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BOX-SPLINE HISTOGRAMS FOR MULTIVARIATE DENSITY ESTIMATION

Abstract. The uniform approach to calculation of MISE for histogram and density box-spline estimators gives us a possibility to obtain estimators of derivatives of densities and the asymptotic constant.

1. Histograms. We have two methods of bandwidth selection for the Rosenblatt–Parzen estimator: cross-validation (unbiased and biased) and plug-in (see for instance [16]). In our paper we present higher order point estimators to obtain plug-in estimates for the bandwidth in the case of histograms. An excellent introduction to histograms is given in the book by Scott [17].

In Section 1 we give a simple introduction to box-spline operators and box-spline histograms based on these operators. From the point of view of estimation of density, two properties of approximation are crucial: the rate of convergence and the so called saturation property (Theorem 3.1). These properties divide the box-spline estimators into three classes. These classes are represented by a histogram, a linear histogram, and a Zwart–Powell histogram (ZP histogram for short). This is the reason why we fix our attention on these three histograms. The basic results are recalled in Section 2. In Section 3 we show how to use the saturation property to estimate derivatives. The presented method is a version of the method from [13], and it is applicable only to box-spline estimators of the type of the ZP histogram. In Section 4 we present a method of estimating the asymptotic constant (see (11)). This method is more general and is applicable to all cases. We present it for the histogram. It seems to be a version of the "no diagonals" estimator [18].

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We consider only dimension $d = 2$ just for simplicity. It is also not complicated to introduce a nonhomogeneous scaling [9], so we omit it. Spline estimators were introduced by Ciesielski [5]. There are a large number of papers concerning methods of bandwidth selection. Let us mention some of them: [1], [11], [14].

We consider a pair of functions (F, G) which are nonnegative, piecewise polynomials with compact support

$$F, G : \mathbb{R}^2 \to \mathbb{R}.$$

The additional assumptions are given below. A box-spline operator under consideration is defined by

$$Qf(x) = \int_{\mathbb{R}^2} K(x,y) f(y)\, dy, \tag{1}$$

with the kernel depending on F, G,

$$K(x,y) = \sum_{\alpha \in \mathbb{Z}^2} F(y-\alpha) G(x-\alpha). \tag{2}$$

The operator Q defines a family of operators Q_h for $h > 0$,

$$Q_h = \sigma_h \circ Q \circ \sigma_{1/h}, \tag{3}$$

where

$$\sigma_h f(x) = f(x_1/h, x_2/h), \qquad x = (x_1, x_2).$$

REMARKS. Since $F, G \geq 0$, it follows that the operators Q_h are positive, i.e. if $f \geq 0$ then $Q_h f \geq 0$. Moreover, we assume that our functions satisfy

$$\sum_{\alpha \in \mathbb{Z}^2} F(\cdot - \alpha) = \frac{1}{\int_{\mathbb{R}^2} G} \qquad \text{a.e.}$$

By this assumption the operators Q_h map densities to densities, i.e. if f is a density, then $\int_{\mathbb{R}^2} Q_h f = 1$. To ensure a good approximation we assume that the Q_h reproduce at least constant polynomials, $Q_h(1) = 1$ (here 1 is the function $1(x) = 1$), or equivalently

$$\sum_{\alpha \in Z^2} G(x - \alpha) = \frac{1}{\int_{\mathbb{R}^2} F} \qquad \text{a.e.}$$

REMARK. In the definition of the kernel K, instead of $\mathbb{Z}^2$ we can take another lattice, for instance $A\mathbb{Z}^2$, where

$$A = \begin{bmatrix} 1 & \sin \pi/6 \\ 0 & \cos \pi/6 \end{bmatrix}. \tag{4}$$

The definition so modified includes for instance the histograms and the linear histogram based on the regular hexagon considered in [17].

Let $X_1, \ldots, X_n$ be a random sample from a distribution with density f. We define a density estimator based on the kernel K by

$$f_{h,n}(x) = \frac{1}{nh^2} \sum_{k=1}^{n} K(x/h, X_k/h). \tag{5}$$

Note that

$$Ef_{h,n} = Q_h f. \tag{6}$$

Now we introduce three examples of box-spline estimators. Let H be the characteristic function of the square $[0,1]^2$, i.e.

$$H(x) = I_{[0,1]^2}(x) = \begin{cases} 1, & x \in [0,1]^2, \\ 0, & x \notin [0,1]^2. \end{cases}$$

In this paper we will consider three examples of pairs (F_i, G_i), $i = 1, 2, 3$.

EXAMPLE 1. *The histogram* corresponds to the choice

$$F_1 = G_1 = H.$$

EXAMPLE 2. *The linear histogram* corresponds to the choice

$$F_2(x_1, x_2) = H(x_1 - 0.5, x_2 - 0.5)$$

and G_2 is the hat function given by

$$G_2(x_1, x_2) = \int_0^1 G_1(x_1 - t, x_2 - t)\, dt.$$

EXAMPLE 3. *The Zwart–Powell histogram* (for short ZP histogram) corresponds to the choice

$$F_3(x_1, x_2) = H(x_1, x_2 - 1)$$

and G_3 is the Zwart–Powell function given by

$$G_3(x_1, x_2) = \int_0^1 G_2(x_1 + t, x_2 - t)\, dt.$$

See [3] for the definition of box-splines and [7] for the definition of box-spline estimators.

2. Asymptotic formulas for MISE. We say that the box-spline operator Q *reproduces the polynomials* of degree less than ϱ if $Q(P) = P$ for all polynomials P with $\deg P < \varrho$. We then say that Q has *polynomial order* ϱ. For a pair of functions (F_j, G_j) introduced in the previous section we will add the superscript j to the operator, i.e. Q^j, and to the kernel i.e. K^j, $j = 1, 2, 3$. Note that the operator Q^1 reproduces only constant polynomials, i.e. $\varrho_1 = 1$, and the operators Q^j, $j = 2, 3$, reproduce the linear and constant polynomials, i.e. $\varrho_j = 2$. The parameter ϱ gives the rate of approximation.

We may check [8] that if Q has polynomial order ϱ, then there is $C > 0$ such that for functions f from the Sobolev space W_2^ϱ,

$$\|Q_h f - f\|_2 \le C h^\varrho |f|_{\varrho,2}, \tag{7}$$

where

$$|f|_{\varrho,2} = \sum_{|\beta|=\varrho} \|D^\beta f\|_2, \qquad \|f\|_2 = \Big(\int_{\mathbb{R}^d} |f|^2 \Big)^{1/2}.$$

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}, \qquad \beta = (\beta_1, \beta_2), \qquad |\beta| = \beta_1 + \beta_2.$$

Recall that the Sobolev space is defined by

$$W_2^\varrho = \Big\{ f \in L^2 : \sum_{|\beta|=\varrho} \|D^\beta f\|_2 < \infty \Big\}.$$

A monomial of degree $|\beta|$ will be denoted by $[]^\beta$, i.e. for $x = (x_1, x_2)$,

$$[]^\beta(x) = x^\beta = x_1^{\beta_1} x_2^{\beta_2}.$$

We assume that $f \in L^2$ to consider the mean integrated square error, given by

$$\mathrm{MISE}(f,h) = E\Big[\int_{\mathbb{R}^2} [f_{h,n} - f]^2 \Big]. \tag{8}$$

Consequently,

$$\mathrm{MISE}(f,h) = E\Big[\int_{\mathbb{R}^2} [f_{h,n} - Q_h f]^2 \Big] + \int_{\mathbb{R}^2} [Q_h f - f]^2. \tag{9}$$

The deterministic part is considered in [8].

THEOREM 2.1. *Assume that Q has polynomial order ϱ. Let $f \in W_2^\varrho(\mathbb{R}^2)$. Then*

$$\lim_{h\to 0^+} \Big\| \frac{Q_h f - f}{h^\varrho} \Big\|_2 = \bigg(\int_{\mathbb{R}^2} \bigg(\int_{[0,1]^2} \Big| \sum_{|\beta|=\varrho} \frac{1}{\beta!} D^\beta f(t) (Q([]^\beta)(x) - []^\beta(x)) \Big|^2 dx \bigg) dt \bigg)^{1/2}. \tag{10}$$

Let us define the asymptotic constant depending on f and the box-spline histogram $(j = 1,2,3)$ by

$$\theta_j = (\mathrm{Asym}(f,j))^2 = \int_{\mathbb{R}^2} \bigg(\int_{[0,1]^2} \Big| \sum_{|\beta|=\varrho_j} \frac{1}{\beta!} D^\beta f(t) (Q^j([]^\beta)(x) - []^\beta(x)) \Big|^2 dx \bigg) dt. \tag{11}$$

Asym$(f,1)$ is known (see for instance [17]):

$$\theta_1 = (\mathrm{Asym}(f,1))^2 = \frac{1}{12}\int_{\mathbb{R}^2}\biggl(\biggl(\frac{\partial f}{\partial x_1}\biggr)^2 + \biggl(\frac{\partial f}{\partial x_2}\biggr)^2\biggr). \tag{12}$$

An easy calculation shows that

$$\begin{aligned}\theta_2 &= (\mathrm{Asym}(f,2))^2 \\ &= \int_{\mathbb{R}^2}\biggl\{\biggl(\frac{\partial^2 f}{\partial x_1^2}\biggr)^2\cdot\frac{49}{2880} + \biggl(\frac{\partial^2 f}{\partial x_2^2}\biggr)^2\cdot\frac{49}{2880} + \biggl(\frac{\partial^2 f}{\partial x_1\partial x_2}\biggr)^2\cdot\frac{1}{90} \\ &\quad + \biggl(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2}\biggr)\cdot\frac{1}{32} + \biggl(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_1\partial x_2}\biggr)\cdot\frac{17}{720} + \biggl(\frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 f}{\partial x_1\partial x_2}\biggr)\cdot\frac{17}{720}\biggr\},\end{aligned} \tag{13}$$

$$\theta_3 = (\mathrm{Asym}(f,3))^2 = \int_{\mathbb{R}^2}\biggl\{\frac{1}{6}\biggl(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2}\biggr)\biggr\}^2. \tag{14}$$

Let us consider the first term of (9). We have the following result (compare [8, Theorem 1.4]).

THEOREM 2.2. *Let the density f be in $W_2^{\varrho}(\mathbb{R}^2)$. If $nh^2 \to \infty$ and $h \to 0$ then*

$$\lim_{nh^2\to\infty} nh^2 E\Bigl[\int_{\mathbb{R}^2}[f_{h,n} - Q_h f]^2\Bigr] = \int_{\mathbb{R}^2}\Bigl[\int_{[0,1]^2}(K(x,y))^2\,dy\Bigr]\,dx. \tag{15}$$

REMARK 1. By (9), (10) and (15) we get

$$\mathrm{MISE}(f,h) \sim \mathrm{AMISE}(f,h)$$

where for the box-spline histograms

$$\mathrm{AMISE}(f,h) = \frac{1}{nh^2}\int_{\mathbb{R}^2}\Bigl[\int_{[0,1]^2}(K^j(x,y))^2\,dy\Bigr]\,dx + h^{2\varrho_j}(\mathrm{Asym}(f,j))^2. \tag{16}$$

So the best choice of the parameter $h > 0$ to minimize (16) is

$$h = \biggl(\frac{\int_{\mathbb{R}^2}[\int_{[0,1]^2}(K^j(x,y))^2\,dy]\,dx}{\varrho_j n(\mathrm{Asym}(f,j))^2}\biggr)^{-1/(2\varrho_j+2)}. \tag{17}$$

Now we have another estimation problem of $\theta_j = (\mathrm{Asym}(f,j))^2$ with a different bandwidth denoted by a. As for the density estimation, the choice of a is crucial to the performance of the estimator $\widehat{\theta}_j(a)$. We use here the notation from [15]. In the next section we construct an estimator g_{an} of the derivatives $D_{Q_3}f$, where (compare (14))

$$D_{Q_3}f = \frac{1}{6}\biggl(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2}\biggr).$$

Hence an estimator of θ_3 is given by

$$\widehat{\theta}_3(a) = \int_{\mathbb{R}^2} (g_{an})^2.$$

In Section 4 we estimate θ_j directly.

3. Choice of bandwidth for estimation of derivatives of density: ZP histogram. The problem of estimation of derivatives in the multivariate case is rather ambiguous. We look for estimators of derivatives which appear in the asymptotic formula. The following theorem ([10, Theorem 2.5]) is important. Let us denote $\breve{F}(x) = F(-x)$.

THEOREM 3.1. *Let $f \in W_2^{\varrho_j}$. If $a \to 0$ then*

$$\frac{Q_a^j f - f}{a^{\varrho_j}} \to D_{Q^j} f \tag{18}$$

weakly in L^2 for $j = 1, 2, 3$, where

$$D_{Q^j} f = \frac{1}{(2\pi i)^{\varrho_j}} \sum_{|\beta|=\varrho_j} \frac{D^\beta f}{\beta!} D^\beta(\widehat{G_j}\widehat{\breve{F}_j})(0).$$

It is crucial for our construction that for $j = 3$ in (18) we have L^2 convergence, but not for $j = 1, 2$ in general. Hence

$$(\mathrm{Asym}(f, 3))^2 = \int_{\mathbb{R}^2} (D_{Q^3} f)^2.$$

It is not difficult to prove (Theorem 3.2 below) that if $a \to 0$, then also

$$\frac{Q_a^3(Q_a^3 f) - Q_a^3 f}{a^2} \to D_{Q^3} f = \frac{1}{6}\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2}\right), \tag{19}$$

in L^2 norm. The property (19) helps us construct an estimator of $D_{Q^3} f$. It leads us to the operator

$$T := Q^3 \circ Q^3 - Q^3.$$

Note that the operator T has the same structure as Q, i.e.

$$Tf(x) = \int_{\mathbb{R}^2} \kappa(x, y) f(y)\, dy,$$

where

$$\kappa(x, y) = \sum_{\alpha \in \mathbb{Z}^2} F_3(y - \alpha)(Q^3(G_3) - G_3)(x - \alpha). \tag{20}$$

We define as above a family of operators T_a for $a > 0$. Now we are ready to define an estimator of the derivatives $D_{Q^3} f$. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density f. Then we define an estimator of $D_{Q^3}f$ by

$$g_{an}(x)=\frac{1}{na^4}\sum_{k=1}^{n}\kappa(x/a,X_k/a). \tag{21}$$

Note that

$$Eg_{an}=\frac{T_a(f)}{a^2}$$

and

$$E\int_{\mathbb{R}^2}|g_{an}-D_{Q^3}f|^2=E\int_{\mathbb{R}^2}\left|g_{an}-\frac{T_a(f)}{a^2}\right|^2+\int_{\mathbb{R}^2}\left|\frac{T_a(f)}{a^2}-D_{Q^3}f\right|^2.$$

THEOREM 3.2. *Let $na^6\to\infty$ and $a\to 0$ and $f\in W_2^4$. Then*

$$\left(\int_{\mathbb{R}^2}\left|\frac{T_a(f)}{a^2}-D_{Q^3}f\right|^2\right)^{1/2}\le C(a|f|_{3,2}+a^2|f|_{4,2}) \tag{22}$$

and

$$\lim_{na^6\to\infty}na^6E\int_{\mathbb{R}^2}\left|g_{an}-\frac{T_a(f)}{a^2}\right|^2=\int_{\mathbb{R}^2}\Big\{\int_{[0,1]^2}\kappa^2(x,y)\,dy\Big\}\,dx.$$

Proof. If we modify slightly the end of the proof of Theorem 2.23 with $\varrho=0$ in [6] and use Lemma 1.1 of [8], we get

$$\left\|\frac{Q_a^3f-f}{a^2}-D_{Q^3}f\right\|_2\le Ca|f|_{3,2}. \tag{23}$$

Now the triangle inequality implies that

$$\begin{aligned}\left\|\frac{T_a(f)}{a^2}-D_{Q^3}f\right\|_2&=\left\|\frac{Q_a^3Q_a^3f-Q_a^3f}{a^2}-D_{Q^3}f\right\|_2\\&\le\left\|\frac{Q_a^3Q_a^3f-Q_a^3f}{a^2}-Q_a^3D_{Q^3}f\right\|_2+\|Q_a^3D_{Q^3}f-D_{Q^3}f\|_2.\end{aligned} \tag{24}$$

Since the operators Q_a^3 are uniformly bounded, applying (23) we get

$$\left\|\frac{Q_a^3Q_a^3f-Q_a^3f}{a^2}-Q_a^3D_{Q^3}f\right\|_2\le C\left\|\frac{Q_a^3f-f}{a^2}-D_{Q^3}f\right\|_2\le Ca|f|_{3,2}.$$

We now turn to the second term of (24). Applying (7) we get

$$\|Q_a^3D_{Q^3}f-D_{Q^3}f\|_2\le Ca^2|D_{Q^3}f|_{2,2}.$$

This finishes the proof of the first inequality.

The proof of the second formula is rather similar to that of [8, Theorem 1.4]. ∎

Now the choice of the estimator of θ_3 is obvious:

$$\widehat{\theta}_3(a) = \int\limits_{\mathbb{R}^2} (g_{an})^2.$$

We can turn to the second strategy of estimating θ_3. Note that

$$(25)\qquad (g_{an})^2(x) = \left(\frac{1}{na^4}\right)^2 \Big(\sum_{k\neq l}^{n} \kappa(x/a, X_k/a)\kappa(x/a, X_l/a) + \sum_{k=1}^{n} (\kappa(x/a, X_k/a))^2\Big)$$

and

$$E(g_{an}(x))^2 = \frac{n^2-n}{n^2}\left(\frac{T_a(f)(x)}{a^2}\right)^2 + \frac{1}{a^8}\,\frac{1}{n}\int\limits_{\mathbb{R}^2} (\kappa(x/a, y/a))^2 f(y)\,dy.$$

Now another estimator ("no diagonals") of θ_3 can be given by the formula

$$\widehat{\theta}_3(a) = \left(\frac{1}{na^4}\right)^2 \int\limits_{\mathbb{R}^2} \sum_{k\neq l}^{n} \kappa(x/a, X_k/a)\kappa(x/a, X_l/a)\,dx.$$

To avoid tedious calculations we propose the following simpler estimator for an even size of a sample:

$$(26)\qquad \widehat{\theta}_3(a) = \frac{1}{(n/2)a^8}\int\limits_{\mathbb{R}^2} \sum_{k=1}^{n/2} \kappa(x/a, X_k/a)\kappa(x/a, X_{n-k}/a)\,dx.$$

This approach with minor changes is applicable to the histogram and the linear histogram. We will see it in the next section for the histogram, i.e. we will construct $\widehat{\theta}_1$.

4. Choice of bandwidth for estimation of the asymptotic constant: histogram. We will explain the estimation of the asymptotic constant in the case of the histogram.

First we construct an operator Q^5 reproducing the polynomials of degree less than or equal to two by the formula

$$Q^5(f) = \sum_{|\gamma|\le 1} a_\gamma Q^3(f(\cdot - \gamma)).$$

Applying (10) and (14) we obtain, for $|\beta| = 2$,

$$Q^3([]^\beta)(x) - x^\beta = A_\beta,$$

where $A_{(1,1)} = 0$, $A_{(2,0)} = 1/3$, $A_{(0,2)} = 1/3$. Consequently, to find the coefficients a_γ we need to solve the system of equations, for all $|\beta| \le 2$,

$$Q^5([]^\beta) = \sum_{|\gamma|\le 1} a_\gamma Q^3((\cdot - \gamma)^\beta) = []^\beta.$$

One of the solutions is $a_{(0,0)} = 4/3$ and $a_{(\gamma_1,\gamma_2)} = -1/12$ for all $|\gamma_1| = |\gamma_2| = 1$ and the other a_γ are zero. Since Q_3 reproduces polynomials of degree less than or equal to two by (7), there is $C > 0$ such that for all $f \in W_2^3$,

$$\|Q_h^5 f - f\|_2 \le Ch^3 |f|_{3,2}.$$

By definition,

$$Q^5(f)(x) = \int_{\mathbb{R}^2} K^5(x,y) f(y)\, dy,$$

where

$$K^5(x,y) = \sum_{\alpha\in\mathbb{Z}^2} F_3(y-\alpha) G_5(x-\alpha), \qquad G_5(x) = \sum_{|\gamma|\le 1} a_\gamma G_3(x-\gamma).$$

Now we consider the operator defined as follows:

$$T^1 := Q^1 \circ Q^5 - Q^5.$$

Using this operator we construct an estimator of θ_1. We can write

$$T^1(f)(x) = \int_{\mathbb{R}^2} \kappa^1(x,y) f(y)\, dy$$

with

$$\kappa^1(x,y) = \sum_{\alpha\in\mathbb{Z}^2} F_3(y-\alpha) K(x-\alpha), \qquad K(x) = Q^1(G_5)(x) - G_5(x).$$

Let $X_1, \ldots, X_n$ be a random sample from a distribution with density f. For simplicity let n be even. To avoid tedious calculations let

$$\widehat{\theta}_1(a) = \frac{1}{(n/2)a^6} \int_{\mathbb{R}^2} \sum_{k=1}^{n/2} \kappa^1(x/a, X_k/a)\kappa^1(x/a, X_{n-k}/a)\, dx, \tag{27}$$

for short $\widehat{\theta}_1 = \widehat{\theta}_1(a)$. By definition,

$$E\widehat{\theta}_1(a) = \int_{\mathbb{R}^2} \left(\frac{T_a^1 f}{a}\right)^2.$$

We have

$$E[\widehat{\theta}_1 - \theta_1]^2 = E[\widehat{\theta}_1 - E\widehat{\theta}_1]^2 + [E\widehat{\theta}_1 - \theta_1]^2.$$

The asymptotic behavior of the deterministic part follows from Theorem 4.1 and the equality

$$[E\widehat{\theta}_1 - \theta_1]^2 = \left|\left\|\frac{T_a^1 f}{a}\right\|_2 - \operatorname{Asym}(f,1)\right|^2 \left|\left\|\frac{T_a^1 f}{a}\right\|_2 + \operatorname{Asym}(f,1)\right|^2.$$

THEOREM 4.1. *Let $f \in W_2^3$. Then*

$$\left|\left\|\frac{T_a^1 f}{a}\right\|_2 - \operatorname{Asym}(f,1)\right| \le C(a|f|_{2,2} + a^2|f|_{3,2}).$$

Proof. From Theorem 8 and Lemma 11 of [9] we infer that there is $C > 0$ such that for $g \in W_2^2$,

$$\left| \left\| \frac{Q_a^1 g - g}{a} \right\|_2 - \operatorname{Asym}(g, 1) \right| \leq Ca|g|_{2,2}.$$

Put $g = Q_a^5 f$. Consequently,

$$\left| \left\| \frac{Q_a^1 Q_a^5 f - Q_a^5 f}{a} \right\|_2 - \operatorname{Asym}(Q_a^5 f, 1) \right| \leq Ch|Q_a^5 f|_{2,2}.$$

From Corollary 2.1 of [10] we obtain, for $|\beta| = 2$,

$$\|D^\beta Q_a^5 f - D^\beta f\|_2 \leq Ca|f|_{3,2}.$$

Then

$$|Q_a^5 f|_{2,2} \leq C(|f|_{2,2} + a|f|_{3,2}).$$

Consequently,

$$\left| \left\| \frac{Q_a^1 Q_a^5 f - Q_a^5 f}{a} \right\|_2 - \operatorname{Asym}(Q_a^5 f, 1) \right| \leq C(a|f|_{2,2} + a^2|f|_{3,2}). \tag{28}$$

From the triangle inequality

$$|\operatorname{Asym}(Q_a^5 f, 1) - \operatorname{Asym}(f, 1)|$$
$$\leq \Bigg(\int\limits_{\mathbb{R}^2} \Bigg(\int\limits_{[0,1]^2} \Bigg| \sum_{|\beta|=1} \frac{1}{\beta!} |D^\beta f(t) - D^\beta Q_a^5 f(t)| (Q^1([]^\beta)(x) - []^\beta(x)) \Bigg|^2 dx \Bigg) dt \Bigg)^{1/2}.$$

By the above mentioned Corollary 2.1 of [10] with $|\beta| = 1$ we have

$$|\operatorname{Asym}(Q_a^5 f, 1) - \operatorname{Asym}(f, 1)| \leq Ca^2|f|_{2,2}. \tag{29}$$

Combining (28) and (29) we finish the proof. ∎

We need the following lemma.

Lemma 4.1. *Let $I := [0,1] \times [1,2]$. Let f be a bounded density such that $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then for fixed $\alpha \in \mathbb{Z}^2$,*

$$\lim_{h \to 0} \frac{1}{h^2} \sum_{\alpha_1 \in \mathbb{Z}^2} \int\limits_{Ih + \alpha_1 h} f(u)\, du \int\limits_{Ih + (\alpha_1 + \alpha)h} f(u)\, du = \int\limits_{\mathbb{R}^2} f^2.$$

The proof is left to the reader. We can reformulate the lemma to obtain the following statement. Let f be a density such that $f \in L^2(\mathbb{R}^2)$. Let $\alpha \in \mathbb{Z}^2$ be fixed. Then for a.e. $x_1, x_2 \in [0,1]^2$,

$$\lim_{h \to 0} \sum_{\alpha_1 \in \mathbb{Z}^2} f(hx_1 + \alpha_1 h) f(hx_2 + \alpha_1 h + \alpha h) h^2 = \int\limits_{\mathbb{R}^2} f^2.$$

We mention this because the convergence of the Riemann sums was observed for Lebesgue-integrable functions in the papers [4] and [12].

Let us note that the support of the function F_3 is equal to I.

THEOREM 4.2. *Let f be a density such that $f \in W_2^2$. If $na^6 \to \infty$ and $a \to 0$ then*

$$\lim na^6 E(\widehat{\theta}_1 - E\widehat{\theta}_1)^2 = 2 \int_{\mathbb{R}^2} f^2 \sum_{\alpha \in \mathbb{Z}^2} b_\alpha^2, \tag{30}$$

where

$$b_\alpha = \int_{\mathbb{R}^2} K(x)K(x-\alpha)\,dx.$$

Proof. Write

$$\begin{aligned}
&E(\widehat{\theta}_1 - E\widehat{\theta}_1)^2 \\
&\quad= E\bigg(\frac{1}{(n/2)a^6} \int_{\mathbb{R}^2} \sum_{k=1}^{n/2} \kappa^1(x/a, X_k/a)\kappa^1(x/a, X_{n-k}/a)\,dx - \int_{\mathbb{R}^2} \Big(\frac{T_a^1 f}{a}\Big)^2\bigg)^2 \\
&\quad= \frac{4}{n^2a^4} E\bigg(\sum_{k=1}^{n/2} \int_{\mathbb{R}^2} \Big(\frac{1}{a^4}\kappa^1(x/a, X_k/a)\kappa^1(x/a, X_{n-k}/a) - (T_a^1 f(x))^2\Big)\,dx\bigg)^2 \\
&\quad= \frac{2}{na^4} E\bigg(\int_{\mathbb{R}^2} \Big(\frac{1}{a^4}\kappa^1(x/a, X_1/a)\kappa^1(x/a, X_2/a) - (T_a^1 f(x))^2\Big)\,dx\bigg)^2 \\
&\quad= \frac{2}{na^4} E\bigg(\int_{\mathbb{R}^2} \frac{1}{a^4}\kappa^1(x/a, X_1/a)\kappa^1(x/a, X_2/a)\,dx\bigg)^2 - \frac{2}{na^4}\Big(\int_{\mathbb{R}^2} (T_a^1 f)^2\Big)^2.
\end{aligned}$$

Only the first term of the last formula is important (let us denote it by P). Using the assumption $f \in W_2^2$ we find that the second term is negligible. Using the kernel representation we get

$$\begin{aligned}
P &= \frac{2}{na^4} E\bigg(\int_{\mathbb{R}^2} \frac{1}{a^4}\kappa^1(x/a, X_1/a)\kappa^1(x/a, X_2/a)\,dx\bigg)^2 \\
&= \frac{2}{na^4} E\bigg(\int_{\mathbb{R}^2} \frac{1}{a^4} \sum_{\alpha_1 \in \mathbb{Z}^2} \sum_{\alpha_2 \in \mathbb{Z}^2} F_3(X_1/a - \alpha_1)F_3(X_2/a - \alpha_2) \\
&\qquad\qquad \times K(x/a - \alpha_1)K(x/a - \alpha_2)\,dx\bigg)^2 \\
&= \frac{2}{na^4} E\bigg(\frac{1}{a^2} \sum_{\alpha_1 \in \mathbb{Z}^2} \sum_{\alpha_2 \in \mathbb{Z}^2} F_3(X_1/a - \alpha_1)F_3(X_2/a - \alpha_2) b_{\alpha_1 - \alpha_2}\bigg)^2 \\
&= \frac{2}{na^8} \sum_{\alpha_1 \in \mathbb{Z}^2} \sum_{\alpha_2 \in \mathbb{Z}^2} \sum_{\alpha_3 \in \mathbb{Z}^2} \sum_{\alpha_4 \in \mathbb{Z}^2} E(F_3(X_1/a - \alpha_1)F_3(X_1/a - \alpha_3)) \\
&\qquad\qquad \times E(F_3(X_2/a - \alpha_2)F_3(X_2/a - \alpha_4)) b_{\alpha_1 - \alpha_2} b_{\alpha_3 - \alpha_4}.
\end{aligned}$$

Observe that since F_3 is the characteristic function of I, if $\alpha_1 \neq \alpha_3$ we have

$$E(F_3(X_1/a - \alpha_1)F_3(X_1/a - \alpha_3)) = 0,$$

while if $\alpha_1 = \alpha_3$,

$$\begin{aligned} E(F_3(X_1/a - \alpha_1))^2 &= \int\limits_{\mathbb{R}^2} (F_3(u/a - \alpha_1))f(u)\,du \\ &= \int\limits_{Ia+\alpha_1 a} f(u)\,du, \end{aligned}$$

where (recall) $I = [0,1] \times [1,2]$. Consequently,

$$P = \frac{2}{na^8} \sum_{\alpha_1 \in \mathbb{Z}^2} \sum_{\alpha_2 \in \mathbb{Z}^2} (b_{\alpha_1 - \alpha_2})^2 \int\limits_{Ia+\alpha_1 a} f(u)\,du \int\limits_{Ia+\alpha_2 a} f(u)\,du.$$

Using Lemma 4.1 finishes the proof since $b_\alpha = 0$ for $|\alpha| > 4$. ■

Note that applying the two last theorems we deduce that to estimate the asymptotic constant the bandwidth is $a_{\mathrm{MISE}} \sim (1/n)^{1/8}$.

5. Simulation results. We show an accuracy of the estimation of the asymptotic constant for the histogram. We take the dimension $d = 1$ and 1000 samples from the distribution of random variables

$$X = \sigma Z + 3\sigma Y,$$

where Z is standard normal $N(0,1)$. The random variable Y is independent of Z and has binomial distribution with $p = 0.5$. We estimate

$$\theta_1 = \frac{1}{12} \int\limits_{\mathbb{R}} (f')^2.$$

In the case of $d = 1$ the formula (27) can be written as

$$\widehat{\theta}_1(a) = \frac{2}{na^3} \sum_{k=1}^{n/2} \sum_{j \in Z} \sum_{|l| \le 4} A_l I_{[1,2]}(X_n/a - j) I_{[1,2]}(X_{n-k}/a - j - l),$$

where $I_{[1,2]}$ is the characteristic function of $[1,2]$,

$$\begin{aligned} A_l &= \int\limits_{\mathbb{R}} K(x+l)K(x)\,dx, \\ K(x) &= -\frac{1}{6}(Q_1(G_3) - G_3)(x-1) \\ &\quad + \frac{4}{3}(Q_1(G_3) - G_3)(x) - \frac{1}{6}(Q_1(G_3) - G_3)(x+1) \end{aligned}$$

and G_3 is the B-spline, i.e.

$$G_3(x) = \frac{1}{2}\sum_{j=0}^{3}(-1)^{r-j}\binom{r}{j}(j-x)_+^2.$$

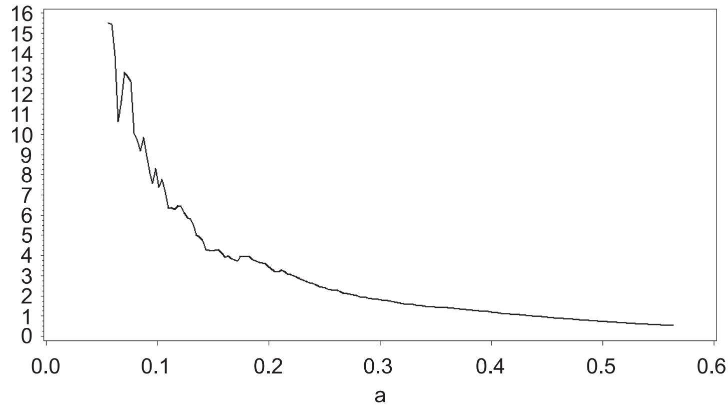
In this case, Q_1 is the orthogonal projection

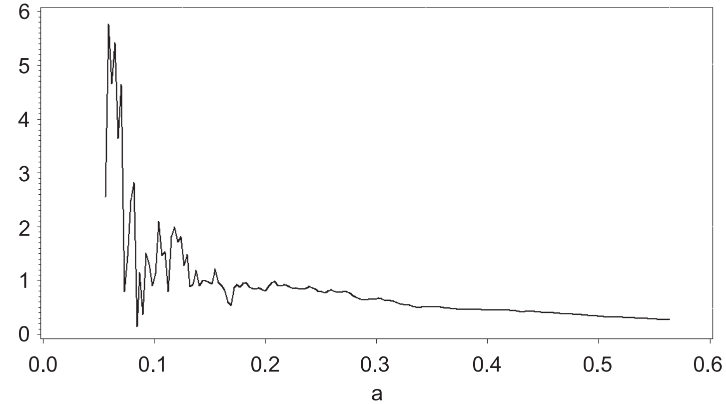
$$Q_1 f(x) = \sum_{k\in Z}\int_k^{k+1} f(u)\,du\, I_{[k,k+1]}(x).$$

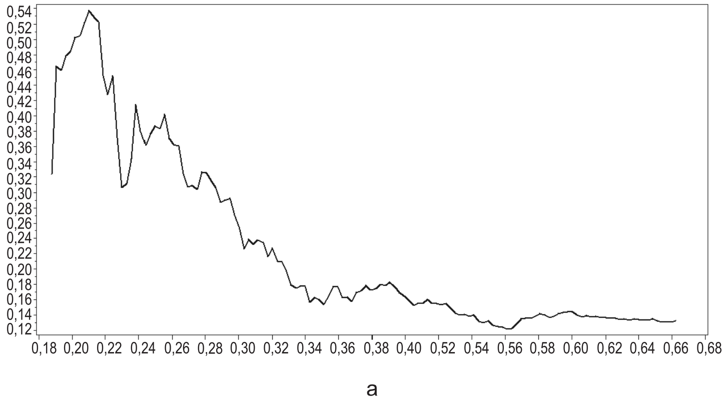
We have

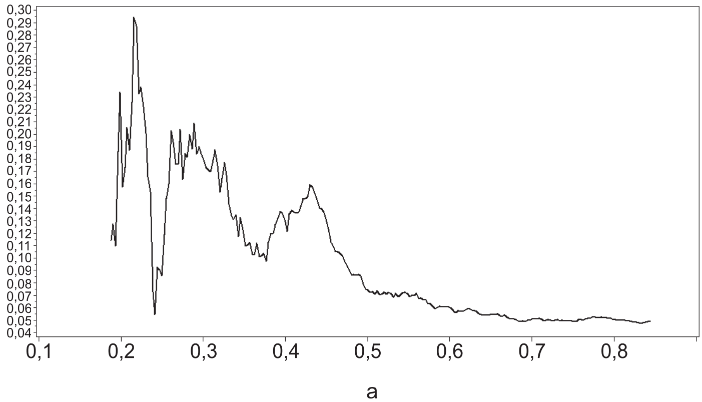
σ	0.1	0.2	0.3	0.4
θ_1	3.709	0.464	0.137	0.058

We show the four functions $\widehat{\theta}_1(a)$ (SAS 9) with respect to different σ from 0.1 to 0.4. The simulations show that a for which $\widehat{\theta}_1(a)$ gives a good estimation of θ_1 lies in the region where the oscillations diminish. It would be interesting to prove it.









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