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IMPLICIT DIFFERENCE METHODS FOR NONLINEAR FIRST ORDER PARTIAL FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. Initial problems for nonlinear hyperbolic functional differential systems are considered. Classical solutions are approximated by solutions of suitable quasilinear systems of difference functional equations. The numerical methods used are difference schemes which are implicit with respect to the time variable. Theorems on convergence of difference schemes and error estimates of approximate solutions are presented. The proof of the stability is based on a comparison technique with nonlinear estimates of the Perron type. Numerical examples are given.

1. Introduction. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. If $A \subset X$ and $\alpha \in C(X,Y)$ then $\alpha|_A$ denotes the restriction of α to the set A. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Suppose that $M = (M_1, \ldots, M_n) \in \mathbb{R}_+^n$, $a > 0$, $\mathbb{R}_+ = [0, +\infty)$, $b = (b_1, \ldots, b_n)$, $b \in \mathbb{R}_+^n$ and $b > Ma$. Let E be the *Haar pyramid*

$$E = \{(t,x) \in \mathbb{R}^{1+n} : t \in [0,a],\ -b + Mt \le x \le b - Mt\}$$

where $x = (x_1, \ldots, x_n)$. Write $E_0 = [-b_0, 0] \times [-b, b]$ where $b_0 \in \mathbb{R}_+$ and

$$B = [-b_0 - a, 0] \times [-2b, 2b],$$
$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n), \qquad 0 \le t \le a.$$

For $(t,x) \in E$ we define

$$D[t,x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \tau \le 0 \text{ and } (t+\tau, x+y) \in E_0 \cup E\}.$$

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Given a function $z\colon E_0 \cup E \to \mathbb{R}^k$ and a point $(t,x) \in E$, we consider the function $z_{(t,x)}\colon D[t,x] \to \mathbb{R}^k$ defined by $z_{(t,x)}(\tau,y) = z(t+\tau, x+y)$, $(\tau,y) \in D[t,x]$. Thus $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$, shifted to the set $D[t,x]$.

Put $\Omega = E \times C(B,\mathbb{R}^k) \times \mathbb{R}^n$ and suppose that $f\colon \Omega \to \mathbb{R}^k$, $f = (f_1,\ldots,f_k)$, is a given function of the variables (t,x,w,q), $w = (w_1,\ldots,w_k)$, $q = (q_1,\ldots,q_n)$. We will say that f satisfies the *condition* (V) if for $(t,x,q) \in E \times \mathbb{R}^n$ and $w, \bar{w} \in C(B,\mathbb{R}^k)$ such that $w|_{D[t,x]} = \bar{w}|_{D[t,x]}$ we have $f(t,x,w,q) = f(t,x,\bar{w},q)$. The condition (V) means that the value of f at $(t,x,w,q) \in \Omega$ depends on (t,x,q) and on the restriction of w to $D[t,x]$ only.

Let us denote by $z = (z_1,\ldots,z_k)$ an unknown function of the variables (t,x). Given $\varphi\colon E_0 \to \mathbb{R}^k$, we consider the system of functional differential equations

$$\partial_t z_i(t,x) = f_i(t,x,z_{(t,x)},\partial_x z_i(t,x)), \quad i = 1,\ldots,k, \tag{1.1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{on } E_0 \tag{1.2}$$

where $\partial_x z_i = (\partial_{x_1} z_i,\ldots,\partial_{x_n} z_i)$. System (1.1) has the property that every equation contains the vector of unknown functions and the derivatives of only one scalar function. We consider classical solutions of (1.1), (1.2) and we assume that f satisfies the condition (V). The Haar pyramid is a natural domain for the existence and uniqueness of classical or generalized solutions for nonlinear hyperbolic systems with initial conditions ([3], [12], [21]).

The following methods of construction of approximate solutions for nonlinear hyperbolic functional differential equations are known: the numerical method of bicharacteristics, the Euler difference method, the Lax scheme. The aim of this paper is to add a new element to the above sequence of numerical methods.

We are interested in establishing a method of numerical approximation of solutions to (1.1), (1.2) by means of solutions of associated systems of implicit difference functional equations and in estimating the difference between the exact and approximate solutions.

We first give some motivations for our investigations. Let $(h_0,h') = h$, $h' = (h_1,\ldots,h_n)$, stand for steps of a mesh. Let $(t^{(r)},x^{(m)})$, $m = (m_1,\ldots,m_n)$, denote nodal points. Let $E_{0.h}$ and E_h be the sets of all nodal points which are elements of E_0 and E respectively. Solutions of difference equations are defined on $E_{0.h} \cup E_h$. Classical difference methods for (1.1), (1.2) consist in replacing the partial derivatives ∂_t and $(\partial_{x_1},\ldots,\partial_{x_n}) = \partial_x$ with difference operators δ_0 and $(\delta_1,\ldots,\delta_n) = \delta$. Moreover, system (1.1) contains the functional variable $z_{(t,x)}$ which is an element of $C(D[t,x],\mathbb{R}^k)$. Then we need an interpolating operator

$$T_h\colon \mathcal{F}(E_{0.h}\cup E_h,\mathbb{R}^k)\to C(E_0\cup E,\mathbb{R}^k). \tag{1.3}$$

Additional assumptions on T_h will be needed in Section 4. System (1.1) leads to the difference functional system

$$\delta_0 z_i^{(r,m)} = f_i(t^{(r)},x^{(m)},(T_h z)_{(t^{(r)},x^{(m)})},\delta z_i^{(r,m)}),\qquad i=1,\dots,k, \tag{1.4}$$

where $\delta z_i=(\delta_1 z_i,\dots,\delta_n z_i)$. Initial conditions are associated with (1.4).

The following examples of systems (1.4) are considered in the literature: the Euler difference method and the Lax scheme (see [12, Chapter 3]). Two types of assumptions are needed in theorems on the stability of difference schemes generated by (1.1), (1.2). Conditions the first type concern the regularity of f and they are the same for both methods. It is required that the function f of variables (t,x,w,q) satisfies nonlinear estimates of the Perron type with respect to w and it is of class C^1 with respect to q and that the functions $\partial_q f_i=(\partial_{q_1}f_i,\dots,\partial_{q_n}f_i)$, $1\le i\le k$, are bounded on Ω. Assumptions the second type in convergence theorems are the Courant–Friedrichs–Levy (CFL) conditions (see [8, Chapter III], [9], [12, Chapter III]). For the analysis of the stability of the Euler difference method we need the following (CFL) conditions:

(i) for each $P=(t,x,w,q)\in\Omega$ we have

$$1-h_0\sum_{j=1}^{n}\frac{1}{h_j}|\partial_{q_j}f_i(P)|\ge 0,\qquad 1\le i\le k, \tag{1.5}$$

(ii) for each $1\le i\le k$, the function

$$\operatorname{sign}\partial_q f_i=(\operatorname{sign}\partial_{q_1}f_i,\dots,\operatorname{sign}\partial_{q_n}f_i) \tag{1.6}$$

is constant on Ω.

The (CFL) conditions for (1.1) and for the Lax difference scheme have the form

$$\frac{1}{n}-\frac{h_0}{h_j}|\partial_{q_j}f_i(P)|\ge 0,\qquad 1\le j\le n,\ 1\le i\le k, \tag{1.7}$$

where $P\in\Omega$.

Note that the assumptions (1.5) and (1.7) require some relations between h_0 and h'. We conclude from condition (ii) that we need more restrictive assumptions on f for the Euler method than for the Lax scheme.

Of course there are systems (1.1) for which both the difference methods are convergent. It follows from the theory of bicharacteristics that in this case the Euler method is more suitable than the Lax scheme. This theoretical observation is confirmed by numerical experiments.

With the above motivation we are interested in proving convergence results for Euler methods and for a possibly large class of nonlinear problems. More precisely, we will show that there are convergent difference methods of the Euler type for which the assumption that the functions (1.6) are constant can be omitted. In other words, we show that we do not need the Lax difference schemes for systems (1.1) with natural regularity assumptions on f. Since we consider implicit difference schemes, we show that the (CFL) condition (1.5) can also be omitted in the convergence analysis.

In recent years, a number of papers concerning numerical methods for first order partial functional differential equations have been published. Explicit difference schemes for initial or initial boundary value problems were studied in the papers [11], [20] and in the monograph [12]. The proofs of convergence were based on functional difference inequalities or on general theorems on error estimates for approximate solutions to functional difference equations of the Volterra type with an unknown function of several variables.

Sufficient conditions for the convergence of implicit difference schemes for initial boundary value problems are given in [4], [13]. It is assumed in [4] that the functions (1.6) are constant on Ω. It follows that the results of [4] are not applicable to (1.1), (1.2). The papers [16], [17] initiated investigations of implicit difference schemes for parabolic equations. Monotone iterative methods and implicit difference schemes for computing approximate solutions of parabolic equations with time delay were analysed in [18], [19], [27]. Implicit difference schemes for nonlinear parabolic functional differential equations with initial boundary conditions of the Dirichlet type were studied in [14]. Convergence theorems were proved by using a comparison technique.

In this paper we propose a new class of difference schemes corresponding to (1.1), (1.2). We approximate the unknown function z and its partial derivatives $\partial_x z_i$, $1 \le i \le k$, by solutions of quasilinear systems of difference functional equations which are implicit with respect to the variable t. In this procedure we linearize (1.1) with respect to the last variable. This method has been used in the existence and uniqueness theory for classical or generalized solutions.

Sufficient conditions for the existence and uniqueness of classical or generalized solutions can be found in the papers [2], [3], [10], [21], [22] and the monograph [12]. We use general ideas concerning the stability of numerical methods for evolution differential or functional differential equations, introduced in [15], [23], [24].

First order partial functional differential equations find applications in different fields of knowledge. Differential-integral systems have been proposed in [1] as simple mathematical models for the nonlinear phenomenon of harmonic generation of laser radiation through piezoelectric crystals for nondispersive materials and of Maxwell–Hopkinson type. Systems of differential equations containing operators acting on the unknown density of populations in dependence on their age, size, and DNA content, are considered in [25]. An equation with a deviated argument ([6]) describes the density of households at a time t, depending on their estate, in the theory of the distribution of wealth. Another system of integral-differential equations is used in mathematical biology to investigate an age-dependent epidemic of a disease with vertical transmissions [7]. The paper [26] deals with integral differential equations motivated by applications in the theory of screening of granular bodies. For further information on applications of functional differential equations see the monographs [12], [28].

2. Discretization of nonlinear systems. We will denote by $\mathcal{F}(X,Y)$ the class of all functions from X into Y where X and Y are arbitrary sets. We will use the symbols $\mathbb{N}$ and $\mathbb{Z}$ to denote the sets of natural numbers and integers respectively. Denote by $M_{k\times n}$ the class of all $k\times n$ matrices with real entries. For $x\in\mathbb{R}^n$, $p\in\mathbb{R}^k$, $U\in M_{k\times n}$ where $x=(x_1,\ldots,x_n)$, $p=(p_1,\ldots,p_k)$, $U=[u_{ij}]_{i=1,\ldots,k,j=1,\ldots,n}$ we put $\|x\|=|x_1|+\cdots+|x_n|$, $\|p\|_\infty=\max\{|p_i|:1\le i\le k\}$ and

$$\|U\|=\max\Big\{\sum_{j=1}^n|u_{ij}|:1\le i\le k\Big\}.$$

The product of two matrices is denoted by $\star$. If $U\in M_{k\times n}$ then U^T denotes the transpose matrix. The scalar product in $\mathbb{R}^n$ is denoted by $\circ$. For $\omega\in C(B,\mathbb{R}^k)$ and $(t,x)\in E_0\cup E$ we put $\|\omega\|_{D[t,x]}=\max\{|\omega(\tau,y)|:(\tau,y)\in D[t,x]\}$. The maximum norm in the space $C(B,\mathbb{R}^k)$ will be denoted by $\|\cdot\|_B$. Let $CL(B,\mathbb{R})$ denote the set of all linear and continuous real functions defined on $C(B,\mathbb{R})$. The norm in $CL(B,\mathbb{R})$ which is generated by the maximum norm in $C(B,\mathbb{R})$ will be denoted by $\|\cdot\|_\star$.

We define a mesh on the set $E_0\cup E$ in the following way. Let (h_0,h'), $h'=(h_1,\ldots,h_n)$, stand for steps of the mesh. For $h=(h_0,h')$ and $(r,m)\in\mathbb{Z}^{1+n}$ where $m=(m_1,\ldots,m_n)$, we define nodal points in the following way: $t^{(r)}=rh_0$, $x^{(m)}=(x_1^{(m_1)},\ldots,x_n^{(m_n)})=(m_1h_1,\ldots,m_nh_n)$. Let $\tilde H$ denote the set of all $h=(h_0,h')$ such that there are $(N_1,\ldots,N_n)=N\in\mathbb{N}^n$ and $K_0\in\mathbb{Z}$ with $(N_1h_1,\ldots,N_nh_n)=b$, $K_0h_0=b_0$ and $h'\le Mh_0$. Define $K\in\mathbb{N}$ by $Kh_0\le a<(K+1)h_0$. For $h\in\tilde H$ we put $\|h\|=h_0+h_1+\cdots+h_n$ and

$$\mathbb{R}_h^{1+n}=\{(t^{(r)},x^{(m)}):(r,m)\in\mathbb{Z}^{1+n}\},\qquad I_h=\{t^{(r)}:0\le r\le K\}.$$

Set $E_{0.h}=E_0\cap\mathbb{R}_h^{1+n}$, $E_h=E\cap\mathbb{R}_h^{1+n}$. For functions $\eta\colon I_h\to\mathbb{R}^n$, $z\colon E_{0.h}\cup E_h\to\mathbb{R}^k$, $u\colon E_{0.h}\cup E_h\to M_{k\times n}$, we write $\eta^{(r)}=\eta(t^{(r)})$, $z^{(r,m)}=z(t^{(r)},x^{(m)})$,

$u^{(r,m)} = u(t^{(r)}, x^{(m)})$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 in the jth place, $1 \le j \le n$. We formulate implicit difference methods of the Euler type for (1.1), (1.2). Write

$$E^+_{i.\epsilon} = \{(t,x) \in E : b_i - M_i t - \epsilon < x_i \le b_i - M_i t\},$$
$$E^-_{i.\epsilon} = \{(t,x) \in E : b_i + M_i t < x_i \le b_i + M_i t + \epsilon\},$$

where $0 < \epsilon < b_i - M_i a$ and $1 \le i \le n$.

Assumption $H_0[f]$. The function $f\colon \Omega \to \mathbb{R}^k$ is continuous, satisfies the condition (V) and

1) the partial derivatives
$$\partial_x f = [\partial_{x_j} f_i]_{i=1,\dots,k,\, j=1,\dots,n}, \qquad \partial_q f = [\partial_{q_j} f_i]_{i=1,\dots,k,\, j=1,\dots,n}$$
exist on Ω and $\partial_x f,\ \partial_q f \in C(\Omega, M_{k\times n})$,
2) there exist the Fréchet derivatives $\partial_w f(P) = [\partial_{w_j} f_i(P)]_{i,j=1,\dots,k}$, and $\partial_{w_j} f_i(P) \in CL(B, \mathbb{R})$ for $1 \le i, j \le k$, $P \in \Omega$,
3) there is $\epsilon > 0$ such that
$$\partial_{q_i} f(t,x,w,q) \le \theta_{[k]} \quad \text{ on } E^+_{i.\epsilon} \times C(B, \mathbb{R}^k) \times \mathbb{R}^n,$$
$$\partial_{q_i} f(t,x,w,q) \ge \theta_{[k]} \quad \text{ on } E^-_{i.\epsilon} \times C(B, \mathbb{R}^k) \times \mathbb{R}^n,$$
where $1 \le i \le k$ and $\partial_{q_i} f = (\partial_{q_i} f_1, \dots, \partial_{q_i} f_k)$, $\theta_{[k]} = (0, \dots, 0) \in \mathbb{R}^k$.

Remark 2.1. Suppose that the function $\partial_q f\colon \Omega \to M_{k\times n}$ satisfies the condition

$$(x_1 \partial_{q_1} f_i(P), \dots, x_n \partial_{q_n} f_i(P)) \le \theta_{[n]}, \qquad 1 \le i \le k,\ P \in \Omega, \tag{2.1}$$

where $\theta_{[n]} = (0, \dots, 0) \in \mathbb{R}^n$. Then condition 3) of Assumption H_0 $[f]$ holds true.

Note that if condition (2.1) is satisfied on $[0,a] \times [-b,b] \times C(B, \mathbb{R}^k) \times \mathbb{R}^k$ then the natural domain for classical solutions of (1.1), (1.2) is the set $E = [0,a] \times [-b,b]$. Then we put $M = \theta_{[n]}$ in the definition of the Haar pyramid. This property of the initial value problem (1.1), (1.2) follows from the theory of bicharacteristics.

Suppose that Assumption $H_0[f]$ is satisfied. Set

$$H = \{h = (h_0, h') \in \tilde{H} : h_j \le \epsilon/2 \text{ for } 1 \le j \le n\}.$$

We write $\partial_x f_i = (\partial_{x_1} f_i, \dots, \partial_{x_n} f_i)$, $\partial_w f_i = (\partial_{w_1} f_i, \dots, \partial_{w_k} f_i)$, where $1 \le i \le k$. Set $E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : (t^{(r+1)}, x^{(m)}) \in E_h\}$ and $E_{r.h} = (E_{0.h} \cup E_h) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$, $0 \le r \le K$. We construct a difference problem corresponding to (1.1), (1.2). Unknown functions in a difference system are denoted by (z,u) where $z = (z_1, \dots, z_k)$, $u = [u_{ij}]_{i=1,\dots,k,\, j=1,\dots,n}$, $u_i = (u_{i1}, \dots, u_{in})$, $1 \le i \le k$. We denote by δ_0 and $(\delta_1, \dots, \delta_n) = \delta$ the difference operators for the variable t and for the spatial variables $(x_1, \dots, x_n) =$

x respectively. Write $\delta_0 u_i = (\delta_0 u_{i1}, \dots, \delta_0 u_{in})$, $\delta z_i = (\delta_1 z_i, \dots, \delta_n z_i)$ and $\delta u_i = [\delta_j u_{is}]_{s,j=1,\dots,n}$ where $1 \le i \le k$. Set

$$T_h u_{(r,m)} = [T_h(u_{ij})_{(r,m)}]_{i=1,\dots,k,\, j=1,\dots,n}, \partial_w f_i(P) \star T_h u_{(r,m)}$$
$$= \Big(\sum_{s=1}^{k} \partial_{w_s} f_i(P) T_h(u_{s1})_{(r,m)}, \dots, \sum_{s=1}^{k} \partial_{w_s} f_i(P) T_h(u_{sn})_{(r,m)}\Big), \quad P \in \Omega,$$

and $P^{(r,m)}[z, u_i] = (t^{(r)}, x^{(m)}, T_h z_{(r,m)}, u_i^{(r,m)})$, $1 \le i \le k$. We consider the system of difference equations

$$\text{(2.2)} \quad \delta_0 z_i^{(r,m)} = f_i(P^{(r,m)}[z, u_i]) + \partial_q f_i(P^{(r,m)}[z, u_i]) \circ (\delta z_i^{(r+1,m)} - u_i^{(r,m)}), \quad 1 \le i \le k,$$

and

$$\text{(2.3)} \quad \delta_0 u_i^{(r,m)} = \partial_x f_i(P^{(r,m)}[z, u_i]) + \partial_w f_i(P^{(r,m)}[z, u_i]) \star T_h u_{(r,m)} + \partial_q f_i(P^{(r,m)}[z, u_i]) \star [\delta u_i^{(r+1,m)}]^T, \quad 1 \le i \le k,$$

with the initial conditions

$$\text{(2.4)} \quad z^{(r,m)} = \varphi_h^{(r,m)}, \quad u^{(r,m)} = \psi_h^{(r,m)} \quad \text{on } E_{0.h}$$

where $\varphi_h \colon E_{0.h} \to \mathbb{R}^k$ and $\psi_h \colon E_{0.h} \to M_{k\times n}$, are given functions. The difference expressions $\delta_0 z_i$ and $\delta_0 u_i$ are defined by

$$\text{(2.5)} \quad \begin{aligned} \delta_0 z_i^{(r,m)} &= \frac{1}{h_0}[z_i^{(r+1,m)} - z_i^{(r,m)}], \\ \delta_0 u_i^{(r,m)} &= \frac{1}{h_0}[u_i^{(r+1,m)} - u_i^{(r,m)}], \quad 1 \le i \le k. \end{aligned}$$

The difference operator δ for the spatial variables is defined in the following way. Suppose that the functions (z, u) are known on the set $E_{r.h}$ where $0 \le r < K$. We put

$$\text{(2.6)} \quad \text{if } \partial_{q_j} f_i(P^{(r,m)}[z, u_i]) \ge 0 \text{ then } \delta_j z_i^{(r+1,m)} = \frac{1}{h_j}[z_i^{(r+1,m+e_j)} - z_i^{(r+1,m)}]$$

and

$$\text{(2.7)} \quad \delta_j u_{is}^{(r+1,m)} = \frac{1}{h_j}[u_{is}^{(r+1,m+e_j)} - u_{is}^{(r+1,m)}], \quad 1 \le s \le n.$$

Moreover we set

$$\text{(2.8)} \quad \text{if } \partial_{q_j} f_i(P^{(r,m)}[z, u_i]) < 0 \text{ then } \delta_j z_i^{(r+1,m)} = \frac{1}{h_j}[z_i^{(r+1,m)} - z_i^{(r+1,m-e_j)}]$$

and

$$\text{(2.9)} \quad \delta_j u_{is}^{(r+1,m)} = \frac{1}{h_j}[u_{is}^{(r+1,m)} - u_{is}^{(r+1,m-e_j)}], \quad 1 \le s \le n.$$

We have $i = 1, \dots, k$, $j = 1, \dots, n$ in (2.6)–(2.9). Note that the difference operators δz_i and δu_i, $1 \le i \le k$, are calculated at the point $(t^{(r+1)}, x^{(m)})$ in (2.2), (2.3). Then the numbers $z_i^{(r+1,m+e_j)}$, $z_i^{(r+1,m-e_j)}$ and the vectors $u_i^{(r+1,m+e_j)}$, $u_i^{(r+1,m-e_j)}$, $1 \le j \le n$, appear on the right hand sides of (2.2), (2.3). It follows that we have obtained implicit difference schemes.

If we apply classical difference methods to problem (1.1), (1.2) then we approximate derivatives with respect to spatial variables by difference expressions which are calculated by using previous values of approximate solutions. In our method we approximate the spatial derivatives of the unknown function in (1.1) by solving suitable difference equations which are generated by the original problem.

Remark 2.2. The above construction of an implicit Euler method has the following property: the definition of the difference expressions δz_i and δu_i, $1 \le i \le k$, at the point $(t^{(r+1)}, x^{(m)})$ depends on the local properties of the functions $\partial_q f_i$, $1 \le i \le k$. Note that we construct the Euler type methods and we do not assume that the functions $(\operatorname{sign} \partial_{q_1} f_i, \dots, \operatorname{sign} \partial_{q_n} f_i)$, $1 \le i \le k$, are constant.

Remark 2.3. Suppose that Assumption $H_0[f]$ is satisfied and $z \in C(E_0 \cup E, \mathbb{R}^k)$, $u \in C(E_0 \cup E, M_{k\times n})$ and $u = [u_{ij}]_{i=1,\dots,k, j=1,\dots,n}$, $u_i = (u_{i1}, \dots, u_{in})$, $1 \le i \le k$. Let us consider the Cauchy problem

$$\eta'(t) = -\partial_q f_i(\tau, \eta(\tau), z_{(\tau,\eta(\tau))}, u_i(\tau, \eta(\tau))), \qquad \eta(t) = x,$$

where $1 \le i \le k$ is fixed and $(t, x) \in E$. The solution

$$g_i[z, u](\cdot, t, x) = (g_{i1}[z, u](\cdot, t, x), \dots, g_{in}[z, u](\cdot, t, x))$$

of the above problem is the ith bicharacteristic of system (1.1) corresponding to (z, u). The following property of bicharacteristics is important in our construction of implicit difference schemes for (1.1), (1.2). The difference operators $(\delta_1, \dots, \delta_n) = \delta$ used in this paper satisfy the following conditions:

(i) if the function $g_{ij}[z, u](\,\cdot, t, x)$ is increasing on $[t - \epsilon_0, t]$, $\epsilon_0 > 0$, then

$$\delta_j z_i(t, x) = \frac{1}{\tau}[z_i(t, x) - z_i(t, x - \tau e_j)],$$

$$\delta_j u_{is}(t, x) = \frac{1}{\tau}[u_{is}(t, x) - u_{is}(t, x - \tau e_j)], \qquad 1 \le s \le n,$$

for some $\tau > 0$,

(ii) if $g_{ij}[z, u](\,\cdot, t, x)$ is decreasing on $[t - \epsilon_0, t]$, $\epsilon_0 > 0$, then

$$\delta_j z_i(t, x) = \frac{1}{\tau}[z_i(t, x - \tau e_j) - z_i(t, x)],$$

$$\delta_j u_{is}(t, x) = \frac{1}{\tau}[u_{is}(t, x - \tau e_j) - u_{is}(t, x)], \qquad 1 \le s \le n,$$

for some $\tau > 0$.

REMARK 2.4. In our theorem on the convergence of the above difference method we will assume that the initial function φ is of class C^1 on E_0 and that the functions φ_h and ψ_h are approximations of φ and $\partial_x\varphi$ respectively.

The difference problem consisting of system (2.2), (2.3) and initial condition (2.4) with the difference operators defined by (2.5)–(2.9) is called a *generalized implicit Euler method* for (1.1), (1.2).

The dfference system (2.2), (2.3) is obtained in the following way. Suppose that Assumption $H_0[f]$ is satisfied and that the derivatives $\partial_x\varphi = [\partial_{x_j}\varphi_i]_{i=1,\dots,k,\,j=1,\dots,n}$ exist on E_0. We apply the method of quasilinearization to problem (1.1), (1.2). It consists in replacing the nonlinear problem (1.1), (1.2) with the following one. Let (z, u) be unknown functions of $(t, x) \in E$ where $z = (z_1, \dots, z_k)$, $u = [u_{ij}]_{i=1,\dots,k,\,j=1,\dots,n}$, $u_i = (u_{i1}, \dots, u_{in})$ for $1 \le i \le k$. Write $U[z, u_i; t, x] = (t, x, z_{(t,x)}, u_i(t, x))$, $1 \le i \le k$. We introduce first an additional unknown function $u = \partial_x z$ in (1.1). Then we consider the following linearization of (1.1) with respect to the last variable:

$$\begin{aligned}(2.10)\quad \partial_t z_i(t,x) &= f_i(U[z,u_i;t,x]) \\ &\quad + \partial_q f_i(U[z,u_i;t,x]) \circ (\partial_x z_i(t,x) - u_i(t,x)), \quad 1 \le i \le k.\end{aligned}$$

It follows from (1.1) that u_i, $1 \le i \le k$, satisfy the functional differential system

$$\begin{aligned}(2.11)\qquad \partial_t u_i(t,x) &= \partial_x f_i(U[z,u_i;t,x]) + \partial_w f_i(U[z,u_i;t,x]) \star u_{(t,x)} \\ &\quad + \partial_q f_i(U[z,u_i;t,x]) \star [\partial_x u_i(t,x)]^T, \quad 1 \le i \le k.\end{aligned}$$

It is natural to consider the following initial condition for the quasilinear system (2.10), (2.11):

$$(2.12)\qquad z(t,x) = \varphi(t,x), \qquad u(t,x) = \partial_x\varphi(t,x) \quad \text{on } E_0.$$

We obtain a generalized implicit Euler method for (1.1), (1.2) by a discretization of (2.10)–(2.12).

Write $S_t = \{x \in \mathbb{R}^n : (t,x) \in E_0 \cup E\}$, $t \in [-b_0, a]$, and $U_x = \{t \in \mathbb{R} : (t,x) \in E_0 \cup E\}$, $x \in [-b, b]$. Suppose that $v\colon E_0 \cup E \to \mathbb{R}$ is a classical solution of (1.1), (1.2). We say that v is of *class* C^* if $v(t,\cdot)\colon S_t \to \mathbb{R}^k$ is of class C^2 for every $t \in [-b_0, a]$ and $v(\cdot, x)\colon U_x \to \mathbb{R}^k$ is of class C^1 for every $x \in [-b, b]$.

The initial value problems (1.1), (1.2) and (2.10)–(2.12) have the following properties ([12, Chapter IV], [22]).

(i) if $v\colon E_0 \cup E \to \mathbb{R}^k$ is a classical solution of (1.1), (1.2) and v is of class C^* then the functions $(v, \partial_x v)$ satisfy (2.10)–(2.12),
(ii) if $(\tilde z, \tilde u)\colon E_0 \cup E \to \mathbb{R}^k \times M_{k\times n}$ is a classical solution of (2.10)–(2.12) then $\partial_x \tilde z = \tilde u$ and $\tilde z$ satisfies (1.1), (1.2) and $\tilde z$ is of class C^*.

There are different ways of constructing implicit difference schemes for first order partial functional differential equations or systems. We present two such constructions. For simplicity we assume that $k = 1$ and that (1.1) does not contain the functional variable. Let us consider the differential equation

$$\partial_t z(t,x) = G(t,x,z(t,x),\partial_x z(t,x)) \tag{2.13}$$

with the initial condition (1.2) for $k = 1$ and $b_0 = 0$. The difference problem

$$\delta_0 z^{(r,m)} = G(t^{(r)}, x^{(m)}, z^{(r+1,m)}, \delta z^{(r+1,m)}), \tag{2.14}$$

$$z^{(0,m)} = \varphi_h^{(0,m)} \quad \text{for } x^{(m)} \in [-b,b], \tag{2.15}$$

can be considered as an implicit method for (2.13), (1.2) with $k = 1$, $b_0 = 0$. Note that the difference operators $(\delta_1,\ldots,\delta_n)$ and the unknown function z are calculated at the point $(t^{(r+1)}, x^{(m)})$.

Let us consider the difference equation

$$\delta_0 z^{(r,m)} = G(t^{(r)}, x^{(m)}, z^{(r,m)}, \delta z^{(r+1,m)}) \tag{2.16}$$

with the initial condition (2.15). In this case the unknown function z is calculated at $(t^{(r)}, x^{(m)})$ and the difference operators $(\delta_1 z,\ldots,\delta_n z)$ at $(t^{(r+1)}, x^{(m)})$. Thus we have obtained an implicit difference scheme. The paper [4] gives sufficient conditions for the convergence of implicit difference methods of the type (2.16), (2.15) for functional differential equations.

The numerical methods (2.14), (2.15) and implicit difference schemes (2.2)–(2.4) exhibit the following differences. It is clear that a nonlinear differential equation leads to a nonlinear algebraic system (2.14). This system requires iterative schemes for the computation of numerical solutions. In the case when (2.13) is a quasilinear differential equation, the corresponding system of the form (2.14) is also nonlinear.

On the other hand, if we consider the difference scheme (2.2)–(2.4) then we obtain a linear system for the unknowns $(z^{(r+1,m)}, u^{(r+1,m)})$. Note that our original problem (1.1), (1.2) is nonlinear.

3. Solutions of implicit Euler schemes. Let us denote by $(z_h,u_h)\colon E_h \to \mathbb{R}^k \times M_{k\times n}$ the solution of (2.2)–(2.4) and suppose that the functions (z_h,u_h) are calculated on the set $E_{r.h}$, $0 \le r < K$. Our implicit difference methods have the following property: we start with the nonlinear system (1.1), and the existence of a solution (z_h,u_h) on $E_{r+1.h}$ is equivalent to the existence of a solution of the system (2.2), (2.3), which is linear. The difference equations

$$z_i^{(r+1,m)} = h_0 \partial_q f_i(P^{(r,m)}[z,u_i]) \circ \delta z_i^{(r+1,m)}, \qquad 1 \le i \le k, \tag{3.1}$$

and

$$u_i^{(r+1,m)} = h_0 \partial_q f_i(P^{(r,m)}[z,u_i]) \star [\delta u_i^{(r+1,m)}]^T, \qquad 1 \le i \le k, \tag{3.2}$$

are the principal parts of (2.2) and (2.3) respectively. We prove a lemma on difference inequalities generated by (3.1), (3.2).

LEMMA 3.1. *Suppose that*

- *Assumption* $H_0[f]$ *is satisfied and condition* (1.3) *holds,*
- $h \in H$ *and* $z_h \in \mathcal{F}(E_{0.h} \cup E_h, \mathbb{R}^k)$, $u_h \in \mathcal{F}(E_{0.h} \cup E_h, M_{k\times n})$ *where* $z_h = (z_{h.1}, \ldots, z_{h.k})$, $u_h = [u_{h.ij}]_{i=1,\ldots,k, j=1,\ldots,n}$ *and* $u_{h.i} = (u_{h.i1}, \ldots, u_{h_in})$, $1 \le i \le k$.

(I) *If the functions* z_h *and* u_h *satisfy the implicit difference inequalities*

$$z_{h.i}^{(r+1,m)} \le h_0 \partial_q f_i(P^{(r,m)}[z_h, u_{h.i}]) \circ \delta z_{h.i}^{(r+1,m)}, \quad 1 \le i \le k, \tag{3.3}$$

and

$$u_{h.i}^{(r+1,m)} \le h_0 \partial_q f_i(P^{(r,m)}[z_h, u_{h.i}]) \star [\delta u_{h.i}^{(r+1,m)}]^T, \quad 1 \le i \le k,$$

then

$$z_{h.i}^{(r,m)} \le 0 \qquad \text{on } E_h \text{ for } 1 \le i \le k, \tag{3.4}$$

$$u_{h.i}^{(r,m)} \le \theta_{[n]} \qquad \text{on } E_h \text{ for } 1 \le i \le k. \tag{3.5}$$

(II) *If the implicit difference inequalities*

$$z_{h.i}^{(r+1,m)} \ge h_0 \partial_q f_i(P^{(r,m)}[z_h, u_{h.i}]) \circ \delta z_{h.i}^{(r+1,m)}, \quad 1 \le i \le k,$$

and

$$u_{h.i}^{(r+1,m)} \ge h_0 \partial_q f_i(P^{(r,m)}[z_h, u_{h.i}]) \star [\delta u_{h.i}^{(r+1,m)}]^T, \quad 1 \le i \le k,$$

are satisfied then $z_{h.i}^{(r,m)} \ge 0$ *and* $u_{h.i}^{(r,m)} \ge \theta_{[n]}$ *on* E_h *for* $1 \le i \le k$.

Proof. Consider case (I). Fix $0 \le r \le K-1$. Write

$$J_{i.+}^{(r,m)} = \{j \in \{1, \ldots, n\} : \partial_{q_j} f_i(P^{(r,m)}[z_h, u_{h.i}]) \ge 0\},$$
$$J_{i.-}^{(r,m)} = \{1, \ldots, n\} \backslash J_{i.+}^{(r,m)}$$

where $1 \le i \le k$. It follows from (3.3) that

$$\begin{aligned} z_{h.i}^{(r+1,m)} \Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f_i(P^{(r,m)}[z_h, u_{h.i}])|\Big] &\le h_0 \sum_{j \in J_{i.+}^{(r,m)}} \frac{1}{h_j} \partial_{q_j} f_i(P^{(r,m)}[z_h, u_{h.i}]) z_{h.i}^{(r+1,m+e_j)} \\ &\quad - h_0 \sum_{j \in J_{i.-}^{(r,m)}} \frac{1}{h_j} \partial_{q_j} f_i(P^{(r,m)}[z_h, u_{h.i}]) z_{h.i}^{(r+1,m-e_j)}. \end{aligned} \tag{3.6}$$

Suppose that $1 \le i \le k$ is fixed and $(t^{(r+1)}, z^{(\tilde{m})}) \in E_h$ is such that

$$z_{h.i}^{(r+1,\tilde{m})} \ge z_{h.i}^{(r+1,m)} \qquad \text{for } x^{(m)} \in [-b + Mt^{(r+1)}, b - Mt^{(r+1)}].$$

If the assertion (3.4) is false then

$$z_{h.i}^{(r+1,\tilde m)} > 0. \tag{3.7}$$

From (3.6) we deduce that

$$z_{h.i}^{(r+1,\tilde m)}\Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f_i(P^{(r,\tilde m)}[z_h, u_{h.i}])|\Big]$$
$$\le h_0 z_{h.i}^{(r+1,\tilde m)} \sum_{j=1}^{m} \frac{1}{h_j} |\partial_{q_j} f_i(P^{(r,\tilde m)}[z_h, u_{h.i}])|.$$

Hence $z_{h.i}^{(r+1,\tilde m)} \le 0$, which contradicts (3.7), and the proof of (3.4) is completed by induction. In a similar way we prove (3.5). Case (II) can be treated in the same way. This completes the proof.

Lemma 3.2. *If Assumption $H_0[f]$ and condition* (1.3) *are satisfied then there exists exactly one solution $(z_h, u_h)\colon E_{0.h} \cup E_h \to \mathbb{R}^k \times M_{k\times n}$ of* (2.2)–(2.4).

Proof. Fix $0 \le r \le K-1$ and suppose that the solution (z_h, u_h) is known on $E_{r.h}$. Consider the linear system

$$\begin{aligned} z_i^{(r+1,m)} &= z_{h.i}^{(r,m)} + h_0 f_i(P^{(r,m)}[z_h, u_{h.i}]) \\ &\quad + h_0 \partial_q f_i(P^{(r,m)}[z_h, u_{h.i}]) \circ (\delta z_i^{(r+1,m)} - u_{h.i}^{(r+1,m)}), \end{aligned} \tag{3.8}$$

$$\begin{aligned} u_i^{(r+1,m)} &= u_{h.i}^{(r,m)} + h_0 \partial_x f_i(P^{(r,m)}[z_h, u_{h.i}]) \\ &\quad + h_0 \partial_w f_i(P^{(r,m)}[z_h, u_{h.i}]) \star (T_h u_{h.i})_{(r,m)} \\ &\quad + h_0 \partial_q f_i(P^{(r,m)}[z_h, u_{h.i}]) \star [\delta u_i^{(r+1,m)}]^T \end{aligned} \tag{3.9}$$

with unknowns $(z_i^{(r+1,m)}, u_i^{(r+1,m)})$ where $x^{(m)} \in [-b+Mt^{(r+1)}, b-Mt^{(r+1)}]$. Suppose that $(t^{(r)}, x^m) \in E'_h$. It follows from condition 3) of Assumption $H_0[f]$ that the expressions $\delta z_i^{(r+1,m)}, \delta u_i^{(r+1,m)}$, $1 \le i \le k$, are well defined. We conclude from Lemma 3.1 that the homogeneous system corresponding to (3.8), (3.9) has exactly one zero solution. Then system (3.8), (3.9) has exactly one solution $(z_{h.i}^{(r+1,m)}, u_{h.i}^{(r+1,m)})$ where $x^{(m)} \in [-b + Mt^{(r+1)}, b - Mt^{(r+1)}]$, and consequently, the functions $(z_{h.i}, u_{h.i})$, $1 \le i \le k$, are defined and they are unique on $E_{r+1.h}$. Since (z_h, u_h) are given on $E_{0.h}$, the proof is completed by induction.

Now we give estimates of solutions to (1.1), (1.2) and (2.2)–(2.4). Write

$$\|\partial_w f(P)\|_* = \max\Big\{\sum_{j=1}^{k} \|\partial_{w_j} f_i(P)\|_* : 1 \le i \le k\Big\}, \qquad P \in \Omega.$$

ASSUMPTION $H_*[f,\varphi]$. The function $f\colon \Omega \to \mathbb{R}^k$ satisfies Assumption $H_0[f]$ and

1) there is $\tilde{A} > 0$ such that $\|\partial_x f(P)\| \le \tilde{A}$ and $\|\partial_w f(P)\|_* \le \tilde{A}$ for $P = (t,x,w,q) \in \Omega$,
2) for $P \in \Omega$ we have $(|\partial_{q_1} f_i(P)|, \ldots, |\partial q_n f_i(P)|) \le M$, $1 \le i \le k$,
3) $\varphi\colon E_0 \to \mathbb{R}^k$ is of class C^1 and
$$c_0 \ge \max\{\|\varphi(t,x)\|_\infty : (t,x) \in E_0\},$$
$$c_1 \ge \max\{\|\partial_x \varphi(t,x)\| : (t,x) \in E_0\}$$
and
$$c_0 \ge \|\varphi_h^{(r,m)}\|_\infty, \qquad c_1 \ge \|\psi_h^{(r,m)}\| \quad \text{for } (t^{(r)}, x^{(m)}) \in E_{0.h},\ h \in H.$$

LEMMA 3.3. *Suppose that Assumption $H_*[f,\varphi]$ is satisfied and $v = (v_1,\ldots,v_k)\colon E_0 \cup E \to \mathbb{R}^k$ is a solution to* (1.1), (1.2) *and v is of class C^* on $E_0 \cup E$. Then*
$$\|v(t,x)\|_\infty \le \bar{\alpha}_0(t), \qquad \|\partial_x v(t,x)\| \le \bar{\alpha}_1(t) \quad \textit{for } (t,x) \in E, \tag{3.10}$$
where
$$\bar{\alpha}_0(t) = c_0 e^{\tilde{A}t} + \frac{\tilde{c}}{\tilde{A}}(e^{\tilde{A}t} - 1) + 2\|M\| c_1 t e^{\tilde{A}t} + \frac{2\|M\|}{\tilde{A}}[1 + \tilde{A}te^{\tilde{A}t} - e^{\tilde{A}t}], \tag{3.11}$$
$$\bar{\alpha}_1(t) = (c_1 + 1)e^{\tilde{A}t} - 1 \tag{3.12}$$
and
$$\tilde{c} = \max\{\|f(t,x,\mathbf{O},\theta_{[n]})\|_\infty : (t,x) \in E\}$$
where $\mathbf{O} \in C(B,\mathbb{R}^k)$ is given by $\mathbf{O}(\tau,y) = 0$ for $(\tau,y) \in B$.

Proof. Let us denote by $g_i[z,u](\cdot,t,x)$ the ith bicharacteristic of (1.1) corresponding to (z,u). It follows that the functions $(v, \partial_x v_i)$, $1 \le i \le k$, satisfy the integral functional system
$$z_i(t,x) = \varphi(0, g_i[z,u](0,t,x)) + \int_0^t f_i(P_i[z,u](\tau,t,x))\,d\tau - \int_0^t \partial_q f_i(P_i[z,u](\tau,t,x)) \circ u_i(\tau, g_i[z,u](\tau,t,x))\,d\tau,$$
$$u_i(t,x) = \partial_x \varphi_i(0, g_i[z,u](0,t,x)) + \int_0^t \partial_x f_i(P_i[z,u](\tau,t,x))\,d\tau + \int_0^t \partial_w f_i(P_i[z,u](\tau,t,x)) \star u_{(\tau, g_i[z,u](\tau,t,x))}\,d\tau, \qquad 1 \le i \le k,$$

and the initial conditions $z(t,x) = \varphi(t,x)$, $u(t,x) = \partial_x\varphi(t,x)$ on E_0 where $P_i[z,u](\tau,t,x) = (\tau, g_i[z,u](\tau,t,x), z_{(\tau,g_i[z,u](\tau,t,x))}, u_i(\tau, g_i[z,u](\tau,t,x)))$, and $1 \le i \le k$. Write

$$\tilde\alpha_0(t) = \max\{\|v(\tau,y)\|_\infty : (\tau,y) \in E_0 \cup E,\ \tau \le t\},$$
$$\tilde\alpha_1(t) = \max\{\|\partial_x v(\tau,y)\| : (\tau,y) \in E_0 \cup E,\ \tau \le t\},$$

where $t \in [0,a)$. It follows that the functions $(\tilde\alpha_0, \tilde\alpha_1)$ satisfy the integral inequalities

$$\tilde\alpha_0(t) \le c_0 + \tilde c t + \int_0^t [\tilde A \tilde\alpha_0(\tau) + 2\|M\|\tilde\alpha_1(\tau)]\,d\tau,$$

$$\tilde\alpha_1(t) \le c_1 + \tilde A t + \tilde A \int_0^t \tilde\alpha_1(\tau)\,d\tau.$$

Hence the functions $(\tilde\alpha_0, \tilde\alpha_1)$ satisfy the equations corresponding to the above integral inequalities. Thus the assertion (3.10) follows.

Assumption $H[T_h]$. The function $T_h\colon \mathcal{F}(E_{0.h} \cup E_h, \mathbb{R}^k) \to C(E_0 \cup E, \mathbb{R}^k)$ is such that

1) for $w, \tilde w\colon E_{0.h} \cup E_h \to \mathbb{R}^k$ we have $\|T_h w - T_h \tilde w\|_{t^{(r)}} \le \|w - \tilde w\|_{r.h}$, $0 \le r \le K$,
2) for each $w\colon E_0 \cup E \to \mathbb{R}^k$ of class C^1 there is $\gamma_*\colon H \to \mathbb{R}_+$ such that

$$\|w - T_h w_h\|_t \le \gamma_*(h), \qquad 0 \le t \le a, \qquad \lim_{h\to 0}\gamma_*(h) = 0$$

where $w_h = w|_{E_{h.0}\cup E_h}$,
3) if $\mathbf{O}_h \in \mathcal{F}(E_{0.h} \cup E_h, \mathbb{R})$ is given by $\mathbf{O}_h(t,x) = 0$ for $(t,x) \in E_{0.h} \cup E_h$ then $(T_h\mathbf{O}_h)(t,x) = 0$ for $(t,x) \in E_0 \cup E$.

Remark 3.1. The above condition 1) states that T_h satisfies the Lipschitz condition with constant $L = 1$. The meaning of condition 2) is that $T_h w_h$ is an approximation of w and the approximation error is estimated by $\gamma_*(h)$. An example of T_h which satisfies the above conditions can be found in [12, Chapter 3].

Lemma 3.4. *Suppose that Assumption $H_*[f,\varphi]$ and condition* (1.3) *are satisfied and $(z_h, u_h)\colon E_{0.h} \cup E_h \to \mathbb{R}^k \times M_{k\times n}$ is a solution to* (2.2)-(2.4) *with δ_0 and δ defined by* (2.5)–(2.9). *Then*

$$\|z_h^{(r,m)}\|_\infty \le \bar\alpha_0^{(r)}, \qquad \|u_h^{(r,m)}\| \le \bar\alpha_1^{(r)} \qquad \text{on } E_h \tag{3.13}$$

where $\bar\alpha_0$, $\bar\alpha_1$ are given by (3.11), (3.12).

Proof. Write

$$\alpha_{h.0}^{(r)} = \max\{\|z_h^{(s,m)}\|_\infty : (t^{(s)}, x^{(m)}) \in E_{r.h}\},$$
$$\alpha_{h.1}^{(r)} = \max\{\|u_h^{(s,m)}\| : (t^{(s)}, x^{(m)}) \in E_{r.h}\}$$

where $0 \le r \le K$. Easy computations show that the functions $(\alpha_{h.0}, \alpha_{h.1})$ satisfy the difference inequalities

$$\frac{1}{h_0}[\alpha_{h.0}^{(r+1)} - \alpha_{h.0}^{(r)}] \le \tilde{c} + \tilde{A}\alpha_{h.0}^{(r)} + 2\|M\|\alpha_{h.1}^{(r)},$$
$$\frac{1}{h_0}[\alpha_{h.1}^{(r+1)} - \alpha_{h.1}^{(r)}] \le \tilde{A}[1 + \alpha_{h.1}^{(r)}], \qquad 0 \le r \le K-1.$$

The functions $(\bar{\alpha}_0, \bar{\alpha}_1)$ satisfy the differential equations

$$y_0'(t) = \tilde{c} + \tilde{A}y_0(t) + 2\|M\|y_1(t), \qquad y_1'(t) = \tilde{A} + \tilde{A}y_1(t).$$

and $\bar{\alpha}_0^{(0)} = \bar{\alpha}_{h.0}^{(0)}$, $\bar{\alpha}_1^{(0)} = \bar{\alpha}_{h.1}^{(0)}$. Thus the assertion (3.13) follows.

4. Convergence of the generalized implicit Euler method. Suppose that Assumption $H_*[f,\varphi]$ is satisfied. Write $d = \bar{\alpha}_0(a)$, $\tilde{d} = \bar{\alpha}_1(a)$ where $(\bar{\alpha}_0, \bar{\alpha}_1)$ are given by (3.11), (3.12) and $\Omega[d,\tilde{d}] = E \times K_{C(B,\mathbb{R}^k)}[d] \times K_{\mathbb{R}^n}[\tilde{d}]$ where

$$K_{C(B,\mathbb{R}^k)}[d] = \{w \in C(B,\mathbb{R}^k) : \|w\|_B \le d\}, \qquad K_{\mathbb{R}^n}[\tilde{d}] = \{q \in \mathbb{R}^n : \|q\| \le \tilde{d}\}.$$

We will need the following assumptions on f.

Assumption $H[f,\varphi]$. The functions $f\colon \Omega \to \mathbb{R}^k$ and $\varphi\colon E_0 \to \mathbb{R}^k$ satisfy Assumption $H_*[f,\varphi]$ and

1) there is $\sigma \in C([0,a] \times \mathbb{R}_+, \mathbb{R}_+)$ such that

- σ is continuous and nondecreasing with respect to both variables,
- $\sigma(t,0) = 0$ for $t \in [0,a]$ and for each $c \ge 1$ the maximal solution of the Cauchy problem

$$\eta'(t) = c[\eta(t) + \sigma(t,\eta(t))], \qquad \eta(0) = 0,$$

is $\bar{\eta}(t) = 0$ for $t \in [0,a]$,

2) the terms

$$\|\partial_x f_i(t,x,w,q) - \partial_x f_i(t,x,\bar{w},\bar{q})\|, \ \|\partial_q f_i(t,x,w,q) - \partial_q f_i(t,x,\bar{w},\bar{q})\|,$$
$$\|\partial_w f_i(t,x,w,q) - \partial_w f_i(t,x,\bar{w},\bar{q})\|$$

are estimated by $\sigma(t, \|w-\bar{w}\|_B + \|q-\bar{q}\|)$ for $1 \le i \le k$ on $\Omega[d,\tilde{d}]$.

Remark 4.1. Theorems on difference methods presented in [4], [13] have the following property: it is assumed that the functions $\partial_x f$, $\partial_q f$, $\partial_w f$ satisfy nonlinear conditions of the Perron type with respect to (w,q) and the estimates are global with respect to (w,q). Note that we have assumed estimates of the Perron type with respect to (w,q), local with respect to (w,q). It is clear that there are differential systems with deviated variables and integral differential systems such that condition 2) holds and global estimates of the Perron type with respect to (w,q) are not satisfied.

Now we formulate the main result of the paper.

THEOREM 4.1. *Suppose that Assumptions $H[f,\varphi]$ and $H[T_h]$ are satisfied and*

1) $\varphi\in C(E_0,\mathbb{R}^k)$ *and* $\varphi(t,\cdot)\colon[-b,b]\to\mathbb{R}^k$ *is of class* C^2 *for every* $t\in[-b_0,0]$ *and* $\varphi(\cdot,x)\colon[-b_0,0]\to\mathbb{R}^k$ *is of class* C^1 *for every* $x\in[-b,b]$,
2) $v\colon E_0\cup E\to\mathbb{R}^k$ *is the solution to* (1.1), (1.2) *and* v *is of class* C^* *on* $E_0\cup E$,
3) $(z_h,u_h)\colon E_{0.h}\cup E_h\to\mathbb{R}^k\times M_{k\times n}$ *is the solution to* (2.2)–(2.4) *with* δ_0, δ *defined by* (2.5)–(2.9) *and there is* $\alpha_0\colon H\to\mathbb{R}_+$ *such that*

$$\|\varphi^{(r,m)}-\varphi_h^{(r,m)}\|+\|\partial_x\varphi^{(r,m)}-\psi_h^{(r,m)}\|\le\alpha_0(h)\quad\text{on } E_{0.h}\tag{4.1}$$

and $\lim_{h\to0}\alpha_0(h)=0$.

Then there exists $\alpha\colon H\to\mathbb{R}_+$ *such that*

$$\|v_h-z_h\|_{r.h}+\|\partial_x v_h-u_h\|_{r.h}\le\alpha_0(h)\quad\text{for } 0\le r\le K,\tag{4.2}$$

and $\lim_{h\to0}\alpha(h)=0$ *where* v_h *and* $\partial_x v_h$ *are the restrictions of* v *and* $\partial_x v$, *respectively, to the set* $E_{0.h}\cup E_h$.

Proof. Write $\chi=\partial_x v$, $\chi=[\chi_{ij}]_{i=1,\dots,k,j=1,\dots,n}$, $\chi_i=(\chi_{i1},\dots,\chi_{in})$, $1\le i\le k$, and

$$\chi_h=\chi|_{E_{0.h}\cup E_h},$$
$$\chi_h=[\chi_{h.ij}]_{i=1,\dots,k,j=1,\dots,n},\ \chi_{h.i}=(\chi_{h.i1},\dots,\chi_{h.in}),\quad 1\le i\le k.$$

Then (v,χ) satisfy the quasilinear system (2.10), (2.11) and the initial condition (2.12). Consider the functions $\xi_h\colon E_{0.h}\cup E_h\to\mathbb{R}^k$ and $\lambda_h\colon E_{0.h}\cup E_h\to M_{k\times n}$ defined by $\xi_h=v_h-z_h$, $\xi_h=(\xi_{h.1},\dots,\xi_{h.k})$, $\lambda_h=\chi_h-u_h$, $\lambda_h=[\lambda_{h.ij}]_{i=1,\dots,k,\,j=1,\dots,n}$ and $\lambda_{h.i}=(\lambda_{h.i1},\dots,\lambda_{h.in})$, $1\le i\le k$. Let $\omega_{h.0},\omega_{h.1}\colon I_h\to\mathbb{R}_+$ be given by

$$\omega_{h.0}^{(r)}=\max\{\|\xi_h^{(i,m)}\|_\infty:(t^{(i)},x^{(m)})\in E_{r.h}\},$$
$$\omega_{h.1}^{(r)}=\max\{\|\lambda_h^{(i,m)}\|_\infty:(t^{(i)},x^{(m)})\in E_{r.h}\}$$

and $\omega_h=\omega_{h.0}+\omega_{h.1}$. We will write a difference inequality for ω_h. We first examine $\omega_{h.0}$. Set $Q^{(r,m)}[v,\chi_i]=(t^{(r)},x^{(m)},v_{(t^r),x^{(m)}},\chi_i^{(r,m)})$, $1\le i\le k$. Let the functions $\Gamma_h,\Lambda_h\colon E'_h\to\mathbb{R}^k$, $\Gamma_h=(\Gamma_{h.1},\dots,\Gamma_{h.k})$, $\Lambda_h=(\Lambda_{h.1},\dots,\Lambda_{h.k})$, be defined by

$$\Gamma_{h.i}^{(r,m)}=\delta_0 v_{h.i}^{(r,m)}-\partial_t v_i^{(r,m)}+\partial_q f_i(Q^{(r,m)}[v,\chi_i])\circ[\partial_x v_i^{(r,m)}-\delta v_{h.i}^{(r+1,m)}]\tag{4.3}$$

and

$$\begin{aligned}(4.4)\quad \Lambda_{h.i}^{(r,m)} &= f_i(Q^{(r,m)}[v,\chi_i]) - f_i(P^{(r,m)}[z_h,u_{h.i})\\ &\quad - \partial_q f_i(Q^{(r,m)}[v,\chi_i]) \circ \chi_i^{(r,m)} + \partial_q f_i(P^{(r,m)}[z_h,u_{h.i}]) \circ u_{h.i}^{(r,m)}\\ &\quad + [\partial_q f_i(Q^{(r,m)}[v,\chi_i]) - \partial_q f_i(P^{(r,m)}[z_h,u_h])] \circ \delta v_{h.i}^{(r+1,m)}\end{aligned}$$

where $1 \le i \le k$. It follows from (2.2), (2.10) that ξ_h satisfies the difference equations

$$(4.5)\quad \delta_0 \xi_{h.i}^{(r,m)} = \partial_q f_i(P^{(r,m)}[z_h,u_{h.i}]) \circ \delta \xi_{h.i}^{(r+1,m)} + \Lambda_{h.i}^{(r,m)} + \Gamma_{h.i}^{(r,m)},\ 1 \le i \le k.$$

We conclude from (2.5)–(2.6) that the above relations are equivalent to

$$\begin{aligned}(4.6)\quad \xi_{h.i}^{(r+1,m)} &\Big[1 + h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])|\Big]\\ &= \xi_{h.i}^{(r,m)} + h_0 \sum_{j\in J_{i.+}^{(r,m)}} \frac{1}{h_j} \partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\, \xi_{h.i}^{(r+1,m+e_j)}\\ &\quad - h_0 \sum_{j\in J_{i.-}^{(r,m)}} \frac{1}{h_j} \partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\, \xi_{h.i}^{(r+1,m-e_j)}\\ &\quad + h_0[\Gamma_{h.i}^{(r,m)} + \Lambda_{h.i}^{(r,m)}], \qquad 1 \le i \le k.\end{aligned}$$

It follows easily that there is $\gamma_0\colon H \to \mathbb{R}_+$ such that

$$(4.7)\qquad \|\Gamma_h^{(r,m)}\| \le \gamma_0(h) \quad \text{on } E_h', \qquad \lim_{h\to 0} \gamma_0(h) = 0.$$

We conclude from Lemma 3.3, Lemma 3.4 and Assumption $H[T_h]$ that

$$\|v_{(t^{(r)},x^{(m)})}\|_{D[t^{(r)},x^{(m)}]} \le d, \quad \|T_h(z_h)_{(r,m)}\|_{D[t^{(r)},x^{(m)}]} \le d,$$
$$\|\chi_i^{(r,m)}\| \le \tilde{d}, \quad \|u_{h.i}^{(r,m)}\| \le \tilde{d}$$

where $(t^{(r)},x^{(m)}) \in E_h'$. Let $\tilde{a} = \max\{d,\tilde{d}\}$. Write $A = \max\{\tilde{A},\|M\|\}$. According to Assumption $H[f,\varphi]$ we have

$$(4.8)\quad \|f_i(Q^{(r,m)}[v,\chi_i]) - f_i(P^{(r,m)}[z_h,u_{h.i}])\| \le A\omega_h^{(r)} \text{ on } E_h' \text{ for } 1 \le i \le k,$$

and

$$(4.9)\qquad \|\partial_q f_i(Q^{(r,m)}[v,\chi_i]) - \partial_q f_i(P^{(r,m)}[z_h,u_{h.i}])\| \le \sigma(t^{(r)},\omega_h^{(r)})$$

on E_h' for $1 \le i \le k$. Then we have

$$(4.10)\qquad |\Lambda_{h.i}^{(r,m)}| \le 2A\omega_h^{(r)} + 2\tilde{a}\sigma(t^{(r)},\omega_h^{(r)}).$$

We see at once that

$$(4.11)\qquad \sum_{j\in J_i^+[r,m]} \frac{1}{h_j}\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\,|\xi_{h.i}^{(r+1,m+e_j)}| - \sum_{j\in J_i^-[r,m]} \frac{1}{h_j}\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\,|\xi_{h.i}^{(r+1,m-e_j)}| \le \omega_{h.0}^{(r+1)} \sum_{j=1}^{n} \frac{1}{h_j}|\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])|.$$

From (4.6), (4.8), (4.10), (4.11) we get

$$(4.12)\qquad \omega_{h.0}^{(r+1)} \le \omega_{h.0}^{(r)} + 2h_0 A\omega_h^{(r)} + 2h_0\tilde{a}\sigma(t^{(r)},\omega_h^{(r)}) + h_0\gamma_0(h)$$

for $0 \le r \le K-1$. Now we write a difference inequality for $\omega_{h.1}$. Let the functions $U_{h.i}, Z_{h.i}\colon E'_h \to \mathbb{R}^n$, $1 \le i \le k$, be defined by

$$(4.13)\qquad U_{h.i}^{(r,m)} = \delta_0\chi_{h.i}^{(r,m)} - \partial_t\chi_i^{(r,m)} + \partial_q f_i(Q^{(r,m)}[v,\chi_i]) \star [\partial_x\chi_i^{(r,m)} - \delta\chi_{h.i}^{(r+1,m)}]^T$$

and

$$(4.14)\qquad Z_{h.i}^{(r,m)} = \partial_x f_i(Q^{(r,m)}[v,\chi_i]) - \partial_x f_i(P^{(r,m)}[z_h,u_{h.i}]) + \partial_w f_i(Q^{(r,m)}[v,\chi_i]) \star (\chi_i)_{(r,m)} - \partial_w f_i(P^{(r,m)}[z_h,u_{h.i}]) \star (T_h u_{h.i})_{(r,m)} + [\partial_q f_i(Q^{(r,m)}[v,\chi_i]) - \partial_q f_i(P^{(r,m)}[z_h,u_{h.i}])] \star [\delta\chi_{h.i}^{(r+1,m)}]^T.$$

Then the functions $\lambda_{h.i}$, $1 \le i \le k$, satisfy the difference equations

$$\delta_0\lambda_{h.i}^{(r,m)} = \partial_q f_i(P^{(r,m)}[z_h,u_{h.i}]) \star [\delta\lambda_{h.i}^{(r+1,m)}]^T + U_{h.i}^{(r,m)} + Z_{h.i}^{(r,m)}.$$

It follows easily that the above relations are equivalent to

$$(4.15)\qquad \lambda_{h.i}^{(r+1,m)}\Big[1 + h_0\sum_{j=1}^{n}\frac{1}{h_j}|\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])|\Big] = \lambda_{h.i}^{(r,m)} + h_0 \sum_{j\in J_{i.+}^{(r,m)}} \frac{1}{h_j}\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\,\lambda_{h.i}^{(r+1,m+e_j)} - h_0 \sum_{j\in J_{i.-}^{(r,m)}} \frac{1}{h_j}\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\,\lambda_{h.i}^{(r+1,m-e_j)} + h_0[U_{h.i}^{(r,m)} + Z_{h.i}^{(r,m)}].$$

It is clear that there is $\gamma\colon H \to \mathbb{R}_+$ such that

$$(4.16)\qquad \|U_{h.i}^{(r,m)}\| \le \gamma(h) \quad \text{on } E'_h \text{ for } 1 \le i \le k \quad \text{and} \quad \lim_{h\to 0}\gamma(h) = 0.$$

The estimates analogous to (4.9) can be obtained for $\partial_x f_i$ and $\partial_w f_i$. It follows from (4.14) and Assumptions $H[f,\varphi]$, $H[T_h]$ that

$$\|Z_h^{(r,m)}\| \le (1+2\tilde{a})\sigma(t^{(r)},\omega_h^{(r)}) + A\omega_h^{(r)}. \tag{4.17}$$

It is easy to see that

$$\begin{aligned}\sum_{j\in J_i^+[r,m]} \frac{1}{h_j}\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\|\lambda_{h.i}^{(r+1,m+e_j)}\| \\ - \sum_{j\in J_i^-[r,m]} \frac{1}{h_j}\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])\|\lambda_{h.i}^{(r+1,m-e_j)}\| \\ \le \omega_{1.h}^{(r+1)} \sum_{j=1}^n \frac{1}{h_j}|\partial_{q_j} f_i(P^{(r,m)}[z_h,u_{h.i}])|.\end{aligned} \tag{4.18}$$

We deduce from (4.15), (4.17), (4.18) that

$$\omega_{h.1}^{(r+1)} \le \omega_{h.1}^{(r)} + h_0 A\omega_h^{(r)} + h_0(1+2\tilde{a})\sigma(t^{(r)},\omega_h^{(r)}) + h_0\gamma(h) \tag{4.19}$$

for $0\le r\le K-1$. Adding inequalities (4.12), (4.19) we get

$$\omega_h^{(r+1)} \le \omega_h^{(r)} + h_0(1+4\tilde{a})\sigma(t^{(r)},\omega_h^{(r)}) + 3h_0A\omega_h^{(r)} + h_0(\gamma_0(h)+\gamma(h)) \tag{4.20}$$

for $0\le r\le K-1$. Consider the Cauchy problem

$$\omega'(t) = 3A\omega(t) + (1+4\tilde{a})\sigma(t,\omega(t)) + (\gamma_0(h)+\gamma(h)), \quad \omega(0)=\alpha_0(h). \tag{4.21}$$

From Assumption $H[f,\varphi]$ we know that there is $\epsilon_0>0$ such that for $\|h\|<\epsilon_0$ there exists a maximal solution $\eta_h\colon [0,a]\to\mathbb{R}_+$ of (4.21) and $\lim_{h\to 0}\eta_h(t)=0$ uniformly on $[0,a]$. The following recurrent inequality is satisfied:

$$\eta_h^{(r+1)} \ge \eta_h^{(r)} + 3h_0A\eta_h^{(r)} + h_0(1+4\tilde{a})\sigma(t^{(r)},\eta_h^{(r)}) + h_0(\gamma_0(h)+\gamma(h)), \tag{4.22}$$

where $0\le r\le K-1$. Since $\omega_h^{(r)}\le\eta_h^{(r)}$, from the above inequality and (4.20) we see that $\omega_h^{(r)}\le\eta_h^{(r)}$ for $0\le r\le K$. We obtain the estimate (4.2) for $\alpha(h)=\eta_h(a)$. This completes the proof.

Remark 4.2. Note that the (CFL) conditions are not assumed in Theorem 4.1.

5. Comments and examples. Suppose that all the assumptions of Theorem 4.1 are satisfied with $\sigma(t,p)=Lp$ on $[0,a]\times\mathbb{R}_+$ where $L>0$. Then

$$\|v_h - z_h\|_{r.h} + \|\partial_x v_h - u_h\|_{r.h} \le \tilde{\alpha}(h), \quad 0\le r\le K, \tag{5.1}$$

where

$$\tilde{\alpha}(h) = \alpha_0(h)e^{\tilde{L}t} + \frac{1}{\tilde{L}}(\gamma_0(h)+\gamma(h))(e^{\tilde{L}t}-1)$$

and $\tilde{L} = 3A + L(1 + 4\tilde{a})$. Inequality (5.1) is obtained by solving the Cauchy problem (4.21) with $\sigma(t, p) = Lp$.

LEMMA 5.1. *Suppose that*

1) *Assumption* $H[f, \varphi]$ *is satisfied with* $\sigma(t, p) = Lp$ *on* $[0, a] \times \mathbb{R}_+$ *where* $L > 0$,
2) *the operator* $T_h \colon \mathcal{F}(E_{0.h} \cup E_h, \mathbb{R}^k) \to C(E_0 \cup E, \mathbb{R})$ *is such that*
   - *for* $w, \tilde{w} \in \mathcal{F}(E_{0.h} \cup E_h, \mathbb{R}^k)$ *we have* $\|T_h w - T_h \tilde{w}\|_{t^{(r)}} \le \|w - \tilde{w}\|_{r.h}$, $0 \le r \le K$,
   - *for each* $w \colon E_0 \cup E \to \mathbb{R}^k$ *of class* C^2 *there is* $C_* \in \mathbb{R}_+$ *such that* $\|w - T_h w_h\|_t \le C_* \|h\|$, $0 \le t \le a$, *where* $w_h = w|_{E_{0.h} \cup E_h}$,
3) $\varphi \colon E_0 \to \mathbb{R}^k$ *is of class* C^3 *and* $v \colon E_0 \cup E \to \mathbb{R}^k$ *is the solution of* (1.1), (1.2) *and* v *is of class* C^3 *on* $E_0 \cup E$,
4) *condition* 2) *of Theorem* 4.1 *is satisfied.*

Then there is $\tilde{C} \in \mathbb{R}_+$ *such that*

$$\|v_h - z_h\|_{r.h} + \|\partial_x v_h - u_h\|_{r.h} \le C_0 \alpha_0(h) + \tilde{C}\|h\|, \qquad 0 \le r \le K, \tag{5.2}$$

where $C_0 = \exp[(3A + L(1 + 4\tilde{a}))a]$.

Proof. There are $c_0, c \in \mathbb{R}_+$ such that conditions (4.7) and (4.16) are satisfied with $\gamma_0(h) = c_0\|h\|$ and $\gamma(h) = c\|h\|$ respectively. Then we obtain (5.2) from (5.1).

REMARK 5.1. The interpolating operator T_h considered in [12, Chapter 3] satisfies condition 2) of Lemma 5.1.

Two models of functional dependence in partial differential equations are used in the literature. There are papers concerning the nonlinear equation

$$\partial_t z(t, x) = G(t, x, (Vz)(t, x), \partial_x z(t, x)) \tag{5.3}$$

with the initial conditions

$$z(t, x) = \varphi(t, x) \quad \text{on } E_0 \tag{5.4}$$

where V is a Volterra type operator and $\varphi \colon E_0 \to \mathbb{R}$, $G \colon E \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. The main assumptions in existence theorems for (5.3), (5.4) concern the operator V. They are formulated in terms of inequalities for norms in some function spaces. A new model of functional dependence for initial value problems with solutions defined on the Haar pyramid is proposed in [21], [22]. Our results on implicit difference methods are based on that idea.

The paper [13] concerns implicit difference methods for (5.3), (5.4). The following property of the operator Vz is important in [13]: the interpolating operator $V_h z$ corresponding to V satisfies the Lipschitz condition with respect to z. Let us consider problem (5.3), (5.4) with

$$(Vz)(t, x) = z(t - \tau, y) \cdot z(t - \tilde{\tau}, y) \tag{5.5}$$

where $\tau, \tilde{\tau} \in [0, b_0]$. The results of [13] are not applicable to (5.3), (5.4) with V defined by (5.5). The interpolating operators $V_h z$ corresponding to (5.5) do not satisfy the Lipschitz condition. Note that Theorem 4.1 can be applied to the above Cauchy problem.

Let us consider problem (5.3), (5.4) with

$$(Vz)(t,x) = \int_{D[t,x]} z^2(\tau, y)\, dy\, d\tau. \tag{5.6}$$

The interpolating operators $V_h z$ corresponding to (5.6) do not satisfy the Lipschitz condition. Thus the results of [13] are not applicable to (5.3), (5.4) with V given by (5.6). Note that Theorem 4.1 can be applied to the above Cauchy problem.

Now we present numerical examples. We apply the results of Section 4 to a differential equation with deviated variables and to a differential integral problem. For $n = 2$ we put $E_0 = \{0\} \times [-b, b] \times [-b, b]$, $M > 0$ and

$$E = \{(t, x, y) \in \mathbb{R}^3 : 0 \le t < a,\ x, y \in [-b + Mt, b - Mt]\}. \tag{5.7}$$

Initial value problems considered here have solutions on E. Let us write $h = (h_0, h_1, h_2)$ and assume $h_1 = h_2$. Let $N_r \in \mathbb{N}$ be defined by $x^{(m_1)}, y^{(m_2)} \in I[r]$ for $m_1, m_2 = -N_r, -N_r+1, \ldots, N_r-1, N_r$, where $I[r] = [-1+2t^{(r)}, 1-2t^{(r)}]$ and $x^{(N_r+1)}, y^{(N_r+1)} \notin I[r]$.

Example 5.1. Consider the differential equation with deviated variables

$$\begin{aligned} &\partial_t z(t,x,y) \\ &= \cos(x\partial_x z(t,x,y) - y\partial_y z(t,x,y)) - x\partial_x z(t,x,y) - y\partial_y z(t,x,y) \\ &\quad + z(t, 0.5(x+y), 0.5(x-y)) + xy(1+2t)z(t,x,y) - \exp(0.25t(x^2-y^2)) \end{aligned} \tag{5.8}$$

with the initial condition

$$z(0,x,y) = 1 \quad \text{on } E_0. \tag{5.9}$$

We put $a = 0.4$, $b = 1$, $M = 2$ in (5.7). The solution of (5.8), (5.9) is known, it is $v(t,x,y) = e^{txy}$. Let us denote by $z_h\colon E_h \to \mathbb{R}$ the solution of the implicit difference problem corresponding to (5.8), (5.9). Write

$$\epsilon_h^{(r)} = \frac{1}{(2N_r+1)^2} \sum_{m_1=-N_r}^{N_r} \sum_{m_2=-N_r}^{N_r} |v_h^{(r,m)} - z_h^{(r,m)}|. \tag{5.10}$$

The number $\epsilon_h^{(r)}$ is the arithmetical mean of the error with fixed $t^{(r)}$. We give experimental values of the above defined errors for $h_0 = 0.01$, $h_1 = h_2 = 0.001$.

$t^{(r)}$	0.15	0.20	0.25	0.30	0.35	0.40
$\epsilon_h^{(r)}$	0.000785	0.000941	0.000975	0.000929	0.000918	0.001150

The results in the table are consistent with our theoretical analysis. We have solved problem (5.8), (5.9) by using an explicit Lax scheme with the same steps of the mesh. In the case in question the (CFL) condition is not satisfied and the errors are greater than 100 for $r \geq 10$.

EXAMPLE 5.2. Consider the differential integral equation

$$\begin{aligned}
(5.11)\qquad & \partial_t z(t,x,y) \\
= & -0.25x\partial_x z(t,x,y) + 0.25x\cos\Big(\partial_x z(t,x,y) + \int_0^x z(t,s,y)\,ds\Big) \\
& -0.25y\partial_y z(t,x,y) - 0.25y\sin\Big(\partial_y z(t,x,y) + \int_0^y z(t,x,s)\,ds\Big) + g(t,x,y)
\end{aligned}$$

where $g(t,x,y) = \cos t\cos x\cos y - x\sin t\sin x\cos y - y\sin t\cos x\sin y - 0.25x$, with the initial condition

$$z(0,x,y) = 0 \quad \text{on } E_0. \tag{5.12}$$

We put $a = 1.5$, $b = 1$, $M = 0.5$ in (5.7). The solution to this problem is $v(t,x,y) = \sin t\cos x\cos y$. Let us denote by $z_h\colon E_h \to \mathbb{R}$ the solution of the implicit difference problem corresponding to (5.11), (5.12). Let $\epsilon_h^{(r)}$ be defined by (5.10). In the table we give experimental values of the above defined errors for $h_0 = 0.02$, $h_1 = h_2 = 0.002$.

$t^{(r)}$	0.54	0.60	0.64	0.70	0.74	0.80
$\epsilon_h^{(r)}$	0.006636	0.007498	0.008044	0.008811	0.009283	0.009927

The results in the table are consistent with our theoretical analysis. We have solved problem (5.8), (5.9) by using an explicit Lax scheme with the same steps of the mesh. In this case the (CFL) condition is not satisfied and the errors are greater than 100 for $r \geq 40$.

Note that we have a little better results for the differential equation with deviated variables than for the differential integral problem. This is due to the fact that we use interpolating values $V_h z_h$ at the points $0.5(x^{(m_1)}+y^{(m_2)})$, $0.5(x^{(m_1)}-y^{(m_2)})$ in the first example and we use interpolating values $V_h z_h$ on the intervals $[0,x^{(m_1)}]$, $[0,y^{(m_2)}]$ in the second example.

The above examples show that there are implicit difference methods for functional differential equations which are convergent and the corresponding explicit difference schemes are not convergent. Difference schemes obtained by the discretization (1.1), (1.2) have the following property: a large number of previous values $z_h^{(r,m)}$, $u_h^{(r,m)}$ must be preserved because they are needed to compute an approximate solution (z_h, u_h) for $t = t^{(r+1)}$.

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