \gamma \le \overline{\gamma^{l-1,r}_{lu}}$, *the game* Γ' *has a mixed Nash equilibrium* (μ^*,ν^*) *of the form* $\mu^* = \delta_l$ *and* $\nu^* = \gamma\delta_r + (1-\gamma)\delta_u$, *which is also a Nash equilibrium in* Γ.

Before formulating our next theorem, for any $(m\times n)$-matrix $C = [c_{ij}]$, define the $(2\times n)$-matrix

$$C^1_m = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{m1} & c_{m2} & \dots & c_{mn}\end{bmatrix}.$$

Then, for any $\Gamma(A,B)_{m\times n}$, we put $\Gamma^1_m = \Gamma(A^1_m, B^1_m)$.

The theorem below generalizes Theorem 6.2 of Radzik [7] to the case of finite non-zero-sum games. It can also be seen as a discrete counterpart of Theorem 3.

THEOREM 8. *Let* $\Gamma = \Gamma(A,B)_{m\times n}$ *be a column-convex game. Then one of the following two cases must occur*:

CASE 1: *There exists a pure Nash equilibrium* (s,r) *in* Γ^1_m.

In this case $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ *is also a pure Nash equilibrium in* Γ.

CASE 2: Γ_m^1 *does not have a pure Nash equilibrium.*

In this case there is a Nash equilibrium in Γ_m^1 *of the form* $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ *and* $\nu^* = \gamma\delta_s + (1-\gamma)\delta_r$ *for some* $0 < \lambda < 1$, $0 \le \gamma \le 1$ *and* $1 \le s < r \le n$, *which is also a Nash equilibrium in* Γ.

Our last result is a modification of Theorem 6 with "concavity" replaced by "convexity".

THEOREM 9. *Let* $\Gamma = \Gamma(A, B)$, *with payoff* $(m \times n)$*-matrices* $A = [a_{ij}]$ *and* $B = [b_{ij}]$, *be a convex game, and let*

$$\Gamma'' = \Gamma\left(\begin{bmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{1n} \\ b_{m1} & b_{mn} \end{bmatrix}\right).$$

Then one of the following two cases must occur:

CASE 1: *There exists a pure Nash equilibrium* (s, r) *in* Γ''.

In this case $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ *is also a pure Nash equilibrium in* Γ.

CASE 2: Γ'' *does not have a pure Nash equilibrium.*

In this case there is a Nash equilibrium in Γ'' *of the form* (μ^*, ν^*), *where* $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ *and* $\nu^* = \gamma\delta_1 + (1-\gamma)\delta_n$, *with* $\lambda = b_{mn}^{11}$ *and* $\gamma = a_{mn}^{11}$, *which is also a Nash equilibrium in* Γ.

4. Auxiliary lemmata. The proofs of Theorems 6–9 will be given in the next section. In view of their complexity, we precede them by seven helpful lemmata.

Throughout this section, Γ denotes any bimatrix game $\Gamma(A, B)$, where $A = [a_{ij}]$ and $B = [b_{ij}]$ are $(m \times n)$-matrices, $m, n \ge 2$.

LEMMA 1. *Any subgame of a column-concave* [*row-concave*] *bimatrix game* Γ *is also a column-concave* [*row-concave*] *game.*

Proof. This is an immediate consequence of Definition 1.

The result of the next lemma in the case of a zero-sum game was proved in [7, Lemma 5.1]. Here we extend it to the non-zero-sum case.

LEMMA 2. *Let* Γ *be a column-concave* [*row-concave*] *game. Then for each* j *there exist natural numbers* $1 \le q \le t \le m$ [*for each* i *there exist natural numbers* $1 \le s \le r \le n$] *such that*

$$a_{1j} < a_{2j} < \cdots < a_{qj} = a_{q+1,j} = \cdots = a_{tj} > a_{t+1,j} > \cdots > a_{mj} \tag{13}$$

$$[b_{i1} < b_{i2} < \cdots < b_{is} = b_{i,s+1} = \cdots = b_{ir} > b_{i,r+1} > \cdots > b_{in}]. \tag{14}$$

When Γ *is a column-convex* [*row-convex*] *game, all the inequalities in* (13) *and* (14) *are reverse.*

Proof. Assume that Γ is column-concave. We easily conclude from (1) that for each j there exist q and t such that the sequence of differences

$$a_{2j}-a_{1j},\ a_{3j}-a_{2j},\ \ldots,\ a_{mj}-a_{m-1,j}$$

has the following property: the first $q-1$ elements are positive, the elements from the qth to the $(t-1)$th are 0, and the remaining ones are negative. (13) is a simple consequence of this property. Inequalities (14) can be proved analogously with the help of (2). The proof of the last part of the lemma is similar. ∎

LEMMA 3. *Let Γ be a column-concave [row-concave] game. For nonnegative numbers $\gamma_1,\ldots,\gamma_n$ $[\lambda_1,\ldots,\lambda_m]$, let $p_k=\sum_{j=1}^{n}\gamma_j a_{kj}$ $[w_l=\sum_{i=1}^{m}\lambda_i a_{il}]$ for $k=1,\ldots,m$ $[l=1,\ldots,n]$. Then there exist natural numbers $1\le s\le r\le m$ [there exist natural numbers $1\le q\le t\le n$] such that*

$$p_1<\cdots<p_s=p_{s+1}=\cdots=p_r>p_{r+1}>\cdots>p_m \tag{15}$$

$$[w_1<\cdots<w_q=w_{q+1}=\cdots=w_t>w_{t+1}>\cdots>w_n]. \tag{16}$$

When Γ is a column-convex [row-convex] game, all the inequalities in (15) *and* (16) *are reverse.*

Proof. To get (15) [(16)], it is enough to multiply (1) [and (2)] by γ_j [and λ_i] and sum the resulting inequalities over all $j=1,\ldots,n$ [all $i=1,\ldots,m$]. The rest of the proof is the same as for the previous lemma. ∎

LEMMA 4. *Let Γ be a concave game and let Γ_{kl}^{ij}, $1\le i<k\le m$, $1\le j<l\le n$, be its subgame. If $(\mu^*,\nu^*)=(\lambda\delta_s+(1-\lambda)\delta_{s+1},\gamma\delta_r+(1-\gamma)\delta_{r+1})$, $0<\lambda,\gamma<1$, $i\le s<k$, $j\le r<l$, is a Nash equilibrium in Γ_{kl}^{ij}, then it is also a Nash equilibrium in Γ.*

Proof. Since $0<\lambda,\gamma<1$ and $(\mu^*,\nu^*)=(\lambda\delta_s+(1-\lambda)\delta_{s+1},\gamma\delta_r+(1-\gamma)\delta_{r+1})$ is a Nash equilibrium in Γ_{kl}^{ij}, by the standard optimality property we have

$$\begin{aligned}\lambda b_{sr}+(1-\lambda)b_{s+1,r}&=\lambda b_{s,r+1}+(1-\lambda)b_{s+1,r+1}\\&\ge\lambda b_{st}+(1-\lambda)b_{s,t+1}\quad\text{for } j\le t\le l\end{aligned} \tag{17}$$

and

$$\begin{aligned}\gamma a_{sr}+(1-\gamma)a_{s,r+1}&=\gamma a_{s+1,r}+(1-\gamma)a_{s+1,r+1}\\&\ge\gamma a_{qr}+(1-\gamma)a_{q,r+1}\quad\text{for } i\le q\le k.\end{aligned} \tag{18}$$

But inequalities (15) and (16) imply that (17) and (18) remain true for $1\le t\le n$ and $1\le q\le m$. Therefore (μ^*,ν^*) is a Nash equilibrium in the entire game Γ. ∎

LEMMA 5. *Let Γ be a concave game and let Γ_{kl}^{ij}, $1\le i<k\le m$, $1\le j<l\le n$, be its subgame. If a pair $(\mu^*,\nu^*)=(\delta_s,\delta_r)$, $i<s<k$,*

$j < r < l$, is a Nash equilibrium in Γ^{ij}_{kl}, then it is also a Nash equilibrium in Γ.

Proof. The proof is the same as for Lemma 4, with (15) and (16) replaced by (13) and (14). ∎

LEMMA 6. *Let Γ be a concave game. Assume that $\Gamma^{l-1,r}_{l+1,r+1}$ is its subgame satisfying the following*:

$$b_{lr} = b_{l,r+1} \tag{19}$$

and

$$\text{(a)}\ \begin{cases} a_{l-1,r} < a_{lr} < a_{l+1,r}, \\ a_{l-1,r+1} > a_{l,r+1} > a_{l+1,r+1}, \end{cases} \quad \textit{or} \quad \text{(b)}\ \begin{cases} a_{l-1,r} > a_{lr} > a_{l+1,r}, \\ a_{l-1,r+1} < a_{l,r+1} < a_{l+1,r+1}. \end{cases} \tag{20}$$

Then for every γ with $\underline{\gamma^{l-1,r}_{l,r+1}} \le \gamma \le \overline{\gamma^{l-1,r}_{l,r+1}}$, there exists a Nash equilibrium (μ^*,ν^*) in $\Gamma^{l-1,r}_{l+1,r+1}$ of the form $\mu^* = \delta_l$, $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$, which is also a Nash equilibrium in the entire game Γ.

Proof. First assume that case (a) of (20) holds. Then, by (1), for some $\theta_{l-1}, \theta_l > 0$,

$$\theta_{l-1}(a_{l,r} - a_{l-1,r}) \ge \theta_l(a_{l+1,r} - a_{lr}) > 0 \tag{21}$$

and

$$0 > \theta_{l-1}(a_{l,r+1} - a_{l-1,r+1}) \ge \theta_l(a_{l+1,r+1} - a_{l,r+1}).$$

The last inequality can be rewritten as

$$\theta_l(a_{l,r+1} - a_{l+1,r+1}) \ge \theta_{l-1}(a_{l-1,r+1} - a_{l,r+1}) > 0. \tag{22}$$

But (21) and (22) lead to

$$\frac{\theta_{l-1}(a_{lr} - a_{l-1,r})}{\theta_{l-1}(a_{l-1,r+1} - a_{l,r+1})} \ge \frac{\theta_l(a_{l+1,r} - a_{l,r})}{\theta_l(a_{l,r+1} - a_{l+1,r+1})} > 0,$$

or equivalently

$$\frac{a_{l-1,r+1} - a_{l,r+1} + a_{l,r} - a_{l-1,r}}{a_{l-1,r+1} - a_{l,r+1}} \ge \frac{a_{l,r+1} - a_{l+1,r+1} + a_{l+1,r} - a_{l,r}}{a_{l,r+1} - a_{l+1,r+1}} > 1.$$

Hence, by (8), we easily get

$$0 < a^{l-1,r}_{l,r+1} \le a^{lr}_{l+1,r+1} < 1.$$

Now, fix any γ in the interval

$$a^{l-1,r}_{l,r+1} \le \gamma \le a^{lr}_{l+1,r+1}. \tag{23}$$

Clearly, $0 < \gamma < 1$. By (8) we can rewrite the second inequality in (23) in the form

$$\gamma(a_{l,r+1} - a_{l+1,r+1} + a_{l+1,r} - a_{lr}) \le a_{l,r+1} - a_{l+1,r+1},$$

which is equivalent to

$$\gamma a_{lr} + (1-\gamma)a_{l,r+1} \ge \gamma a_{l+1,r} + (1-\gamma)a_{l+1,r+1}. \tag{24}$$

Similarly, the first inequality of (23) implies

$$\gamma a_{l-1,r} + (1-\gamma)a_{l-1,r+1} \le \gamma a_{lr} + (1-\gamma)a_{l,r+1}. \tag{25}$$

From inequalities (24) and (25) we know that $\mu^* = \delta_l$ is the best strategy for Player I against the strategy $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$ of Player II in $\Gamma^{l-1,r}_{l+1,r+1}$, and because of (19), ν^* is also the best strategy for Player II against Player I's strategy μ^* in $\Gamma^{l-1,r}_{l+1,r+1}$. Therefore (μ^*, ν^*) is a Nash equilibrium in $\Gamma^{l-1,r}_{l+1,r+1}$, whence, by Lemmata 2 and 3, it is also a Nash equilibrium in Γ. To end case (a) notice that (23) is equivalent to

$$\underline{\gamma^{l-1,r}_{l,r+1}} \le \gamma \le \overline{\gamma^{l-1,r}_{l,r+1}}. \tag{26}$$

Case (b) of (20) is symmetric to case (a), in the sense that one of them becomes the other after interchanging $l-1$ with $l+1$. Consequently, we can repeat the considerations of case (a), getting inequality (23) in the form $a^{l+1,r}_{l,r+1} \le \gamma \le a^{lr}_{l-1,r+1}$. But this is also equivalent to (26), since, by (8), $a^{l+1,r}_{l,r+1} = a^{lr}_{l+1,r+1}$ and $a^{lr}_{l-1,r+1} = a^{l-1,r}_{l,r+1}$. ∎

LEMMA 7. *Assume that Γ is a concave bimatrix game which does not satisfy the assumptions of Cases* 1 *and* 2 *of Theorem* 6. *Then this game has a $(k\times 2)$-subgame of type $\Gamma^{sr}_{s+k-1,r+1}$ or a $(2\times k)$-subgame of type $\Gamma^{sr}_{s+1,r+k-1}$ without pure Nash equilibria.*

Proof. This is an immediate consequence of [4, Theorem 6] and Lemma 2. ∎

LEMMA 8. *Let $\Gamma = \Gamma(A,B)_{m\times n}$ be any bimatrix game. Then there is a Nash equilibrium (μ^*, ν^*) in Γ with supports of μ^* and ν^* consisting of at most* $\min(m,n)$ *elements.*

Proof. This follows from the well known theorem of Vorob'ev–Kuhn for bimatrix games (see [2, Lemmata 1 and 2] or [9]). ∎

5. Proof of the theorems

Proof of Theorem 6. Assume that there is no pure Nash equilibrium in Γ. We will show the validity of the statements in the remaining three Cases 2–4.

Proof of the statement in Case 2. It follows directly from Lemma 4 and Remark 3.

Proof of the statement in Case 3. Assume that a $(k \times 2)$-subgame $\Gamma^{sr}_{s+k-1,r+1}$ of Γ satisfies (9) and does not have any pure Nash equilibrium. Suppose that $a_{tr} = a_{t+1,r}$ for some t with $s \le t < s+k-1$. Then by (13), we easily conclude that

$$a_{sr} \le a_{s+1,r} \le \cdots \le a_{tr} = a_{t+1,r} \ge a_{t+2,r} \ge \cdots \ge a_{s+k-1,r}.$$

But this together with (9) implies that (t, r) and $(t+1, r)$ are pure Nash equilibria of $\Gamma^{sr}_{s+k-1,r+1}$, which contradicts the assumption. Therefore,

$$a_{tr} \ne a_{t+1,r} \quad \text{for all } t \text{ with } s \le t < s+k-1. \tag{27}$$

Now assume that $a_{sr} < a_{s+1,r}$. Then, in view of (13) and (27), only two subcases can happen:

$$a_{sr} < a_{s+1,r} < \cdots < a_{s+k-1,r} \tag{28}$$

or, for some i with $s < i < s+k-1$,

$$a_{sr} < a_{s+1,r} < \cdots < a_{ir} > a_{i+1,r} > \cdots > a_{s+k-1,r}. \tag{29}$$

But (29) is not possible, because then (i, r) would be a pure Nash equilibrium in $\Gamma^{sr}_{s+k-1,r+1}$. Therefore (28) must hold if $a_{sr} < a_{s+1,r}$.

In the second subcase $a_{sr} > a_{s+1,r}$, we see directly from (13) that

$$a_{sr} > a_{s+1,r} > \cdots > a_{s+k-1,r} \tag{30}$$

Exactly in the same way we can show that there are only two other possibilities, described by (28) or (30) with r replaced by $r+1$.

Summarizing, we easily conclude that one of the following two conditions must be satisfied:

$$\begin{gathered} \begin{cases} a_{sr} < a_{s+1,r} < \cdots < a_{s+k-1,r}, \\ a_{s,r+1} > a_{s+1,r+1} > \cdots > a_{s+k-1,r+1} \end{cases} \\ \text{or} \\ \begin{cases} a_{sr} > a_{s+1,r} > \cdots > a_{s+k-1,r}, \\ a_{s,r+1} < a_{s+1,r+1} < \cdots < a_{s+k-1,r+1}. \end{cases} \end{gathered} \tag{31}$$

Otherwise all the inequalities in one of them would be of the same type, "$<$" or "$>$", easily implying the existence of a pure Nash equilibrium in $\Gamma^{sr}_{s+k-1,r+1}$.

Now, take any l with $s < l < s+k-1$. Then (9), (31) and Lemma 6 immediately imply the validity of the statement in Case 3.

Proof of the statement in Case 4. This case is symmetric to Case 3 and is omitted.

To complete the proof of Theorem 6, it should be shown that only Cases 1–4 are possible. Assume then that Cases 1 and 2 do not hold.

By Lemma 7, we can assume that Γ has a minimal $(k \times 2)$-subgame of the form $\Gamma^{sr}_{s+k-1,r+1}$ without pure Nash equilibria (the other possibility with a $(2 \times k)$-subgame $\Gamma^{sr}_{s+1,r+k-1}$ is symmetric). Therefore, all the proper subgames of $\Gamma^{sr}_{s+k-1,r+1}$ of size $k' \times 2$ have pure Nash equilibria.

For any l with $s < l < s+k-1$, consider two proper subgames of $\Gamma^{sr}_{s+k-1,r+1}$, namely $\Gamma^{sr}_{l,r+1}$ and $\Gamma^{lr}_{s+k-1,r+1}$. Both should have pure Nash equilibria. If the first of them has one outside its last row, then, by (13), it is also a pure Nash equilibrium in $\Gamma^{sr}_{s+k-1,r+1}$, contradicting the assumption. The same arguments imply that $\Gamma^{lr}_{s+k-1,r+1}$ cannot have a pure Nash equilibrium outside its first row. Hence, $\Gamma^{sr}_{l,r+1}$ and $\Gamma^{lr}_{s+k-1,r+1}$ have pure Nash equilibria in the lth row of Γ. It is easy to conclude now that they have different pure equilibria, (l,r) and $(l,r+1)$ (otherwise, $\Gamma^{sr}_{s+k-1,r+1}$ would have a pure Nash equilibrium). However, this implies $b_{lr} = b_{l,r+1}$ for $s < l < s+k-1$, which ends the proof of Theorem 6. ■

Proof of Theorem 7. We begin by showing that one of Cases 1–3 must occur. Assume that Cases 1 and 2 do not hold. Since Γ is a column-concave game, the part of Lemma 2 with (13) holds.

Therefore each subgame $\Gamma^{l1}_{l+1,n}$, $1 \le l < m$, has a pure Nash equilibrium, say $(x(l), y(l))$. If $x(1) = 1$ or $x(m-1) = m$, then $(x(1), y(1))$ or $(x(m-1), y(m-1))$, respectively, would be a pure Nash equilibrium in the entire game Γ, because of (13). Therefore $x(1) = 2$ and $x(m-1) = m-1$. Consequently, there exists $1 \le l < m$ such that $x(l-1) = x(l) = l$. Set $y(l-1) = r$ and $y(l) = u$. Then, because (l,r) is a pure Nash equilibrium in $\Gamma^{l-1,1}_{ln}$ and (l,u) is a pure Nash equilibrium in $\Gamma^{l1}_{l+1,n}$, we have

$$b_{lr} = b_{lu} \ge b_{lj} \quad \text{for all } 1 \le j \le n, \tag{32}$$

and

$$a_{lr} \ge a_{l-1,r} \quad \text{and} \quad a_{lu} \ge a_{l+1,u}.$$

But, because Case 1 does not hold, neither (l,r) nor (l,u) can be a pure Nash equilibrium in the entire game Γ. Hence, using (32) and (13), we easily deduce

$$a_{l-1,r} < a_{lr} < a_{l+1,r} \quad \text{and} \quad a_{l-1,u} > a_{lu} > a_{l+1,u}. \tag{33}$$

If $r < u$ then (33) is equivalent to (12)(a), while if $u < r$ then (33) is equivalent to (12)(b). Therefore only Cases 1, 2 or 3 can hold.

Now we will show the validity of the statements in Cases 2 and 3 of the theorem.

Proof of the statement in Case 2. Let the assumptions of this case be satisfied and consider the subgame $\Gamma^{s1}_{s+1,n}$. From Lemma 8 it follows that in this subgame there exists a Nash equilibrium (μ^*, ν^*) of the form $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ and $\nu^* = \gamma\delta_r + (1-\gamma)\delta_u$ for some $0 \le \lambda, \gamma \le 1$ and $1 \le r < u \le n$. Suppose that $\lambda = 0$ or $\lambda = 1$. Then, using the properties of Nash equilibrium, we easily deduce that $\Gamma^{s1}_{s+1,n}$ has a pure Nash equilibrium, contradicting the assumption. Therefore $0 < \lambda < 1$. Hence, by the optimality of Nash equilibrium we have

$$\gamma a_{sr} + (1-\gamma)a_{su} = \gamma a_{s+1,r} + (1-\gamma)a_{s+1,u},$$

which together with (15) implies that μ^* is the best strategy for Player I against ν^* of Player II in the entire game Γ. The Nash optimality of ν^* against μ^* in Γ follows directly from its optimality in $\Gamma_{s+1,n}^{s1}$. Thus (μ^*, ν^*) is a Nash equilibrium in Γ.

Proof of the statement in Case 3. The game Γ' is column-concave and has only two columns, hence it is concave. Since (11) holds, Lemma 6 can be applied to conclude that the pair (μ^*, ν^*) described in Case 3 of the theorem is a Nash equilibrium in Γ'. The fact that (μ^*, ν^*) is an equlibrium in the entire game $\Gamma(A, B)$ follows easily from (11) and (15) taken for $p_k = \gamma a_{kr} + (1-\gamma)a_{ku}$, $k = 1, \ldots, m$. The simple details are omitted. ∎

Proof of Theorem 8. We apply Lemma 8 to get the existence of a Nash equilibrium (μ^*, ν^*) in Γ_m^1 of the form $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ and $\nu^* = \gamma\delta_s + (1-\gamma)\delta_r$ for some λ, γ, s and r with $0 \le \lambda, \gamma \le 1$ and $1 \le s \le r \le n$.

By optimality of (μ^*, ν^*), we easily get

$$\begin{aligned}\lambda b_{1s} + (1-\lambda)b_{ms} &= \lambda b_{1r} + (1-\lambda)b_{mr} \\ &\ge \lambda b_{1j} + (1-\lambda)b_{mj} \quad \text{for all } 1 \le j \le n.\end{aligned} \tag{34}$$

Now, if we put $p_i = \gamma a_{is} + (1-\gamma)a_{ir}$, $i = 1, \ldots, m$, then, by the "convex" part of Lemma 3, for some t and u we have

$$p_1 > \cdots > p_t = p_{t+1} = \cdots = p_u < p_{u+1} < \cdots < p_m.$$

But the last inequalities, (34) and the definition of (μ^*, ν^*) immediately imply that (μ^*, ν^*) is a Nash equilibrium in Γ. If it is a pure Nash equilibrium, then we have Case 1 of the theorem, otherwise we have Case 2, completing the proof. ∎

Proof of Theorem 9. In the game Γ' there is a Nash equilibrium (μ^*, ν^*) with $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ and $\nu^* = \gamma\delta_1 + (1-\gamma)\delta_n$, where $0 \le \lambda, \gamma \le 1$. Let $p_i = \gamma a_{i1} + (1-\gamma)a_{in}$, $i = 1, \ldots, m$, and $w_j = \lambda b_{1j} + (1-\lambda)b_{mj}$, $j = 1, \ldots, n$. Then, by the "convex" part of Lemma 3, for some r, s, t and u we have

$$p_1 > p_2 > \cdots > p_r = p_{r+1} = \cdots = p_s < p_{s+1} < \cdots < p_m, \tag{35}$$

$$w_1 > w_2 > \cdots > w_t = w_{t+1} = \cdots = w_u < w_{u+1} < \cdots < w_n. \tag{36}$$

But (35) implies that μ^* is the best strategy for Player I against the strategy ν^* of Player II in the entire game Γ. Similarly (36) implies that ν^* is the best strategy for Player II against μ^* of Player I in Γ. Hence, (μ^*, ν^*) is also a Nash equilibrium in Γ. If it is a pure Nash equilibrium, then we have Case 1 of the theorem, otherwise Case 2 holds. ∎

Acknowledgments. The author wishes to thank Professors T. Radzik and A. S. Nowak for their remarks and comments.

References

[1] I. Glicksberg, *A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points*, Proc. Amer. Math. Soc. 38 (1952), 170–174.
[2] H. W. Kuhn, *An algorithm for equilibrium points in bimatrix games*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1657–1662.
[3] J. Nash, *Non-cooperative games*, Ann. of Math. 54 (1951), 286–295.
[4] W. Połowczuk, *Pure Nash equilibria in finite two-person non-zero-sum games*, Internat. J. Game Theory 32 (2003), 229–240.
[5] T. Radzik, *Saddle point theorems*, ibid. 20 (1991), 23–32.
[6] —, *Nash equilibria of discontinuous non-zero-sum two-person games*, ibid. 21 (1993), 429–437.
[7] —, *Characterization of optimal strategies in matrix games with convexity properties*, ibid. 29 (2000), 211–227.
[8] L. Shapley, *Some topics in two-person games*, in: Advances in Game Theory, Ann. of Math. Stud. 52, Princeton Univ. Press, 1964, 1–28.
[9] N. N. Vorob'ev, *Equilibrium points in bimatrix games*, Teor. Veroyatn. i Primenen. 3 (1958), 318–331 (in Russian).

Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: Wojciech.Polowczuk@pwr.wroc.pl

Received on 29.9.2003;
revised version on 14.11.2005 (1707)