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WEAK SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN A Y-SHAPED DOMAIN

Abstract. We prove the existence of weak solutions to the Navier–Stokes equations describing the motion of a fluid in a Y-shaped domain.

1. Introduction. We consider the inflow-outflow problem in a reverse Y-shaped domain, with one inflow and two outflows. This can be treated as a simple model of the blood flow in veins or arteries. The motion of the fluid is described by the Navier–Stokes equations with boundary slip conditions. The domain $\Omega \subset \mathbb{R}^3$ is given by $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with the boundary $\partial\Omega = S = \sum_i S_0^i \cup S_i^i$ where Ω_i, $i = 1,2,3$, is a cylindrical type domain. To simplify the notation, we often omit the obvious index i so that $S_i^i \equiv S_i$. We denote by $\overline{n}$ the unit outward vector normal to the boundary S and by $\overline{\tau}_j$, $j = 1,2$, vectors tangent to S. We introduce the velocity vector $v(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t)) \in \mathbb{R}^3$ with $v_i(x,t) = v(x,t)|_{\Omega_i}$, the velocity defined on Ω_i, and the pressure $p = p(x,t) \in \mathbb{R}^1$. The domain Ω and the velocity vectors are presented in Figure 1.

The problem reads

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f \qquad && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v|_{t=0} = v(0), &&\\
&v\cdot\overline{n}|_{S_0^i} = 0, &&\\
&v\cdot\overline{n}|_{S_1} = -a_1, &&
\end{aligned}
\tag{1.1}
$$

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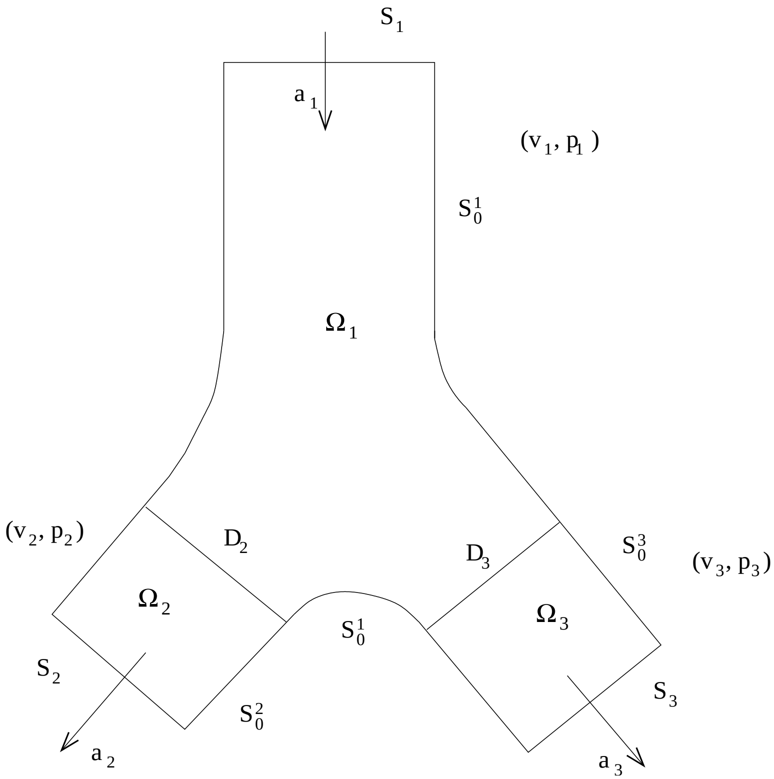

Fig. 1. Y-shaped domain

$$
\begin{aligned}
&v\cdot\overline{n}|_{S_i}=a_i, \qquad i=2,3,\\
&\nu\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_j+\gamma v\cdot\overline{\tau}_j=0, \quad j=1,2, \quad \text{on } S_0^i,\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_j=0, \quad j=1,2, \qquad\qquad \text{on } S_i,\ i=1,2,3,
\end{aligned}
\tag{1.1 [cont.]}
$$

where $f=f(x,t)=(f^1(x,t),f^2(x,t),f^3(x,t))\in\mathbb{R}^3$ is the external force, ν is the constant viscosity coefficient, $\gamma>0$ is the slip coefficient, and the stress tensor $\mathbb{T}$ and the dilatation tensor $\mathbb{D}$ are given as

$$
\mathbb{D}(v)=\{v^i_{,x_j}+v^j_{,x_i}\}_{i,j=1,2,3}, \qquad \mathbb{T}(v,p)=\nu\mathbb{D}(v)-pI.
$$

The inflow a_1 and outflows a_2,a_3 satisfy the compatibility condition

$$
\int_{S_1}a_1=\int_{S_2}a_2+\int_{S_3}a_3.
$$

We set $\overline{n}_i=\overline{n}|_{\Omega_i}$. We define the artificial boundaries $D_i=\Omega_1\cap\Omega_i$, $i=2,3$. Then

$$
\begin{aligned}
&v_1=v_i,\\
&\overline{n}_1\cdot\mathbb{T}(v_1,p_1)=\overline{n}_1\cdot\mathbb{T}(v_i,p_i) \quad \text{on } D_i,\ i=2,3,\ j=1,2.
\end{aligned}
\tag{1.2}
$$

In Section 2, we prove some a priori energy type estimates. This is motivated by considerations from [Z1]. Section 3 is devoted to the proof of existence of weak solutions to the problem (1.1) by the Galerkin method (see [L, Chapter 6, Section 7]). The last part is the Appendix where the properties of solutions in the neighborhood of the transmission sections D_2 and D_3 are examined.

2. Problem reformulation and a priori estimates. To obtain energy type estimates we need to work with a function v which satisfies the homogeneous Dirichlet boundary condition. To reformulate the problem (1.1) we introduce a new function α satisfying

$$\alpha_1 \cdot \overline{n}_1|_{S_1} = -a_1, \qquad \alpha_i \cdot \overline{n}_1|_{S_i} = -a_i, \qquad i = 2, 3,$$

and next, we define functions u_i on Ω_i by

$$u_i = v_i - \alpha_i, \qquad i = 1, 2, 3.$$

Thus we have

$$\operatorname{div} u_i = -\operatorname{div} \alpha_i, \qquad u_i \cdot \overline{n}_i|_{S_i} = 0.$$

Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a solution to the problem

$$\begin{aligned} &\Delta\varphi_i = -\operatorname{div} \alpha_i && \text{in } \Omega_i, \\ &\overline{n}_i \cdot \nabla\varphi_i = 0 && \text{on } S_i \text{ and } S_0^i, \\ &\int_{\Omega_i} \varphi_i \, dx = 0, && \\ &\varphi_1 = \varphi_i && \text{on } D_i,\ i = 2, 3, \\ &\frac{\partial}{\partial n_1}\varphi_1 = \frac{\partial}{\partial n_1}\varphi_i && \text{on } D_i,\ i = 2, 3, \end{aligned} \tag{2.1}$$

where n_i is the curvilinear coordinate along the curve tangent to $\overline{n}_i$, $i = 1, 2, 3$. We claim

LEMMA 2.1. *For every extension function α such that $\alpha_i \in H^1(\Omega_i)$, $i = 1, 2, 3$, there exists a solution $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ to the problem* (2.1) *and the following bound holds*:

$$\sum_{i=1}^{3} \|\nabla\varphi_i\|_{H^2(\Omega_i)} \le c \sum_{i=1}^{3} \int \|\alpha_i\|_{H^1(\Omega_i)}. \tag{2.2}$$

For convenience of the reader, we sketch the proof of this technical result in the Appendix.

Therefore, we can define new functions

$$w_i = v_i - \alpha_i - \nabla\varphi_i \equiv v_i - \delta_i$$

satisfying the following system:

$$
(2.3)\qquad
\begin{aligned}
&w_{i,t} + w_i\cdot\nabla w_i + w_i\cdot\nabla\delta_i + \delta_i\cdot\nabla w_i - \operatorname{div}\mathbb{T}(w_i,p_i)\\
&\quad = f_i - \delta_{i,t} - \delta_i\cdot\nabla\delta_i + \nu\operatorname{div}\mathbb{D}(\delta_i) \equiv F_i \quad &&\text{in } \Omega_i,\\
&\operatorname{div} w_i = 0 &&\text{in } \Omega_i,\\
&w_i\cdot\overline{n}|_{S_i} = 0,\\
&w_i\cdot\overline{n}|_{S_0^i} = 0,\\
&\nu\overline{n}\cdot\mathbb{D}(w_i)\cdot\overline{\tau}_j + \gamma w_i\cdot\overline{\tau}_j\\
&\quad = -\nu\overline{n}\cdot\mathbb{D}(\delta_i)\cdot\overline{\tau}_j - \gamma\delta_i\cdot\overline{\tau}_j \equiv B_{0j}^i,\ j=1,2, &&\text{on } S_0^i,\\
&\overline{n}\cdot\mathbb{D}(w_i)\cdot\overline{\tau}_j = -\overline{n}\cdot\mathbb{D}(\delta_i)\cdot\overline{\tau}_j \equiv B_{ij}^i,\ j=1,2, &&\text{on } S_i,
\end{aligned}
$$

and the transmission conditions

$$
(2.4)\qquad w_1 = w_i \quad\text{and}\quad \frac{\partial}{\partial n_1}w_1 = \frac{\partial}{\partial n_1}w_i \quad \text{on } D_i,\ i=2,3.
$$

Now, we introduce weak solutions to (2.3)–(2.4).

Definition 2.1. A *weak solution* to (2.3)–(2.4) is a triple (w_1, w_2, w_3) satisfying the identities

$$
(2.5)\qquad \sum_{i=1}^3\Big(\int_{\Omega_i^T} w_{i,t}\varphi\,dx\,dt + \int_{\Omega_i^T} H(w_i)\varphi\,dx\,dt + \nu\int_{\Omega_i^T}\mathbb{D}(w_i)\mathbb{D}(\varphi)\,dx\,dt
$$
$$
+\gamma\sum_{j=1}^2\int_{S_0^T} w_i\cdot\overline{\tau}_j\varphi\cdot\overline{\tau}_j\,dS_0^i\,dt - \sum_{j=1}^2\sum_{\sigma=0,i}\int_{S_\sigma^{iT}} B_{\sigma j}^i\varphi\cdot\overline{\tau}_j\,dS_\sigma^i\Big) = \sum_{i=1}^3\int_{\Omega_i^T} F_i\cdot\varphi\,dx\,dt,
$$

where $H(w) = w\cdot\nabla w + w\cdot\nabla\delta + \delta\cdot\nabla w$, for any sufficiently smooth function φ with $\operatorname{div}\varphi = 0$, $\varphi\cdot\overline{n}|_S = 0$.

We introduce some useful notation:

$$
|u|_{p,Q} = \sum_{i=1}^3 \|u\|_{L_p(Q_i)}, \qquad Q\in\{\Omega^T, S^T, \Omega, S\},\ p\in[1,\infty],
$$

$$
\|u\|_{s,Q} = \sum_{i=1}^3 \|u\|_{H^s(Q_i)}, \qquad Q\in\{\Omega, S\},\ s\in\mathbb{R}_+\cup\{0\},
$$

$$
|u|_{p,q,Q^T} = \sum_{i=1}^3 \|u\|_{L_q(0,T;L_p(Q_i))}, \qquad Q\in\{\Omega, S\},\ p,q\in[1,\infty],
$$

and a space natural for the study of the Navier–Stokes equations:

$$V_2^0(\Omega^T)=\Big\{u:\|u\|_{V_2^0(\Omega^T)}=\operatorname*{ess\,sup}_{t\in(0,T)}\|u\|_{L_2(\Omega)}+\Big(\int_0^T\|\nabla u\|^2_{L_2(\Omega)}\,dt\Big)^{1/2}<\infty\Big\}.$$

We will need the following result:

Lemma 2.2 (Korn inequality). *Assume that*

$$E_\Omega(w)=\sum_{i,j=1}^3\int_\Omega(w^i_{x_j}+w^j_{x_i})^2\,dx<\infty \tag{2.6}$$

and

$$\sum_{j=1}^2|w\cdot\overline{\tau}_j|^2_{2,S_0}<\infty,\qquad w\cdot\overline{n}|_S=0,\qquad \operatorname{div}w|_\Omega=0. \tag{2.7}$$

Then there exists a constant c independent of w such that

$$\|w\|^2_{H^1(\Omega)}\le c\Big(E_\Omega(w)+\sum_{j=1}^2|w\cdot\overline{\tau}_j|^2_{L_2(S_0)}\Big)\equiv cE. \tag{2.8}$$

Proof. We have

$$\begin{aligned}E_\Omega(w)&=2\sum_{i,j=1}^3\Big(\int_\Omega(w^i_{x_j})^2\,dx+\int_\Omega w^i_{x_j}\cdot w^j_{x_i}\,dx\Big)\\&=2\sum_{i,j=1}^3\Big(\int_\Omega(w^i_{x_j})^2\,dx+\int_\Omega(w^i_{x_j}\cdot w^j)_{x_i}\,dx\Big)\\&=2\sum_{i,j=1}^3\Big(\int_\Omega(w^i_{x_j})^2\,dx+\int_S w^i_{x_j}\cdot w^j\cdot n_i\,dS\Big)\\&=2\sum_{i,j=1}^3\Big(\int_\Omega(w^i_{x_j})^2\,dx-\int_S w^i\cdot w^j\cdot n_{i,x_j}\,dS\Big),\end{aligned}$$

where $\int_\Omega f$ denotes $\sum_{i=1}^3\int_{\Omega_i}f_i$. This implies

$$|\nabla w|^2_{2,\Omega}\le cE. \tag{2.9}$$

For a non-axially symmetric function w we can use the results in [Z2] to find that

$$|w|^2_{2,\Omega}\le\delta|\nabla w|^2_{2,\Omega}+M(\delta)E_\Omega(w) \tag{2.10}$$

so that (2.9) and (2.10) yield (2.8). ∎

We will show the following a priori estimate for w.

LEMMA 2.3. *Assume that* $a_1 \in L_6(0,T;L_3(S_1))$, $\nabla\alpha \in L_2(0,T;L_3(\Omega))$ *and* $w_i(0) \in L_2(\Omega_i)$, $i = 1,2,3$. *Let*

$$\Gamma^2(t) = |f|^2_{6/5,\Omega} + |\alpha_t|^2_{6/5,\Omega} + |\nabla\alpha_t|^2_{6/5,\Omega} + |\alpha|^2_{2,S_0} + |\nabla\alpha|^2_{2,\Omega}(1 + |\alpha|^2_{W^1_3(\Omega)})$$

with

$$\int_0^T \Gamma^2(t)\,dt < \infty.$$

Then

$$|w|^2_{V^0_2(\Omega^t)} \le ce^{c(|a_1|^6_{3,6,S^t_1} + |\nabla\alpha|^2_{3,2,\Omega^t})}\Big(\int_0^t \Gamma^2(t')\,dt' + |w(0)|^2_{2,\Omega}\Big). \tag{2.11}$$

Proof. With $\varphi = w$ and $w\cdot\overline{n}|_S = 0$ we get by definition of weak solutions

$$\sum_{i=1}^3\Big(\frac{1}{2}\frac{d}{dt}|w_i|^2_{2,\Omega_i} + \int_{\Omega_i}|\delta_i\cdot\nabla w_i\cdot w_i + w_i\cdot\nabla\delta_i\cdot w_i|\,dx + \nu|\mathbb{D}(w_i)|^2_{2,\Omega_i} + \gamma|w_i\cdot\overline{\tau}_j|^2_{2,S^i_0}\Big) = \sum_{i=1}^3\Big(\sum_{j=1}^2\sum_{\sigma=0,i}\int_{S^i_\sigma} B^i_{\sigma j}w_i\cdot\overline{\tau}_j\,dS^i_\sigma + \int_{\Omega_i} F_i\cdot w_i\,dx\Big). \tag{2.12}$$

Now, we analyze the second term on the l.h.s. We have

$$\int_{\Omega_i}\delta_i\cdot\nabla w_i\cdot w_i\,dx = \int_{\Omega_i}(\alpha_i + \nabla\varphi_i)\cdot\nabla w_i\cdot w_i\,dx = \int_{\Omega_i}\alpha_i\cdot\nabla w_i\cdot w_i\,dx + \int_{\Omega_i}\nabla\varphi_i\cdot\nabla w_i\cdot w_i\,dx \equiv I_1 + I_2$$

so that

$$I_1(w_i) = \frac{1}{2}\int_{\Omega_i}\alpha_i\cdot\nabla(w_i^2)\,dx = \frac{1}{2}\int_{\Omega_i}\operatorname{div}(\alpha_i w_i^2)\,dx - \frac{1}{2}\int_{\Omega_i}\operatorname{div}\alpha_i\cdot w_i^2\,dx \equiv I^a_1 + I^b_1.$$

Next, we calculate

$$I^a_1(w_1) = -\frac{1}{2}\int_{S_1}a_1w_i^2\,dS_1 + \frac{1}{2}\int_{D_2}\alpha\cdot\overline{n}_1w_2^2\,dD_2 + \frac{1}{2}\int_{D_3}\alpha\cdot\overline{n}_1w_3^2\,dD_3,$$

$$I^a_1(w_i) = \frac{1}{2}\int_{S_i}a_iw_i^2\,dS_i - \frac{1}{2}\int_{D_i}\alpha\cdot\overline{n}_1w_i^2\,dD_i, \quad i = 2,3.$$

Thus,

$$\sum_{i=1}^3 I^a_1(w_i) = \frac{1}{2}\Big(-\int_{S_1}a_1w_1^2\,dS_1 + \int_{S_2}a_2w_2^2\,dS_2 + \int_{S_3}a_3w_3^2\,dS_3\Big),$$

and we can estimate

$$-\sum_{i=1}^{3} I_1^a(w_i) \le \varepsilon_1^a \|w_1\|_{1,\Omega_1}^2 + c(1/\varepsilon_1^a)|a_1|_{3,S_1}^6 |w_1|_{2,\Omega_1}^2.$$

For I_1^b we have

$$\Big|\sum_{i=1}^{3} I_1^b(w_i)\Big| \le \varepsilon_1^b |w|_{6,\Omega}^2 + c(1/\varepsilon_1^b)|\nabla\alpha|_{3,\Omega}^2 |w|_{2,\Omega}^2$$

so

$$\Big|\sum_{i=1}^{3} I_1(w_i)\Big| \le \varepsilon_1(|w|_{6,\Omega}^2 + \|w\|_{1,\Omega}^2) + c(1/\varepsilon_1)(|a_1|_{3,S_1}^6 + |\nabla\alpha|_{3,\Omega}^2)|w|_{2,\Omega}^2.$$

Also we obtain

$$\sum_{i=1}^{3} I_2(w_i) = \frac{1}{2}\sum_{i=1}^{3}\int_{\Omega_i} \nabla\varphi_i\cdot\nabla(w_i^2) = -\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega_i} \Delta\varphi_i w_i^2 = \frac{1}{2}\sum_{i=1}^{3}\int_{\Omega_i} \operatorname{div}\alpha_i w_i^2.$$

Consequently,

$$\Big|\sum_{i=1}^{3} I_2(w_i)\Big| \le \varepsilon_2 |w|_{6,\Omega}^2 + c(1/\varepsilon_2)|\nabla\alpha|_{3,\Omega}^2 |w|_{2,\Omega}^2.$$

Next, we consider the expression

$$\int_{\Omega_i} w_i\cdot\nabla\delta_i\cdot w_i = \int_{\Omega_i} w_i\cdot\nabla\alpha\cdot w_i + \int_{\Omega_i} w_i\cdot\nabla(\nabla\varphi_i)\cdot w_i \equiv I_3 + I_4$$

with

$$\Big|\sum_{i=1}^{3} I_3(w_i)\Big| \le \varepsilon_3 |w|_{6,\Omega}^2 + c(1/\varepsilon_3)|\nabla\alpha|_{3,\Omega}^2 |w|_{2,\Omega}^2,$$

$$\Big|\sum_{i=1}^{3} I_4(w_i)\Big| \le \varepsilon_4 |w|_{6,\Omega}^2 + c(1/\varepsilon_4)|\nabla\alpha|_{3,\Omega}^2 |w|_{2,\Omega}^2.$$

We sum (2.12) over $i = 1, 2, 3$, and use the above estimates. Then, we use the imbedding inequality

$$|u|_{6,\Omega} \le c\|u\|_{W_2^1(\Omega)},$$

and the Korn inequality (2.8):

$$\|w\|_{H^1(\Omega)} \le c\Big(\int_{\Omega_i} \mathbb{D}(w)^2 + \sum_{j=1}^{2} |w\cdot\overline{\tau}_j|_{2,S_0}^2\Big),$$

to obtain

$$\tag{2.13} \begin{aligned} &\frac{1}{2}\frac{d}{dt}|w|_{2,\Omega}^2 + |w|_{H^1(\Omega)}^2 \\ &\quad \le cA|w|_{2,\Omega}^2 + \int_\Omega F\cdot w + \sum_{i=1}^3\sum_{j=1}^2\sum_{\sigma=0,i}\int_{S_\sigma^i} B_{\sigma,j}^i w_i\cdot\overline{\tau}_j\,dS_\sigma^i \\ &\quad \equiv cA|w|_{2,\Omega}^2 + J, \end{aligned}$$

where $A = |a_1|_{3,S_1}^6 + |\nabla\alpha|_{3,\Omega}^2$. We now deal with the r.h.s. of the above inequality. The last two terms have the form

$$\begin{aligned} J = &\sum_{i=1}^3\sum_{j=1}^2\int_{\Omega_i}(f_i - \delta_{i,t} - \delta_i\cdot\nabla\delta_i)w_i\,dx + \nu\int_{\Omega_i}\operatorname{div}\mathbb{D}(\delta_i)\cdot w_i\,dx \\ &- \int_{S_0^i}(\nu\overline{n}_i\mathbb{D}(\delta_i)\overline{\tau}_j w_i\cdot\overline{\tau}_j + \gamma\delta_i\cdot\overline{\tau}_j\cdot w_i\cdot\overline{\tau}_j)\,dS_0^i - \int_{S_i}\nu\overline{n}_i\mathbb{D}(\delta_i)\overline{\tau}_j w_i\cdot\overline{\tau}_j\,dS_i. \end{aligned}$$

To simplify we only study the second term of J:

$$\begin{aligned} \int_{\Omega_1}\operatorname{div}\mathbb{D}(\delta_1)\cdot w_1 &= \sum_{\sigma=0,1}\sum_{j=1}^2\int_{S_\sigma^1}\overline{n}_1\mathbb{D}(\delta_1)\overline{\tau}_j w_1\cdot\overline{\tau}_j - \int_{\Omega_1}\mathbb{D}(\delta_1)\cdot\mathbb{D}(w_1) \\ &\quad + \sum_{k=2,3}\sum_{j=1}^2\int_{D_k}\overline{n}_1\mathbb{D}(\delta_k)\overline{\tau}_j w_k\cdot\overline{\tau}_j, \\ \int_{\Omega_i}\operatorname{div}\mathbb{D}(\delta_i)\cdot w_i &= \sum_{\sigma=0,i}\sum_{j=1}^2\int_{S_\sigma^i}\overline{n}_i\mathbb{D}(\delta_i)\overline{\tau}_j w_i\cdot\overline{\tau}_j - \int_{\Omega_i}\mathbb{D}(\delta_i)\cdot\mathbb{D}(w_i) \\ &\quad + \sum_{j=1}^2\int_{D_i}\overline{n}_i\mathbb{D}(\delta_i)\overline{\tau}_j w_i\cdot\overline{\tau}_j \\ &= \sum_{\sigma=0,i}\sum_{j=1}^2\int_{S_\sigma^i}\overline{n}_i\mathbb{D}(\delta_i)\overline{\tau}_j w_i\cdot\overline{\tau}_j - \int_{\Omega_i}\mathbb{D}(\delta_i)\cdot\mathbb{D}(w_i) \\ &\quad - \sum_{j=1}^2\int_{D_i}\overline{n}_1\mathbb{D}(\delta_i)\overline{\tau}_j w_i\cdot\overline{\tau}_j, \qquad i = 2,3, \end{aligned}$$

to obtain

$$\begin{aligned} J &= \sum_{i=1}^3\Big(\int_\Omega(f_i - \delta_{i,t} - \delta_i\cdot\nabla\delta_i)\cdot w_i - \gamma\int_{S_0^i}\delta_i\cdot\overline{\tau}_j\cdot w_i\cdot\overline{\tau}_j\,dS_0^i - \nu\int_\Omega\mathbb{D}(\delta_i)\cdot\mathbb{D}(w_i)\Big) \\ &\equiv \sum_{i=1}^3(J_1^i + J_2^i + J_3^i) \equiv J_1 + J_2 + J_3. \end{aligned}$$

Then

$$|J_1| \le \varepsilon_5 |w|^2_{6,\Omega} + c(1/\varepsilon_5)(|f|^2_{6/5,\Omega} + |\delta_t|^2_{6/5,\Omega} + |\delta\cdot\nabla\delta|^2_{6/5,\Omega}),$$

where

$$|\delta_t|_{6/5,\Omega} \le |\alpha_t|_{6/5,\Omega} + |\nabla\varphi_t|_{6/5,\Omega} \le |\alpha_t|_{6/5,\Omega} + \Big|\int_\Omega \nabla G\nabla\alpha_t\Big|_{6/5,\Omega}$$
$$\le c(|\alpha_t|_{6/5,\Omega} + |\nabla\alpha_t|_{6/5,\Omega}),$$

and G is the Green function for the problem for φ. Similarly,

$$|\delta\cdot\nabla\delta|_{6/5,\Omega} \le |\delta|_{3,\Omega}|\nabla\delta|_{2,\Omega} \le c|\alpha|_{W^1_3(\Omega)}|\nabla\alpha|_{2,\Omega}.$$

We examine J_2 and J_3 to get

$$|J_2| \le \varepsilon_6|w|^2_{H^1(\Omega)} + c(1/\varepsilon_6)\Big(|\alpha|^2_{2,S_0} + \sum_{j=1}^2 |\overline{\tau}_j\cdot\nabla\varphi|^2_{2,S_0}\Big)$$
$$\le \varepsilon_6|w|^2_{H^1} + c(1/\varepsilon_6)(|\alpha|^2_{2,S_0} + |\nabla\alpha|^2_{2,\Omega}),$$

$$|J_3| \le \varepsilon_7|w|^2_{H^1(\Omega)} + c(1/\varepsilon_7)|\mathbb{D}(\delta)|^2_{2,\Omega}$$
$$\le \varepsilon_7|w|^2_{H^1(\Omega)} + c(1/\varepsilon_7)(|\nabla\alpha|_{2,\Omega} + |\nabla\nabla\varphi|_{2,\Omega})^2$$
$$\le \varepsilon_7|w|^2_{H^1(\Omega)} + c(1/\varepsilon_7)|\nabla\alpha|^2_{2,\Omega}.$$

The above estimates yield

$$|J| \le \varepsilon|w|_{H^1(\Omega)} + c(1/\varepsilon)[|f|^2_{6/5,\Omega} + |\alpha_t|^2_{6/5,\Omega} + |\nabla\alpha_t|^2_{6/5,\Omega} + |\alpha|^2_{2,S_0}$$
$$+ |\nabla\alpha|^2_{2,\Omega}(1 + |\alpha|^2_{W^1_3(\Omega)})]$$
$$= \varepsilon|w|_{H^1(\Omega)} + c(1/\varepsilon)\Gamma^2(t).$$

Then from (2.13) we obtain

$$\frac{1}{2}\frac{d}{dt}|w|^2_{2,\Omega} + |w|^2_{H^1(\Omega)} \le c(A|w|^2_{2,\Omega} + \Gamma^2(t)). \tag{2.14}$$

If we set $A(t) = |a_1|^6_{3,6,S_1^t} + |\nabla\alpha|^2_{3,2,\Omega^t}$, this can be rewritten as

$$\frac{d}{dt}(|w|^2_{2,\Omega}e^{-cA(t)}) + |w|^2_{H^1(\Omega)}e^{-cA(t)} \le c\Gamma^2(t)e^{-cA(t)} \tag{2.15}$$

and integrated in time:

$$|w(t)|^2_{2,\Omega} + e^{cA(t)}\int_0^t |w(t')|^2_{H^1(\Omega)}e^{-cA(t')}\,dt'$$
$$\le ce^{cA(t)}\Big(\int_0^t \Gamma^2(t')e^{-cA(t')}\,dt' + |w(0)|^2_{2,\Omega}\Big).$$

We can estimate the r.h.s. and simplify as follows:

$$|w(t)|_{2,\Omega}^2 + \int_0^t |w(t')|_{H^1(\Omega)}^2\,dt' \le ce^{cA(t)}\Big(\int_0^t \Gamma^2(t')\,dt' + |w(0)|_{2,\Omega}^2\Big).$$

Omitting the first term on the l.h.s. we get

$$\int_0^t |w(t')|_{H^1(\Omega)}^2\,dt' \le ce^{cA(t)}\Big(\int_0^t \Gamma^2(t')\,dt' + |w(0)|_{2,\Omega}^2\Big). \tag{2.16}$$

On the other hand, we can omit the second term in (2.15) to obtain

$$\frac{d}{dt}(|w|_{2,\Omega}^2 e^{-cA(t)}) \le c\Gamma^2(t), \tag{2.17}$$

and integrate in time to get the estimate for $|w|_{2,\Omega}$. Together with (2.16) this gives the result. ∎

We have $v = w + \delta = w + \alpha + \nabla\varphi$ where

$$\begin{aligned}|\delta|_{V_2^0(\Omega^T)}^2 &\le |\delta|_{2,\infty,\Omega^T}^2 + \int_0^T \|\delta(t')\|_{1,\Omega}^2\,dt' \\ &\le |\alpha|_{2,\infty,\Omega^T}^2 + |\nabla\alpha|_{2,\infty,\Omega^T}^2 + \int_0^T \|\alpha(t')\|_{1,\Omega}^2\,dt'.\end{aligned}$$

Thus, we have the following corollary:

Lemma 2.4. *Let the assumptions of Lemma 2.3 be satisfied and*

$$\Lambda(T) = c(|\alpha|_{2,\infty,\Omega^T}^2 + |\nabla\alpha|_{2,\infty,\Omega^T}^2) + \int_0^T \|\alpha(t')\|_{1,\Omega}^2\,dt' < \infty. \tag{2.18}$$

Then

$$|v|_{V_2^0(\Omega^T)}^2 \le ce^{c(|a_1|_{3,6,S_1^T}^6 + |\nabla\alpha|_{3,2,\Omega^T}^2)}\Big(\int_0^T \Gamma^2(t')\,dt' + |v(0)|_{2,\Omega}^2\Big) + \Lambda(T). \tag{2.19}$$

3. Weak solutions to (2.3). In this section, we follow the ideas from [L, Chapter 6, Section 7]. We will use the Galerkin method to prove the existence of weak solutions to the problem (2.3). Namely, we introduce the sequence of approximating functions w_N given as

$$w^N(x,t) = \sum_{k=1}^N C_{kN}(t)a^k(x),$$

where $\{a^k\}_{k=1}^\infty$ is a system of orthogonal functions in $L_2(\Omega) \cap J_2^0(\Omega)$. Here, $J_2^0(\Omega) = \{f \in H^1(\Omega) : \operatorname{div} f = 0\}$ and $\{a^k\}_{k=1}^\infty$ is a fundamental system in $H^1(\Omega)$ with $\sup_{x\in\Omega}|a^k(x)| < \infty$, $\sup_{x\in\partial\Omega}|a^k(x)| < \infty$. The coefficients

$C_{kN}(0)$ are defined by

$$C_{kN}|_{t=0} = (w_0, a_k), \qquad k = 1, \dots, N,$$

and the functions w^N satisfy the following system with test functions a^k:

$$\sum_{i=1}^{3} \bigg\{ \int\limits_{\Omega_i} \Big(\frac{1}{2} \frac{d}{dt} w_i^N a^k + w_i^N \cdot \nabla w_i^N a^k + \delta_i \cdot \nabla w_i^N \cdot w_i^N + w_i^N \cdot \nabla \delta_i \cdot w_i^N + \nu \mathbb{D}(w_i^N) \mathbb{D}(a^k) \Big)\, dx + \gamma \int\limits_{S_0^i} w_i^N \cdot \overline{\tau}_j a^k \overline{\tau}_j \, dS_0^i \bigg\} = \sum_{i=1}^{3} \Big(\sum_{j=1}^{2} \sum_{\sigma=0,i} \int\limits_{S_\sigma^i} B_{\sigma j}^i a^k \cdot \overline{\tau}_j \, dS_\sigma^i + \int\limits_{\Omega_i} F_i \cdot a^k \, dx \Big)$$

for $k = 1, \dots, N$. Thus, w^N would be a weak solution to (2.3).

With $(f,g) = \int_\Omega fg\,dx$ and $(f,g)_S = \int_S fg\,dS$ this can be rewritten as

$$\sum_{i=1}^{3} \{ (w_{i,t}^N, a^k) + (w_i^N \cdot \nabla w_i^N, a^k) + (\delta_i \cdot \nabla w_i^N, a^k) + (w_i^N \cdot \nabla \delta_i, a^k) + \nu (\mathbb{D}(w_i^N), \mathbb{D}(a^k)) + \gamma (w_i^N \cdot \overline{\tau}_j, a^k \cdot \overline{\tau}_j)_{S_0^i} \} = \sum_{i=1}^{3} \Big[\sum_{j=1}^{2} \sum_{\sigma=0,i} (B_{\sigma j}^i, a^k \cdot \overline{\tau}_j)_{S_\sigma^i} + (F_i, a^k) \Big], \qquad k = 1, \dots, N.$$

Thus,

$$\Big(\frac{d}{dt} w^N, a^k \Big) + (w^N \cdot \nabla w^N, a^k) + (\delta \cdot \nabla w^N, a^k) + (w^N \cdot \nabla \delta, a^k) + \nu (\mathbb{D}(w^N), \mathbb{D}(a^k)) + \gamma (w^N \cdot \overline{\tau}_j, a^k \cdot \overline{\tau}_j)_{S_0} = \sum_{i=1}^{3} \sum_{j=1}^{2} \sum_{\sigma=0,i} (B_{\sigma j}, a^k \cdot \overline{\tau}_j)_{S_\sigma^i} + (F, a^k), \qquad k = 1, \dots, N. \tag{3.1}$$

The above equations are in fact a system of ordinary differential equations for the functions $C_{kN}(t)$. The properties of the sequence a^k imply

$$|w^N(\cdot, t)|_{2,\Omega}^2 = \sum_{k=1}^{N} C_{kN}^2(t).$$

On the other hand, we can obtain a priori bounds for the approximate solutions w^N of the same form as in (2.11):

$$|w^N|_{V_2^0(\Omega^T)}^2 = \sup_{0 \le t \le T} |w^N|_{2,\Omega} + \int\limits_0^T |\nabla w^N|_{2,\Omega} \, dt' \le c e^{c(|a_1|_{3,6,S_1^T}^6 + |\nabla \alpha|_{3,2,\Omega^T}^2)} \Big(\int\limits_0^T \Gamma^2(t') \, dt' + |w(0)|_{2,\Omega}^2 \Big) \le C. \tag{3.2}$$

Therefore, $\sup_{0\le t\le T}|C_{kN}(t)|$ is bounded on $[0,T]$ and w^N are well defined for all times t.

Define now $\psi_{N,k}\equiv(w^N(x,t),a^k(x))$. This sequence is uniformly bounded by (3.2). We can also show that it is equicontinuous. Namely, we integrate (3.1) with respect to t from t to $t+\Delta t$ to obtain

$$
\begin{aligned}
&|\psi_{N,k}(t+\Delta t)-\psi_{N,k}(t)|\\
&\quad\le \sup_{x\in\Omega}|a^k(x)|\int_t^{t+\Delta t}(|w^N\cdot\nabla w^N|_{2,\Omega}+|\delta\cdot\nabla w^N|_{2,\Omega}|w^N\cdot\nabla\delta|_{2,\Omega}+|F|_{2,\Omega})\,dt'\\
&\qquad+\nu|\nabla a^k|_{2,\Omega}\int_t^{t+\Delta t}|\nabla w^N|_{2,\Omega}\,dt'\\
&\qquad+\gamma\sup_{x\in S}|a^k(x)|\int_t^{t+\Delta t}\Big(|w^N\cdot\overline{\tau}_j|_{2,S_0}+\sum_{i=1}^3\sum_{j=1}^2\sum_{\sigma=0,i}|B_{\sigma j}|_{2,S_\sigma^i}\Big)\,dt'\\
&\quad\le \sup_{x\in\Omega}|a^k(x)|\sqrt{\Delta t}\,(\sup_{x\in\Omega}|w^N|_{2,\Omega}(|\nabla w^N|_{2,\Omega^T}+|\nabla\delta|_{2,\Omega^T})\\
&\qquad+\sup_{x\in\Omega}|\delta|_{2,\Omega}|\nabla w^N|_{2,\Omega^T})\\
&\qquad+\sup_{x\in\Omega}|a^k(x)|\int_t^{t+\Delta t}|F|_{2,\Omega}\,dt'+\nu|\nabla a^k|_{2,\Omega}\sqrt{\Delta t}\,|\nabla w^N|_{2,\Omega^T}\\
&\qquad+\gamma\sup_{x\in S}|a^k(x)|\Big(\sqrt{\Delta t}\,|\nabla w^N|_{2,\Omega^T}+\int_t^{t+\Delta t}\sum_{j=1}^2|B_j|_{2,S}\Big)\,dt'\\
&\quad\le C(k)\Big(\sqrt{\Delta t}+\int_t^{t+\Delta t}(|F|_{2,\Omega}+\sum_{j=1}^2|B_j|_{2,S})\,dt'\Big).
\end{aligned}
$$

We can see that for given k and $N\ge k$ the r.h.s. tends to zero as $\Delta t\to 0$ uniformly in N. Thus, one can choose a subsequence N_m such that $\psi_{N_m,k}$ converges as $m\to\infty$ uniformly to some continuous function ψ_k for any given k. Since the limit function w is defined as

$$
w(x,t)=\sum_{k=1}^\infty\psi_k(t)a^k(x),
$$

we conclude that $(w^{N_m}-w,\psi)$ tends to zero as $m\to\infty$ uniformly with respect to $t\in[0,T]$ for any $\psi\in J_2^0(\Omega)$, and $w(x,t)$ is continuous in t in the weak topology. Moreover, estimate (3.2) remains true for the limit function w.

We will show that $\{w^{N_m}\}$ converges strongly in $L_2(\Omega^T)$. To this end, we need to apply the following version of the Friedrichs lemma: for any $\varepsilon > 0$, there exists N_ε such that for any $u \in W_2^1(\Omega)$,

$$\|u\|_{2,\Omega}^2 \le \sum_{k=1}^{N_\varepsilon}(u, a^k) + \varepsilon\|\nabla u\|_{2,\Omega}^2.$$

This in terms of $u = w^{N_m} - w^{N_l}$ reads

$$\|w^{N_m} - w^{N_l}\|_{2,\Omega^T}^2 \le \sum_{k=1}^{N_\varepsilon}\int_0^T (w^{N_m} - w^{N_l}, a^k)\,dt + \varepsilon\|\nabla w^{N_m} - \nabla w^{N_l}\|_{2,\Omega^T}^2.$$

By (3.2), we have

$$\|\nabla w^{N_m} - \nabla w^{N_l}\|_{2,\Omega^T}^2 \le 2C^2$$

for some constant C. The above integral, for given N_ε, can be arbitrarily small provided m and l are sufficiently large, so it tends to zero as $m, l \to \infty$. Therefore, $\{w^{N_m}\}$ converges strongly in $L_2(\Omega^T)$.

We summarize the above convergence properties of the sequence $\{w^{N_m}\}$:

(i) $w^{N_m} \to w$ strongly in $L_2(\Omega^T)$ for some w,

(ii) $w^{N_m} \to w$ weakly in $L_2(\Omega)$ uniformly with respect to $t \in [0,T]$,

(iii) $\nabla w^{N_m} \to \nabla w$ weakly in $L_2(\Omega^T)$.

For given $\Phi^k = \sum_{j=1}^k d_j(t)a^j(x)$, the sequence $\{w^{N_m}\}$ satisfies the identities

$$\int_\Omega\bigg(\frac{d}{dt}w^{N_m}\Phi^k + (w^{N_m}\cdot\nabla w^{N_m} + \delta\cdot\nabla w^{N_m} + w^{N_m}\cdot\nabla\delta)\Phi^k + \nu\mathbb{D}(w^{N_m})\mathbb{D}(\Phi^k)\bigg)\,dx$$

$$+ \gamma\int_{S_0} w^{N_m}\cdot\overline{\tau}_j\Phi^k\cdot\overline{\tau}_j\,dS_0 = \sum_{i=1}^3\sum_{j=1}^2\sum_{\sigma=0,i}\int_{S_\sigma^i} B_{\sigma j}\Phi^k\cdot\overline{\tau}_j\,dS_\sigma^i + \int_\Omega F\Phi^k\,dx.$$

Then we can pass to the limit as $m \to \infty$ to obtain the identity for w. The conditions $\operatorname{div} w^N = 0$, $w^N\cdot\overline{n}|_{S^T} = 0$ stay true for the limit function w as well.

It remains to consider the limit $\lim_{t\to 0} w(x,t)$. We note that the w^{N_m} satisfy the relation (2.12) (if we use the test function w^{N_m}). This yields

$$|w^{N_m}|_{2,\Omega} \le |w_0|_{2,\Omega} + \int_0^t (|F|_{2,\Omega} + |B|_{2,S})\,dt'.$$

In the limit $m \to \infty$ we obtain

$$|w|_{2,\Omega} \le |w_0|_{2,\Omega} + \int_0^t (|F|_{2,\Omega} + |B|_{2,S})\,dt',$$

which implies

$$\overline{\lim_{t\to 0}}\,|w|_{2,\Omega} \le |w_0|_{2,\Omega}.$$

On the other hand, since w^{N_m} tends to w as $m\to\infty$, we have $|w^{N_m}-w_0|_{2,\Omega}\to 0$. Therefore, $|w^{N_m}-w_0|\to 0$ weakly in $L_2(\Omega)$ as $t\to 0$ and

$$|w_0|_{2,\Omega} \le \underline{\lim}_{t\to 0}\,|w|_{2,\Omega}.$$

We conclude that the limit $\lim_{t\to 0}|w|_{2,\Omega}$ exists and is equal to $|w_0|_{2,\Omega}$ where the convergence is strong, in the $L_2(\Omega)$ norm.

Consequently, we have proved the following result.

THEOREM 1. *Let the assumptions of Lemma* 2.3 *be satisfied. Then there exists a weak solution w to problem* (2.3) *such that w is weakly continuous with respect to t in $L_2(\Omega)$ norm and w converges to w_0 as $t\to 0$ strongly in $L_2(\Omega)$ norm.*

4. Appendix: sketch of proof of Lemma 2.1. We discuss the properties of the functions φ_i, $i=1,2,3$, solving problem (2.1). To this end, we need the notion of a regularizer and a partition of unity for the domain Ω. Namely, let us define two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k\in\mathcal{M}\cup\mathcal{N}$, such that $\overline{\omega^{(k)}}\subset\Omega^{(k)}\subset\Omega$, $\bigcup_k\omega^{(k)}=\bigcup_k\Omega^{(k)}=\Omega$, $\overline{\Omega^{(k)}}\cap S=\emptyset$ for $k\in\mathcal{M}$ and $\overline{\Omega^{(k)}}\cap S\ne\emptyset$ for $k\in\mathcal{N}$. We assume that at most a finite number of $\Omega^{(k)}$ have nonempty intersection.

We will treat in more detail only the local problem on some sufficiently small subset $\Omega^N\subset\Omega$ such that $\Omega^N\cap D_2\ne\emptyset$ and $\Omega^N\cap S_0^i\ne\emptyset$, $i=1,2$. The case of a domain that intersects D_3 is analogous and subsets that lie entirely (i.e. with their closures) in one of Ω^i, $i=1,2,3$, are much easier to treat.

First, we straighten the boundary $(S_0^1\cup S_0^2)\cap\Omega^N$ and by the reflection technique we transform the problem on Ω^N to an equivalent problem on some subset Ω^M where $\Omega^M\cap D_2\ne\emptyset$ and $\operatorname{int}\{\Omega^M\}\cap S_0^i\ne\emptyset$, $i=1,2$ (see Figure 2.)

The system (2.1) now reads

$$\begin{aligned} -\Delta\varphi_1 &= \operatorname{div}\alpha_1 \quad && \text{in } \Omega^M\cap\Omega_1,\\ -\Delta\varphi_2 &= \operatorname{div}\alpha_2 \quad && \text{in } \Omega^M\cap\Omega_2,\\ \frac{\partial\varphi_1}{\partial n_1}\Big|_{D_2} &= \frac{\partial\varphi_2}{\partial n_1}\Big|_{D_2}. \end{aligned}$$

Here, we denote in fact by φ the new function $\varphi\zeta$ where ζ is a smooth function with compact support in Ω^M. In the new coordinates the local problem on

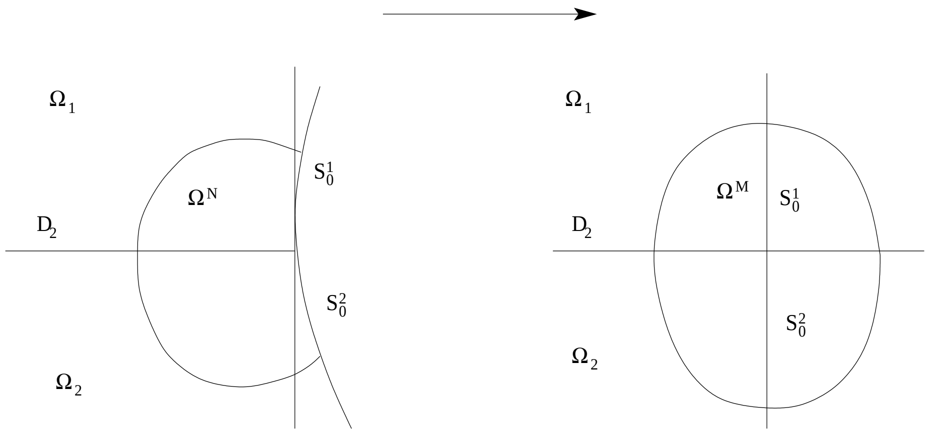

Fig. 2. Transformation from Ω^N to Ω^M

Ω^M takes the following form in a half-space:

$$
\begin{aligned}
-\Delta\varphi_1 &= \operatorname{div}\alpha_1 \quad \text{for } x_3 > 0,\\
-\Delta\varphi_2 &= \operatorname{div}\alpha_2 \quad \text{for } x_3 < 0,\\
\frac{\partial\varphi_1}{\partial x_3}\Big|_{x_3=0} &= \frac{\partial\varphi_2}{\partial x_3}\Big|_{x_3=0},
\end{aligned}
\tag{4.1}
$$

and it is completed with the conditions at infinity:

$$
\begin{aligned}
\varphi_1 &\to 0 \quad \text{as } x_3 \to \infty,\\
\varphi_2 &\to 0 \quad \text{as } x_3 \to -\infty.
\end{aligned}
\tag{4.2}
$$

We introduce new functions $u_i = \varphi_i - \widetilde{\varphi}_i$ where $\widetilde{\varphi}_i$ satisfy the first two equations of the system (4.1). Therefore, we consider the equivalent problem

$$
\begin{aligned}
&-\Delta u_1 = 0 \quad \text{for } x_3 > 0,\\
&-\Delta u_2 = 0 \quad \text{for } x_3 < 0,\\
&\frac{\partial u_1}{\partial x_3} - \frac{\partial u_2}{\partial x_3}\Big|_{x_3=0} = \frac{\partial\widetilde{\varphi}_2}{\partial x_3} - \frac{\partial\widetilde{\varphi}_1}{\partial x_3}\Big|_{x_3=0} \equiv -\psi_1,\\
&u_1 - u_2|_{x_3=0} = \widetilde{\varphi}_2 - \widetilde{\varphi}_1 \equiv \psi_2,\\
&u_1 \to 0 \quad \text{as } x_3 \to \infty,\\
&u_2 \to 0 \quad \text{as } x_3 \to -\infty.
\end{aligned}
$$

Applying the Fourier transform (with respect to $x' = (x_1, x_2)$), i.e.

$$\widetilde{u}(\xi, x_3) = \int_{\mathbb{R}^2} e^{-i\xi x'} u(x', x_3)\, dx',$$

where $\xi = (\xi_1, \xi_2)$ and $\xi \cdot x' = \xi_1 x_1 + \xi_2 x_2$, we obtain the problem

$$
\begin{aligned}
&\xi^2 \widetilde{u}_1 - \frac{\partial^2 \widetilde{u}_1}{\partial x_3^2} = 0 \quad \text{for } x_3 > 0,\\
&\xi^2 \widetilde{u}_2 - \frac{\partial^2 \widetilde{u}_2}{\partial x_3^2} = 0 \quad \text{for } x_3 < 0,\\
&\frac{\partial \widetilde{u}_1}{\partial x_3} - \frac{\partial \widetilde{u}_2}{\partial x_3}\bigg|_{x_3=0} = -\widetilde{\psi}_1,\\
&\widetilde{u}_1 - \widetilde{u}_2|_{x_3=0} = \widetilde{\psi}_2,\\
&\widetilde{u}_1 \to 0 \quad \text{as } x_3 \to \infty,\\
&\widetilde{u}_2 \to 0 \quad \text{as } x_3 \to -\infty.
\end{aligned}
\tag{4.3}
$$

We can easily find the solutions

$$\widetilde{u}_1 = c_1 e^{-|\xi| x_3}, \qquad \widetilde{u}_2 = c_2 e^{|\xi| x_3},$$

where

$$c_1 + c_2 = \widetilde{\psi}_1, \qquad c_1 - c_2 = \widetilde{\psi}_2,$$

thus

$$c_1 = \frac{1}{2}(\widetilde{\psi}_1 + \widetilde{\psi}_2), \qquad c_2 = \frac{1}{2}(\widetilde{\psi}_1 - \widetilde{\psi}_2).$$

We want to use $\widetilde{u}_i$ to estimate the H^2 norm of u_i. By way of example, we examine $\widetilde{u}_1$. We observe that

$$
\begin{aligned}
&\int_0^\infty |\widetilde{u}_1|^2 = \int_0^\infty c_1^2 e^{-2|\xi| x_3}\, dx_3 \le \frac{c}{|\xi|},\\
&\left\| \frac{d}{dx_3} \widetilde{u}_1 \right\|_{L_2}^2 = \int_0^\infty \left| \frac{d}{dx_3} \widetilde{u}_1 \right|^2 = c_1^2 \int_0^\infty |\xi|^2 e^{-2|\xi| x_3}\, dx_3 \le c|\xi|,\\
&\left\| \frac{d^2}{dx_3^2} \widetilde{u}_1 \right\|_{L_2}^2 \le c|\xi|^3.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\sum_{i=1}^{2} \|u_i\|_{H^2}^2 &= \int \left([(1+\xi^2)\widetilde{u}]^2 + \left| \frac{d^2}{dx_3^2} \widetilde{u} \right|^2 \right) d\xi\\
&\le \int \left((1+\xi^2)^2 \frac{|\widetilde{\psi}|^2}{|\xi|} + |\xi|^3 |\widetilde{\psi}|^2 \right) d\xi_1\, d\xi_2\\
&\le c\|\widetilde{\psi}\|_{H^{3/2}(\mathbb{R}^2)}^2 \le c\|\widetilde{\alpha}\|_{H^1(\mathbb{R}^3)}.
\end{aligned}
$$

Hence, by the regularizer technique and the a priori estimate on φ,

$$\sum_{i=1}^{3} \int_{\Omega_i} |\nabla \varphi_i|^2 \le c \sum_{i=1}^{3} \int_{\Omega_i} |\nabla \alpha_i|^2$$

we deduce the statement of Lemma 2.1 and the estimate (2.2).

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