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CONVERGENCE OF OPTIMAL STRATEGIES IN A DISCRETE TIME MARKET WITH FINITE HORIZON

Abstract. A discrete-time financial market model with finite time horizon is considered, together with a sequence of investors whose preferences are described by a convergent sequence of strictly increasing and strictly concave utility functions. Existence of unique optimal consumption-investment strategies as well as their convergence to the limit strategy is shown.

Introduction. Recently, in a number of papers the following question was considered: does convergence of investors’ preferences imply the convergence of their optimal strategies? In [2] a model with complete Brownian market model was described, while in [1] a discrete time model with finite horizon and utility functions defined on the whole real line was studied. Both papers gave a positive answer to the above problem under suitable assumptions.

In the present paper we prove a similar result for a discrete time market model with a finite horizon. We assume weaker regularity conditions on utility functions: strict concavity and strict monotonicity. The utility functions considered are defined on the positive axis.

In the first section we describe our model of financial market. Then we consider a one-step model and utilizing ideas from [4], we establish a few useful technical results. Finally, we prove the existence of optimal strategies for our model and their convergence together with the convergence of the investors’ preferences.

1. Market model. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a discrete-time filtered probability space with finite time horizon $T \in \mathbb{N}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prices of

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$d$ risky securities available on the market are represented by a $d$-dimensional, almost surely positive adapted process $S_t = (S_{t,1}, \ldots, S_{t,d})'$, $0 \leq t \leq T$. For $t = 0, \ldots, T - 1$ we define
\[
\zeta_{t,i} = \frac{S_{t+1,i}}{S_{t,i}}, \quad i = 1, \ldots, d,
\]
and $\zeta_t = (\zeta_{t,1}, \ldots, \zeta_{t,d})'$. Let $D_t(\omega)$ be the smallest linear subspace containing the support of the regular conditional distribution of $\zeta_t$ with respect to $\mathcal{F}_t$ (it exists, cf. [6, Theorem 2.7.5]). Throughout the paper we assume that there are no redundant assets on the market, thus we have the following non-degeneracy assumption:

**Assumption 1.1.** $D_t$ is almost surely equal to $\mathbb{R}^d$ for $0 \leq t \leq T - 1$.

Let $\Delta_0 = \{ \nu \in \mathbb{R}^d : \nu_i \geq 0, \sum_{i=1}^d \nu_i \leq 1 \}$, and $\Delta = \{ \nu \in \Delta_0 : \sum_{i=1}^d \nu_i = 1 \}$. We denote by $\langle \cdot , \cdot \rangle$ the usual scalar product in $\mathbb{R}^d$. Denote by $X_t$ the wealth process at time $t$ before consumption and possible transactions. Let $\pi_{t,i}$ and $\overline{\pi}_{t,i}$ be the portions of the wealth $X_t$ invested in the $i$th asset at time $t$ before and respectively after consumption and possible transactions. We do not allow short selling or short borrowing, so $\pi_{t} = (\pi_{t,1}, \ldots, \pi_{t,d})' \in \Delta$ and $\overline{\pi}_{t} = (\overline{\pi}_{t,1}, \ldots, \overline{\pi}_{t,d})' \in \Delta_0$.

At time $t = 0, \ldots, T - 1$, the investor who owns initial wealth $X_t$ invested in portfolio $\pi_t$, consumes a part $\alpha_t \in [0, 1]$ of his wealth and changes his portfolio composition to $\overline{\pi}_{t}$, according to the equation
\[
X_{t} = X_{t_{\alpha} \sum_{i=1}^{d} \overline{\pi}_{t,i} ,}
\]
which implies that we are interested only in $\mathcal{F}_t$-measurable strategies such that $(\alpha_t, \overline{\pi}_{t}) \in [0, 1] \times \Delta_0$ a.s. and
\[
\alpha_t + \sum_{i=1}^{d} \overline{\pi}_{t,i} = 1 \quad \text{a.s.}
\]
(1.1)

Denote the set of such strategies by $\mathcal{A}_t$.

At time $t + 1$, due to price changes, the investor’s wealth changes to
\[
X_{t+1} = \sum_{i=1}^{d} X_t \overline{\pi}_{t,i} \zeta_{t,i} = X_t \langle \overline{\pi}_{t}, \zeta_t \rangle.
\]
(1.2)

Equation (1.2) describes the dynamics of the control system we are dealing with: $X_t$ is regarded as a state of the system, and $(\alpha_t, \overline{\pi}_{t}) \in [0, 1] \times \Delta_0$ are its control parameters, constrained by (1.1) describing the admissible strategies. The initial condition is given by the endowment $x := X_0 > 0$. 

We consider a sequence of investors with preferences described by utility functions $U^n_t: (0, \infty) \to \mathbb{R}$, $0 \leq t \leq T$, $n \in \mathbb{N} := \mathbb{N} \cup \{\infty\}$.

**Assumption 1.2.** The functions $U^n_t$ are strictly increasing and strictly concave for $t \in \{0, \ldots, T\}$ and $n \in \mathbb{N}$. Moreover, for all $t \in \{0, \ldots, T\}$ and $x \in (0, \infty)$,

$$U^n_t(x) \to U^\infty_t(x) \quad \text{as } n \to \infty.$$

We are interested in maximization of the expected utility from consumption and terminal wealth, that is, we want to maximize the following reward functional:

\begin{equation}
J_T^n(x, (\alpha, \bar{\pi})) = \mathbb{E} \left( \sum_{t=0}^{T-1} U^n_t(X_t\alpha_t) + U^n_T(X_T) \right).
\end{equation}

For our dynamic programming problem to be well posed and finite, we assume that the following conditions are satisfied:

**Assumption 1.3.** For all $n \in \mathbb{N}$, $k \in \{1, \ldots, T\}$ and $x > 0$,

$$\mathbb{E}(U^n_k)^+(x \prod_{i=0}^{k-1} \max \{\zeta_{t,i} : i = 1, \ldots, d\}) < \infty,$$

$$\mathbb{E}(U^n_k)^-(x \prod_{i=0}^{k-1} \min \{\zeta_{t,i} : i = 1, \ldots, d\}) < \infty.$$

**Remark 1.4.** One can consume all or nothing of the wealth, so we need values of utility functions at 0. We deal with that problem by putting $U(0) := \lim_{x \to 0^+} U(x)$; if this limit is finite, the continuity and concavity properties are kept, and if not (e.g. for a logarithmic function), the agent will not choose such a strategy to maximize utility.

### 2. One-step case.
We start with the case $T = 1$. Let $u, v: (0, \infty) \to \mathbb{R}$ be strictly increasing functions, $u$ strictly concave and $v$ concave. Let $\mathcal{H}$ be a sub-$\sigma$-field of $\mathcal{F}$, let $\zeta = (\zeta_1, \ldots, \zeta_d)'$ be an $\mathbb{R}^d$-valued random variable with non-degenerate (in the sense of Assumption 1.1) conditional distribution with respect to $\mathcal{H}$, and let $\mathbb{E}(\cdot | \mathcal{H})$ denote conditional expectation with respect to $\mathcal{H}$. Denote by $\mathcal{A}$ the set of admissible strategies: $\mathcal{H}$-measurable random variables such that $(\alpha, \bar{\pi}) \in [0, 1] \times \Delta_0$ a.s. and $\alpha + \sum_{j=1}^d \bar{\pi}_j = 1$.

Define the value function by

$$w(x) := \text{ess sup}_{(\alpha, \bar{\pi}) \in \mathcal{A}} \left\{ u(x\alpha) + \mathbb{E}(v(x\bar{\pi}, \zeta)) \mid \mathcal{H} \right\}, \quad x > 0.$$

Analogously to Assumption 1.3, we introduce
Assumption 2.1. For all \( x > 0 \),
\[
\mathbb{E}v^+(x \max_{i \in \{1, \ldots, d\}} \zeta_i) < \infty \quad \text{and} \quad \mathbb{E}v^-(x \min_{i \in \{1, \ldots, d\}} \zeta_i) < \infty.
\]

The following technical lemmas are crucial:

Lemma 2.2. There exists an almost surely continuous, strictly concave and strictly increasing (with respect to every coordinate) version of
\[
[0, \infty)^d \setminus \{0\} \ni \pi \mapsto \mathbb{E}(v(\langle \pi, \zeta \rangle) \mid \mathcal{H}).
\]

Proof. Let \( \kappa \) denote the regular conditional distribution of \( \zeta \) given \( \mathcal{H} \). Then
\[
\mathbb{E}(v(\langle \pi, \zeta \rangle) \mid \mathcal{H}) = \int v(\langle \pi, x \rangle) \kappa(dx) \quad \text{a.s.,}
\]
and we take the right side as a definition of our version. By a routine calculation one checks it has the desired properties. We will show concavity. Fix \( \pi^1, \pi^2 \in [0, \infty)^d \setminus \{0\}, \pi^1 \neq \pi^2 \) and \( t \in (0, 1) \). Then
\[
t \mathbb{E}(v(\langle \pi^1, \zeta \rangle) \mid \mathcal{H}) + (1 - t) \mathbb{E}(v(\langle \pi^2, \zeta \rangle) \mid \mathcal{H})
\]
\[
= \int [tv(\langle \pi^1, x \rangle) + (1 - t)v(\langle \pi^2, x \rangle)] \kappa(dx)
\]
\[
< \int v(\langle t\pi^1 + (1 - t)\pi^2, x \rangle) \kappa(dx)
\]
\[
= \mathbb{E}(v(\langle t\pi^1 + (1 - t)\pi^2, \zeta \rangle) \mid \mathcal{H}) \quad \text{a.s.}
\]
The strict inequality is justified by Assumption 1.1. \( \blacksquare \)

Proposition 2.3. For every \( x \in (0, \infty) \) there exists a unique optimal pair \( (\hat{\alpha}, \hat{\pi}) \in \mathcal{A} \) such that
\[
(2.1) \quad w(x) = u(x\hat{\alpha}) + \mathbb{E}(v(x\langle \hat{\pi}, \zeta \rangle) \mid \mathcal{H}) \quad \text{a.s.}
\]

Proof. We take the version of conditional expectation with the properties stated in Lemma 2.2, and consider the mapping
\[
\Phi: [0, 1] \times \Delta_0 \times \Omega \ni (\alpha, \pi, \omega) \mapsto u(x\alpha) + \mathbb{E}(v(x\langle \pi, \zeta \rangle) \mid \mathcal{H})(\omega) \in \mathbb{R}
\]
which is continuous except on a \( \mathbb{P} \)-zero set \( N \). Since the set
\[
(2.2) \quad \left\{ (\alpha, \pi) \in [0, 1] \times \Delta_0 : \alpha + \sum_{j=1}^d \pi_j = 1 \right\}
\]
is compact, for any \( \omega \in \Omega \setminus N \) there is a pair \( (\hat{\alpha}(\omega), \hat{\pi}(\omega)) \) attaining the supremum of \( \Phi \).

Suppose that there are two such pairs, say \( (\alpha^1, \pi^1), (\alpha^2, \pi^2) \in \mathcal{A} \). Take any \( t \in (0, 1) \). Putting \( \alpha = t\alpha^1 + (1 - t)\alpha^2, \pi = t\pi^1 + (1 - t)\pi^2 \) we have
\( \alpha \in (0,1), \ \bar{\pi} \in \Delta_0 \) a.s. Since \( \sum_{i=1}^{m} \bar{\pi}_i = 1 - \alpha \), it follows that \( (\alpha, \bar{\pi}) \in \mathcal{A} \) and
\[
 w(x) = tw(x) + (1-t)w(x) \\
= t[u(x\alpha^1) + \mathbb{E}(v(x\bar{\pi}^1, \zeta)) | \mathcal{H}] \\
+ (1-t)[u(x\alpha^2) + \mathbb{E}(v(x\bar{\pi}^2, \zeta)) | \mathcal{H}] \\
\leq u(x\alpha) + \mathbb{E}(v(tx\bar{\pi}^1, \zeta) + (1-t)x\bar{\pi}^2, \zeta) | \mathcal{H}) \\
= u(x\alpha) + \mathbb{E}(v(x\bar{\pi}, \zeta)) | \mathcal{H}) \leq w(x) \quad \text{a.s.}
\]

Both \( u \) and \( v \) are strictly concave, thus the above inequality turns into an equality iff \( \alpha^1 = \alpha^2 \) and \( \langle \bar{\pi}^1, \zeta \rangle = \langle \bar{\pi}^2, \zeta \rangle \) a.s. From the assumption we made on the support of the distribution of \( \zeta \), that implies \( \bar{\pi}_i^1 = \bar{\pi}_i^2 \) a.s., \( i = 1, \ldots, d \), hence the proof of uniqueness is finished.

The optimal pair \( (\hat{\alpha}, \hat{\pi}) \) is an \( \mathcal{H} \)-measurable random variable, since for any open ball \( B \subset \mathbb{R}^{d+1} \),
\[
(\hat{\alpha}, \hat{\pi})(\omega) \in B \iff \bigvee_{(\alpha^*, \pi^*) \in C \cap B} \bigwedge_{(\alpha, \pi) \in C \setminus B} \Phi(\alpha^*, \pi^*)(\omega) > \Phi(\alpha, \pi)(\omega)
\]

where \( C \) denotes a countable dense subset of \( (2.2) \), and therefore
\[
\{ (\hat{\alpha}, \hat{\pi}) \in B \} = \bigcup_{(\alpha^*, \pi^*) \in C \cap B} \bigcap_{(\alpha, \pi) \in C \setminus B} \{ \Phi(\alpha^*, \pi^*) > \Phi(\alpha, \pi) \} \in \mathcal{H}. \quad \blacksquare
\]

**Lemma 2.4.** There is a version of the value function \( w \) which is almost surely strictly increasing and strictly concave.

**Proof.** For every \( q \in (0, \infty) \cap \mathbb{Q} \) fix a version of \( w(q) \), which by Assumption 2.1 is almost surely finite. Fix \( x, y \in (0, \infty) \cap \mathbb{Q} \). It is obvious that if \( y < x \) then \( w(y) < w(x) \) a.s. To show strict concavity, fix \( t \in (0,1) \cap \mathbb{Q} \) and let \((\alpha^x, \bar{\pi}^x), (\alpha^y, \bar{\pi}^y) \in \mathcal{A}\) be optimal pairs for \( x \) and \( y \) respectively. Put \( z = tx + (1-t)y, \beta = tx/z, \alpha = \beta \alpha^x + (1-\beta)\alpha^y, \bar{\pi} = \beta \bar{\pi}^x + (1-\beta)\bar{\pi}^y. \)

Obviously \( \alpha \in [0,1], \beta \in (0,1) \) a.s. Since \( \sum_{i=1}^{d} \bar{\pi}_i = 1 - \alpha \), we obtain
\[
tx\bar{\pi}^x + (1-t)y\bar{\pi}^y = z(\beta \bar{\pi}^x + (1-\beta)\bar{\pi}^y) = z\bar{\pi},
\]
and since \( u \) and \( v \) are strictly concave and \( \zeta \) is almost surely positive, we have
\[
 tw(x) + (1-t)w(y) = t[u(x\alpha^x) + \mathbb{E}(v(x\bar{\pi}^x, \zeta)) | \mathcal{H}] \\
+ (1-t)[u(y\alpha^y) + \mathbb{E}(v(y\bar{\pi}^y, \zeta)) | \mathcal{H}] \\
\leq u(z\alpha) + \mathbb{E}(v(z\bar{\pi}, \zeta) | \mathcal{H})) \leq w(z) \quad \text{a.s.}
\]

and moreover this inequality turns into an equality iff
\[
x\alpha^x = y\alpha^y \quad \text{and} \quad x\langle \bar{\pi}^x, \zeta \rangle = y\langle \bar{\pi}^y, \zeta \rangle \quad \text{a.s.}
\]

Once again using our assumption on the distribution of \( \zeta \), this implies
\[
x\bar{\pi}^x_i = y\bar{\pi}^y_i, \quad i = 1, \ldots, d,
\]
and summing those equalities up for \( i = 1, \ldots, d \) we obtain
\[
x[1 - \alpha^x] = y[1 - \alpha^y],
\]
hence also \( x = y \). This shows in particular that for all \( x, y \in (0, \infty) \cap \mathbb{Q} \), \( x \neq y \), we have
\[
w\left( \frac{x + y}{2} \right) > \frac{w(x) + w(y)}{2} \quad \text{a.s.}
\]
We can now extend this version of \( w \) to a function which is almost surely strictly increasing and strictly continuous for all \( x \in (0, \infty) \). Finally, from monotone convergence, for fixed \( x \in (0, \infty) \) and a sequence of rationals \( q_n \uparrow x \) we have
\[
w(x) = \lim_{n \to \infty} w(q_n) = \lim_{n \to \infty} \ess sup_{(\pi, \alpha) \in \mathcal{A}} \{ u(q_n \alpha) + \mathbb{E}(v(q_n \langle \pi, \zeta \rangle) | \mathcal{H}) \}
\]
\[
= \ess sup_{(\pi, \alpha) \in \mathcal{A}} \{ u(x \alpha) + \mathbb{E}(v(x \langle \pi, \zeta \rangle) | \mathcal{H}) \}. \quad \blacksquare
\]

**Proposition 2.5.** There exists a selector of optimal strategies
\[
(0, \infty) \ni x \mapsto (\tilde{\alpha}, \tilde{\pi})(x) \in \mathcal{A}
\]
which is continuous for almost all \( \omega \).

**Proof.** We fix a version of conditional expectation with the properties stated in Lemma 2.2. The random function
\[
w(x, (\alpha, \pi)) := u(x \alpha) + \mathbb{E}(v(x \langle \pi, \zeta \rangle) | \mathcal{H})
\]
is then almost surely continuous, jointly for all arguments. Suppose there exists \( x \in (0, \infty) \) and a sequence \( x_n \in (0, \infty), n \in \mathbb{N} \), such that \( x_n \to x \) and \( (\tilde{\alpha}, \tilde{\pi})(x_n) \not\sim (\tilde{\alpha}, \tilde{\pi})(x) \). Since all \( (\tilde{\alpha}, \tilde{\pi})(x_n) \) belong to the compact set (2.2), we may choose, using Lemma 2 from [3], a random subsequence \( (\tilde{\alpha}, \tilde{\pi})(x_{n_k}) \) converging to some \( (\tilde{\alpha}, \tilde{\pi}) \). Condition (1.1) holds for all \( k \in \mathbb{N} \), so letting \( k \to \infty \), we get \( (\tilde{\alpha}, \tilde{\pi}) \in \mathcal{A} \). By continuity,
\[
\lim_{k \to \infty} w(x_{n_k}, (\tilde{\alpha}, \tilde{\pi})(x_{n_k})) = w(x, (\tilde{\alpha}, \tilde{\pi})) =: \tilde{w},
\]
\[
\lim_{n \to \infty} w(x_n, (\tilde{\alpha}, \tilde{\pi})(x)) = w(x, (\tilde{\alpha}, \tilde{\pi})(x)) =: w,
\]
and if \( (\tilde{\alpha}, \tilde{\pi}) \neq (\tilde{\alpha}, \tilde{\pi})(x) \), then \( \tilde{w} < w \). If we fix \( \varepsilon \in (0, (w - \tilde{w})/2) \), then for \( k \) large enough
\[
w(x_{n_k}, (\tilde{\alpha}, \tilde{\pi})(x_{n_k})) > w - \varepsilon > \tilde{w} + \varepsilon,
\]
while from (2.3),
\[
w(x_{n_k}, (\tilde{\alpha}, \tilde{\pi})(x_{n_k})) < \tilde{w} + \varepsilon.
\]
Inequalities (2.5) and (2.6) lead to
\[
w(x_{n_k}, (\tilde{\alpha}, \tilde{\pi})(x)) > w(x_{n_k}, (\tilde{\alpha}, \tilde{\pi})(x_{n_k}))
\]
contradicting the optimality of \( (\tilde{\alpha}, \tilde{\pi})(x_{n_k}) \). \( \blacksquare \)
3. Convergence of optimal strategies. We are now going to use the
results of the previous section in the general case. We define the Bellman
functions:

\[ V^n_T(x) := U_T(x), \]
\[ (3.1) \quad V^n_t(x) := \text{ess sup}_{(\alpha, \pi) \in \mathcal{A}} \left\{ U^n_t(\alpha x) + \mathbb{E}(V^n_{t+1}(x(\pi, \zeta_t)) \mid \mathcal{F}_t) \right\}, \]

for \( x \in (0, \infty) \) and \( t = 0, \ldots, T - 1 \).

**Theorem 3.1.** For all \( n \in \mathbb{N} \) and \( t = 0, \ldots, T \):

(i) the function \( V^n_t \) has a version which is strictly increasing and strictly
concave almost surely,

(ii) there exists a unique \( \mathcal{B}(0, \infty) \otimes \mathcal{F}_t \)-measurable function \((\hat{\alpha}^n_t, \hat{\pi}^n_t) \in \mathcal{A}_t \)
such that for all \( x \in (0, \infty) \),

\[ V^n_t(x) = U^n_t(\hat{x}^n_t(x)) + \mathbb{E}(V^n_{t+1}(x(\hat{\pi}^n(x), \zeta_t)) \mid \mathcal{F}_t). \]

**Proof.** Fix \( n \in \mathbb{N} \) and use backward induction. It is clear that \( V^n_T \) is
strictly concave and strictly increasing since \( U^n_T \) is. Then decreasing \( t \) from
\( T - 1 \) to 0 and applying Lemma 2.4 and Proposition 2.3 with \( w := V^n_T, u := U^n_t, v := V^n_{t+1}, \mathcal{A} := \mathcal{A}_t, \mathcal{H} := \mathcal{F}_t \)
and \( \zeta := \zeta_t \), we find that \( V^n_t \) has a strictly increasing and strictly concave version, and there is a unique
optimal strategy \((\hat{\alpha}^n_t, \hat{\pi}^n_t) := (\hat{\alpha}, \hat{\pi}) \) which is \( \mathcal{F}_t \)-measurable for all \( x \in (0, \infty) \)
and almost surely continuous, hence \( \mathcal{B}(0, \infty) \otimes \mathcal{F}_t \)-measurable. This proves
the theorem. \( \blacksquare \)

In this section we will make repeated use of the following elementary fact.
It may be derived e.g. from pages 90 and 248 of [5], but we include an easy
proof for completeness.

**Lemma 3.2.** Let \( U \subset \mathbb{R} \) be an open set and \( f_n : U \to \mathbb{R} \) be a sequence of
increasing functions such that \( f_n \) converges pointwise on \( U \) to a continuous
function \( f \). Then \( f_n \) converges to \( f \) uniformly on each compact subset of \( U \).

**Proof.** First notice that \( f \) is increasing, being the limit of a sequence
of increasing functions. Fix a compact set \( C \subset U \) and an arbitrary \( \varepsilon > 0 \).
Without loss of generality, we may assume that \( C = [a, b] \) is an interval. On \( C \),
the function \( f \) is uniformly continuous, hence we can find \( x_0, \ldots, x_k \in C \) with \( a := x_0 < x_1 < \cdots < x_{k-1} < x_k =: b \) such that \( |f(x_i) - f(x_{i-1})| < \varepsilon / 2 \) for
\( i \in \{1, \ldots, k\} \). Let \( N_i \in \mathbb{N} \) be such that \( |f_n(x_i) - f(x_i)| < \varepsilon / 2 \) for \( n \geq N_i \),
and define \( N := \max\{N_i : i \in \{0, \ldots, k\}\} \). Then for any \( x \in A \) there is
\( i \in \{0, \ldots, k - 1\} \) such that \( x \in [x_i, x_{i+1}] \), and for \( n \geq N \) we have
\[ f(x) - \varepsilon \leq f(x_{i+1}) - \varepsilon \leq f_n(x_{i+1}) - \varepsilon / 2 \leq f_n(x) \leq f_n(x_i) + \varepsilon / 2 \leq f(x_i) + \varepsilon \leq f(x) + \varepsilon. \]

Since \( x \in C \) was arbitrary, the assertion follows. \( \blacksquare \)
Now we are ready to prove the convergence of optimal strategies. Again we will start with the one-step case.

**Proposition 3.3.** Assume that for every $n \in \mathbb{N}$ functions $u^n$, $v^n$ are strictly increasing and strictly concave, and moreover $\lim_{n \to \infty} u^n(x) = u^\infty(x)$ and $\lim_{n \to \infty} v^n(x) = v^\infty(x)$ for all $x \in (0, \infty)$. Let $(\hat{\alpha}^n, \hat{\pi}^n)$ denote the optimal strategy fulfilling (2.1) with $u$ and $v$ replaced by $u^n$ and $v^n$. Then, for every $x \in (0, \infty)$,

$$\lim_{n \to \infty} (\hat{\alpha}^n, \hat{\pi}^n)(x) = (\hat{\alpha}^\infty, \hat{\pi}^\infty)(x) \quad \text{a.s.}$$

**Proof.** Suppose that, on the contrary, the convergence fails for some $x \in (0, \infty)$. Since $[0, 1] \times \Delta_0$ is compact, by the use of Lemma 2 from [3] we choose a random subsequence $(n_k \in \mathbb{N} : k \in \mathbb{N})$ such that $\lim_{k \to \infty} (\hat{\alpha}^{n_k}, \hat{\pi}^{n_k})(x) = (\hat{\alpha}, \hat{\pi}) \in \mathcal{A}$, $(\hat{\alpha}, \hat{\pi}) \neq (\hat{\alpha}^\infty, \hat{\pi}^\infty)$. Define

$$w^n(\alpha, \pi) := u^n(x \alpha) + \mathbb{E}v^n(x(\pi, \zeta) | \mathcal{H}), \quad (\alpha, \pi) \in \mathcal{A}, n \in \mathbb{N},$$

with a continuous version of the conditional expectation. Then the functions $w^n$ depend continuously on $\hat{\pi}$ and $\alpha$, the uniform convergence of $w^n$ and $v^n$ on compact sets gives

$$\lim_{k \to \infty} w^{n_k}(\alpha^{n_k}, \hat{\pi}^{n_k}) = w^\infty(\hat{\alpha}, \hat{\pi}) \quad \text{a.s.,}$$

and by our hypothesis

$$\tilde{w} := w^\infty(\hat{\alpha}, \hat{\pi}) < w^\infty(\hat{\alpha}^\infty, \hat{\pi}^\infty) =: w.$$

Fix $\varepsilon \in (0, (w - \tilde{w})/2)$. Since pointwise convergence ensures

$$\lim_{n \to \infty} w^n(\hat{\alpha}^\infty, \hat{\pi}^\infty) = w,$$

for $k$ large enough we have

$$w^k(\hat{\alpha}^\infty, \hat{\pi}^\infty) > w - \varepsilon > \tilde{w} + \varepsilon,$$

while from (3.2) we get

$$w^{n_k}(\alpha^{n_k}, \hat{\pi}^{n_k}) < \tilde{w} + \varepsilon.$$

Combining (3.3) and (3.4) we obtain $w^{n_k}(\hat{\alpha}^\infty, \hat{\pi}^\infty) > w^{n_k}(\hat{\alpha}^{n_k}, \hat{\pi}^{n_k})$, contradicting the optimality of $(\hat{\alpha}^{n_k}, \hat{\pi}^{n_k})$. ■

Now we can prove the main theorem.

**Theorem 3.4.** Let $((\hat{\alpha}_t^n, \hat{\pi}_t^n) : t = 0, \ldots, T - 1)$ be optimal strategies maximizing (1.3) with the corresponding functions $(U^n_0, \ldots, U^n_T)$, $n \in \mathbb{N}$. Then for every $x \in (0, \infty)$,

$$\lim_{n \to \infty} (\hat{\alpha}_t^n, \hat{\pi}_t^n)(x) = (\hat{\alpha}_t^\infty, \hat{\pi}_t^\infty)(x) \quad \text{a.s.,} \quad t = 0, \ldots, T - 1.$$

**Proof.** The assertion follows from the foregoing proposition applied consecutively to the Bellman functions (3.1) with $u^n := U_t^n$ and $v^n := V_{t+1}^n$ for $t = T - 1, \ldots, 0$. We only need to check that $\lim_{n \to \infty} V_t^n(x) = V_t^\infty(x)$
for $x \in (0, \infty)$ and $t = T, \ldots, 1$. For $t = T$ this is obvious since $V^n_T = U^n_T$, $n \in \mathbb{N}$. If we have proved that $(\widehat{\alpha}^n_t, \widehat{\pi}^n_t) \to (\widehat{\alpha}^\infty_t, \widehat{\pi}^\infty_t)$ for some $t \leq T$, then from uniform convergence on compact sets and the Lebesgue Theorem, for all $x \in (0, \infty)$,

$$
\lim_{n \to \infty} V^n_t(x) = \lim_{n \to \infty} (U^n_t(x\widehat{\alpha}^n_t(x)) + \mathbb{E}(V^n_{t+1}(x\widehat{\pi}^n_t(x, \zeta_t) | \mathcal{F}_t))
= U^\infty_t(x\widehat{\alpha}^\infty_t(x)) + \mathbb{E}(V^\infty_{t+1}(x\widehat{\pi}^\infty_t(x, \zeta_t) | \mathcal{F}_t)
= V^\infty_t(x).
$$

References


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