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PROPERTIES OF THE INDUCED SEMIGROUP OF AN ARCHIMEDEAN COPULA

Abstract. It is shown that to every Archimedean copula H there corresponds a one-parameter semigroup of transformations of the interval $[0,1]$. If the elements of the semigroup are diffeomorphisms, then it determines a special function v_H called the vector generator. Its knowledge permits finding a pseudoinverse $y = h(x)$ of the additive generator of the Archimedean copula H by solving the differential equation $dy/dx = v_H(y)/x$ with initial condition $(dh/dx)(0) = -1$. Weak convergence of Archimedean copulas is characterized in terms of vector generators. A new characterization of Archimedean copulas is also given by using the notion of a projection of a copula.

1. Introduction. An increasing role in mathematical statistics is played by a class of copulas called Archimedean. Basic properties of Archimedean copulas were studied by Genest and MacKay (1986a, 1986b). Marshall and Olkin (1988) emphasized the relevance of Archimedean copulas in the context of mixture models. Genest and Rivest (1993) studied dependence model selection and fitting of copulas within this class. Bilodeau (1989) investigated conditions under which a parametric family of Archimedean copulas has the regression dependence (RD) property. See also Nelsen (1998) for a general overview of the field, containing in particular much information on Archimedean copulas. The present work provides new tools for the study of Archimedean copulas.

An *Archimedean copula* is a two-dimensional copula of the form

$$H(u_1,u_2) = g^{[-1]}(g(u_1)+g(u_2)) \quad \text{for all } (u_1,u_2) \in [0,1]^2, \tag{1}$$

where $g^{[-1]}$ is a pseudoinverse of the convex and strictly decreasing function

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$g : [0,1] \to [0,\infty]$ satisfying $g(1) = 0$. The function $g^{[-1]}$ is an extension of the usual inverse g^{-1} to the whole interval $[0,\infty]$. The pseudoinverse is defined as follows:

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if } 0 \le x \le g(0), \\ 0 & \text{if } x > g(0). \end{cases}$$

The pseudoinverse of g coincides with the usual inverse if and only if $g(0) = \infty$.

The function g appearing in (1) is called the *additive generator* of the copula H. The set of all additive generators is denoted by $\boldsymbol{\mathcal{G}}$. A generator $g \in \boldsymbol{\mathcal{G}}$ is called *strict* if it is unbounded. Otherwise it is *nonstrict.* In what follows, for convenience, we denote the pseudoinverse of the additive generator g by h.

In the next section we study Archimedean copulas whose additive generators and the corresponding pseudoinverses satisfy certain regularity conditions. We assume that the pseudoinverse h of the additive generator g has a continuous second derivative, except possibly at $x = g(0)$. The set of all such generators is denoted by $\overset{*}{\boldsymbol{\mathcal{G}}}$, and the corresponding set of pseudoinverses by $\overset{*}{\boldsymbol{\mathcal{H}}}$.

The additive generator is uniquely determined up to a multiplicative constant. Therefore we may assume that $\overset{*}{\boldsymbol{\mathcal{G}}}$ is the set of all those additive generators g that satisfy $(dg/dt)(1) = \varepsilon$, where $\varepsilon = -1$ or 0. The sets of generators and of their pseudoinverses corresponding to the specific values of ε will be denoted by $\overset{*}{\boldsymbol{\mathcal{G}}}{}^0$, $\overset{*}{\boldsymbol{\mathcal{G}}}{}^-$ and $\overset{*}{\boldsymbol{\mathcal{H}}}{}^0$, $\overset{*}{\boldsymbol{\mathcal{H}}}{}^-$. The families of Archimedean copulas generated by $\overset{*}{\boldsymbol{\mathcal{G}}}{}^0$ and $\overset{*}{\boldsymbol{\mathcal{G}}}{}^-$ will be denoted by $\boldsymbol{\mathcal{H}}^0$ and $\boldsymbol{\mathcal{H}}^-$, respectively.

For convenience, we set $\boldsymbol{\mathcal{H}} := \boldsymbol{\mathcal{H}}^0 \cup \boldsymbol{\mathcal{H}}^-$, $\mathbb{I} = [0,1]$, $\mathbb{T} = [0,\infty]$.

Example 1. The functions

1) $g_\theta^{(1)}(t) = \dfrac{1}{\theta}(t^{-\theta} - 1)$ (for $\theta > 0$),

2) $g_\theta^{(2)}(t) = \dfrac{1}{\theta}\arcsin(1 - t^\theta)$ (for $\theta \in (0,1]$),

3) $g_\theta^{(3)}(t) = \theta(1 - t^{1/\theta})$ (for $\theta \ge 2$),

4) $g_\theta^{(4)}(t) = \cot^\theta\left(\dfrac{1}{2}\pi t\right)$ (for $\theta > 1$),

5) $g^{(5)}(t) = \dfrac{1-t}{t^2}$

are additive generators of the following Archimedean copulas:

1) $H_\theta^{(1)}(u_1,u_2) = [u_1^{-\theta} + u_2^{-\theta} - 1]^{-1/\theta}$ for all $(u_1,u_2) \in \mathbb{I}^2$.

We have $\{g_\theta^{(1)} : \theta > 0\} \subset \overset{*}{\boldsymbol{\mathcal{G}}}{}^-$.

2) $H_\theta^{(2)}(u_1,u_2) =$

$$\begin{cases} [1-\sin(\arcsin(1-u_1^\theta)+\arcsin(1-u_2^\theta))]^{1/\theta} & \text{if } (u_1,u_2)\in \mathrm{H}_\theta^{(2)},\\ 0 & \text{if } (u_1,u_2)\in \mathbb{I}^2-\mathrm{H}_\theta^{(2)}, \end{cases}$$

where $\mathrm{H}_\theta^{(2)} = \{(u_1,u_2) : \arcsin(1-u_1^\theta)+\arcsin(1-u_2^\theta) \le \frac{1}{2}\pi\}$. We have $\{g_\theta^{(2)} : \theta \in (0,1]\} \subset \overset{*}{\boldsymbol{\mathcal{G}}}^-$.

3) $$H_\theta^{(3)}(u_1,u_2) = \begin{cases} [u_1^{1/\theta}+u_2^{1/\theta}-1]^\theta & \text{if } (u_1,u_2)\in \mathrm{H}_\theta^{(3)},\\ 0 & \text{if } (u_1,u_2)\in \mathbb{I}^2-\mathrm{H}_\theta^{(3)}, \end{cases}$$

where $\mathrm{H}_\theta^{(3)} = \{(u_1,u_2) : u_1^{1/\theta}+u_2^{1/\theta} \ge 1\}$. We have $\{g_\theta^{(3)} : \theta \ge 2\} \subset \overset{*}{\boldsymbol{\mathcal{G}}}^-$.

4) $$H_\theta^{(4)}(u_1,u_2) = \frac{2}{\pi}\operatorname{arccot}\left(\left[\cot^\theta\left(\frac{1}{2}\pi u_1\right)+\cot^\theta\left(\frac{1}{2}\pi u_2\right)\right]^{1/\theta}\right)$$

for all $(u_1,u_2)\in\mathbb{I}^2$. We have $\{g_\theta^{(4)} : \theta > 1\} \subset \overset{*}{\boldsymbol{\mathcal{G}}}^0$.

5) $$H^{(5)}(u_1,u_2) = \frac{u_1u_2[u_1^2u_2^2+4(u_1^2+u_2^2-u_1u_2(u_1+u_2))]^{1/2}-u_1^2u_2^2}{2(u_1^2+u_2^2-u_1u_2(u_1+u_2))}$$

for all $(u_1,u_2)\in\mathbb{I}^2$. The generator $g^{(5)}$ is an element of the family defined by $g_\theta^{(5)}(t) = (1-t)/t^\theta$ for $\theta \ge 0$. The generator $g_0^{(5)}$ belongs to $\boldsymbol{\mathcal{G}}$. It generates the copula $H^-(u_1,u_2) = \max(u_1+u_2-1,0)$. For $\theta > 0$, $g_\theta^{(5)} \in \overset{*}{\boldsymbol{\mathcal{G}}}^-$. We are able to give analytic formulas for the copulas $H_\theta^{(5)}$ only for a few parameters θ. This is due to the fact that only in these cases can we find the pseudoinverses $h_\theta^{(5)}$.

The family $\{H_\theta^{(1)}\}$ was discussed by Clayton (1978), Oakes (1982, 1986), Cox and Oakes (1984), and Cook and Johnson (1981, 1986). The families $\{g_\theta^{(2)} : \theta \in (0,1]\}$ and $\{g_\theta^{(3)} : \theta \ge 2\}$ are known. They can be found e.g. in Nelsen (1998). The families $\{g_\theta^{(4)} : \theta > 1\}$ and $\{g_\theta^{(5)} : \theta \ge 0\}$ are new to the best of our knowledge.

To end this section, let us recall that if for some continuous increasing bijection f of the interval $\mathbb{I}$ onto itself, the composition $-\ln\circ f$ is an additive generator of an Archimedean copula H, then f is called the *multiplicative generator* of the copula H. The corresponding copula then has the form

$$H(u_1,u_2) = f^{-1}(f(u_1)f(u_2)) \quad \text{for all } (u_1,u_2)\in\mathbb{I}^2.$$

Let $\boldsymbol{X} : (\Omega,\mathcal{A},P) \to \mathbb{R}^2$ be a two-dimensional random vector defined on a probability space $(\Omega,\mathcal{A},P)$. Let F and F_j be the distribution functions of the random vector $\boldsymbol{X} = (X_1,X_2)$ and of the coordinates X_j for $j=1,2$, respectively. We assign to $\boldsymbol{X}$ the random vector $\boldsymbol{U} = (U_1,U_2)$ by $U_j = F_j(X_j)$ for $j=1,2$. If F is continuous, then the distribution function H

of $\boldsymbol{U}$ has a unique representation $H(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))$ for all $(u_1, u_2) \in \mathbb{I}^2$. The inverse of F_j is defined by $F_j^{-1}(u_j) = \inf\{x : F_j(x) \geq u_j\}$ for $j = 1, 2$. The coordinates of $\boldsymbol{U}$ are uniformly distributed on $[0, 1]$. The distribution function H is called a *copula*, and the random vector $\boldsymbol{U}$ is the *uniform representation* of $\boldsymbol{X}$.

2. Vector generators. In this section we show that to every Archimedean copula there corresponds a one-parameter semigroup of bijections. It turns out that if the elements of the semigroup are diffeomorphisms, then the semigroup determines a special function called a generator. Its knowledge enables finding the pseudoinverse of the additive generator of the Archimedean copula.

Let $\widetilde{f}_t : [0, \infty) \to [0, \infty)$ be defined by $\widetilde{f}_t(u) = e^t u$ (multiplication by e^t), $t \in \mathbb{T}$. Let h be the pseudoinverse of the additive generator $g \in \boldsymbol{\mathcal{G}}$ of an Archimedean copula H. We define a family $\{f_t : t \in \mathbb{T}\}$ of functions as follows:

$$f_t(u) = h \circ \widetilde{f}_t \circ g(u) \quad \text{for all } u \in \mathbb{I}. \tag{2}$$

We indicate some simple properties of the family (2). First assume that the generator g is strict. Then:

(a_1) The function f_t is a continuous increasing bijection of $\mathbb{I}$ onto itself satisfying $f_t(0) = 0$, $f_t(1) = 1$.

(a_2) For $t > 0$, $f_t(u) < u$ for all $u \in (0, 1)$.

(a_3) If $s, t \in \mathbb{T}$, then $f_{t+s}(u) = f_t \circ f_s(u) = f_s \circ f_t(u)$, $f_0 = \mathrm{Id}$.

Only (a_2) is not evident, although its proof is very simple. Applying the (decreasing) function h to both sides of the inequality $e^t g(u) > g(u)$, $u \in (0, 1)$, $t > 0$, we obtain $f_t(u) = h \circ [e^t g(u)] < h \circ g(u) = u$. Thus the family $\{f_t : t \in \mathbb{T}\}$ is a one-parameter semigroup of transformations of $\mathbb{I}$.

For a nonstrict generator the situation is more complex. The functions f_t of the family (2) then have the following properties:

(A_1) The function f_t is an increasing bijection of its support $[\alpha_t, 1]$ onto $\mathbb{I}$, where $\alpha_t = h(e^{-t} g(0))$. Moreover, $f_t(\alpha_t) = 0$, $f_t(1) = 1$.

(A_2) For $t > 0$, $f_t(u) < u$ for all $u \in (0, 1)$.

(A_3) For every $t \in \mathbb{T}$ there are subsets U_t, V_t of $\mathbb{I}$ such that $f_t : U_t \to V_t$ is bijective and for all $s, t \in \mathbb{T}$, $f_{t+s}(u) = f_t \circ f_s(u)$ for all those u that satisfy $u \in U_s$, $u \in U_{t+s}$ and $f_s(u) \in U_t$. Moreover, $f_0 = \mathrm{Id}$.

So in contrast to the preceding case, we can now call the family (2) a local one-parameter semigroup of transformations of $\mathbb{I}$.

Suppose that $\{f_t : t \in \mathbb{T}\}$ is a family of transformations with the properties (a_1)–(a_3) or (A_1)–(A_3) and satisfying: (1) for all $t \in \mathbb{T}$ the functions f_t are diffeomorphisms, (2) for each fixed $u \in \mathbb{I}$ the function $t \mapsto f_t(u)$ is differentiable.

Let $v : \mathbb{I} \to \mathbb{R}^1$ be defined by

$$v(u) = \lim_{t \to 0_+} \frac{f_t(u) - u}{t}. \tag{3}$$

If $h \in \overset{*}{\boldsymbol{\mathcal{H}}}^-$ in (2), then the function v for the family (2) has the form

$$v(u) = \left[\frac{dh}{dx} \circ g(u)\right] g(u) = g(u) \left[\frac{dg(u)}{du}\right]^{-1}. \tag{4}$$

The function v is called the *generator* of the one-parameter semigroup of diffeomorphisms (2). For simplicity, v will be briefly called the generator. Note that v is always defined on the whole interval $\mathbb{I}$, whether g is a strict generator or not.

EXAMPLE 2. The generators corresponding to the Archimedean copulas (or additive generators) of Example 1 are the following:

1) $v_\theta^{(1)}(u) = \dfrac{1}{\theta} u(u^\theta - 1) \quad \text{for } \theta > 0,$

2) $v_\theta^{(2)}(u) = -\dfrac{1}{\theta} u^{1-\theta} [1 - (1 - u^\theta)^2]^{1/2} \arcsin(1 - u^\theta) \quad \text{for } \theta \in (0, 1],$

3) $v_\theta^{(3)}(u) = \theta[u - u^{1-1/\theta}] \quad \text{for } \theta \geq 2,$

4) $v_\theta^{(4)}(u) = -\dfrac{1}{\pi\theta} \sin(\pi u) \quad \text{for } \theta > 1,$

5) $v_\theta^{(5)}(u) = \dfrac{u(1-u)}{u(\theta - 1) - \theta} \quad \text{for } \theta > 0,$

LEMMA 1. *If* $h \in \overset{*}{\boldsymbol{\mathcal{H}}}^-$, *then the generator defined by* (4) *has the following properties*:

(1) $v : \mathbb{I} \to [-1, 0]$, $v(0) = v(1) = 0$,
(2) v *is continuously differentiable on the interval* $(0, 1]$,
(3) $dv/du < 1$ *for all* $u \in [0, 1)$ *and* $(dv/du)(1) = 1$.

Proof. The identity $f_t(0) \equiv 0$ yields $v(0) = 0$. The properties (3) follow directly from the representation

$$\frac{dv}{du}(u) = 1 - g(u) \frac{d^2 g(u)}{du^2} \left[\frac{dg(u)}{du}\right]^{-2}.$$

The other properties are obvious. ∎

REMARK 1. A simple application of the mean value theorem shows that property (3) is equivalent to the inequality $u - 1 \leq v(u) \leq 0$ for all $u \in \mathbb{I}$.

The set of all functions $v : \mathbb{I} \to [-1, 0]$ with the properties listed in Lemma 1 will be denoted by $\boldsymbol{\mathcal{V}}^-$. The generators corresponding to the family $\boldsymbol{\mathcal{H}}^0$ are formally of the form (4). But they fail property (3) of Lemma 1. The set of all functions v with properties (1) and (2) will be denoted by $\boldsymbol{\mathcal{V}}^0$.

The proof of the main theorem of this section requires some notions from the theory of ordinary differential equations. Some facts needed here are taken from Arnold's book (1971), in a simplified or modified form.

- An ordered pair consisting of a set $\mathbb{I}$ and a one-parameter semigroup of transformations $\{f_t : t \in \mathbb{T}\}$ of $\mathbb{I}$ is called a *phase flow.* The set $\mathbb{I}$ is called a *phase space.*
- The map $T : \mathbb{T} \to \mathbb{I}$ defined by $T(t) = f_t(u)$ is the *trajectory* of the phase flow corresponding to u. The image of T is a *phase curve.*
- The *generator* v of the phase curve is the right hand derivative of the map T at $t = 0$, that is, the function defined by (3).
- The function $df_t(u)/dt$ is called a *vector field* in the domain $\mathbb{I}$. By the equality $df_t(u)/dt = v(f_t(u))$, the vector field $df_t(u)/dt$ is identified with the generator v.

The theorem below concerns formula (4). For $x = g(u)$ the formula takes the form

$$\frac{dh(x)}{dx} = \frac{v \circ h(x)}{x}.$$

In the proof of the theorem we write $\dot{z}$ for the derivative dz/dt.

THEOREM 1. *Let $v \in \boldsymbol{\mathcal{V}}^-$. Then the function $y = h(x)$ which solves the differential equation*

$$\frac{dy}{dx} = \frac{v(y)}{x} \tag{5}$$

with initial condition $(dy/dx)(0) = -1$ is a pseudoinverse in the family $\overset{*}{\boldsymbol{\mathcal{H}}}{}^-$.

Proof. We consider two cases: when the support of h is unbounded (strict generator) and bounded (nonstrict generator).

Case 1. The solution $y = y(x)$ is a phase curve of the vector field $\dot{x} = x$, $\dot{y} = v(y)$ in the plane $\mathbb{R}^2$. In the neighbourhood of $(x = 0, y = 1)$ the vector field $v(y)$ has the expansion $v(y) = y - 1 + O((y-1)^2)$. Hence

$$\dot{x} = x, \qquad \frac{d}{dx}(y-1) = y - 1 + O((y-1)^2).$$

Its linear part has the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The point $(0,1)$ is an unstable focus. This means that from $(0,1)$ in a given direction there emanates exactly one phase curve. Let z be the slope of the direction passing through $(0,1)$, $z = (y-1)/x$. Then the phase curve corresponding to $z = -1$ is our solution. In the variables z, x the differential equation (5) takes the form $dz/dx = A(z,x)$, where A is a differentiable function. The last equation with initial condition $z(0) = -1$ has the solution $z(x) = -1 + \cdots$. It yields $y(x) = -1 + z(x)x = 1 - x + \cdots$. Note that the function $y(x)$ is decreasing, since the right hand side of (5) is nonpositive. To check that $d^2y/dx^2 \geq 0$ we differentiate both sides of (5):

$$\frac{d^2y}{dx^2} = \frac{v(y)}{x^2}\left[\frac{dv}{du}\circ h(x) - 1\right].$$

Hence the condition $dv/du \le 1$ ($v \in \boldsymbol{\mathcal{V}}^-$) implies $d^2y/dx^2 \ge 0$.

Case 2. The only difference is in the phase portrait of the vector field $\dot{x} = x$, $\dot{y} = v(y)$. The phase curves in $\{(x,y) : x \ge 0, y \ge 0 \wedge y \le 1\}$ emanating from $(0,1)$ are no longer asymptotic to the y axis. This axis is tangent to them at all points $\ge x_0$, where x_0 is the right end point of the support of the solution $y = h(x)$. Indeed, the differentiability of h implies that $dh(x)/dx = 0$ for $x \ge x_0$. ∎

Theorem 1 enables one to produce new families of additive generators of Archimedean copulas.

EXAMPLE 3. • It is easy to check that the functions

$$v_\theta^{(6)}(u) = 2\int_0^u \Phi(\theta\Phi^{-1}(t))\,dt - u$$

for $\theta \ge 1$ belong to the family $\boldsymbol{\mathcal{V}}^-$. Clearly, $(dv_\theta^{(6)}/du)(1) = 1$, and the condition $dv_\theta^{(6)}(u)/du < 1$ for all $u \in [0,1)$ follows from the convexity of $v_\theta^{(6)}$, which is equivalent to the inequality

$$\frac{d^2v_\theta^{(6)}(u)}{du^2} = 2\theta e^{\frac{1}{2}(1-\theta^2)[\Phi^{-1}(u)]^2} > 0.$$

The function Φ appearing here is the standard normal cumulative distribution function, $\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^t e^{-s^2/2}\,ds$.

The additive generators $g_\theta^{(6)}$ corresponding to the solutions $h_\theta^{(6)}$ of the differential equation (5) for $v = v_\theta^{(6)}$ cannot be given explicitly, except for the case $\theta = 1$. The generator $g_1^{(6)}$ coincides with the generator $g_1^{(1)}$ of Example 1, $v_1^{(6)}(u) = u^2 - u$.

• It is easy to check that the functions

$$v_\theta^{(7)}(u) = -\frac{u(1-u)}{[u^{1/\theta} + (1-u)^{1/\theta} - u^{1/\theta}(1-u)^{1/\theta}]^{1/\theta}}$$

for $\theta > 0$ are in $\boldsymbol{\mathcal{V}}^-$. Clearly,

$$\frac{dv_\theta^{(7)}(u)}{du}(1) = 1, \qquad \frac{dv_\theta^{(7)}(u)}{du} < 1 \quad \text{for all } u \in [0,1).$$

Solving the differential equation (5) for $v = v_\theta^{(7)}$ we obtain the family of additive generators $g_\theta^{(7)}$ for $\theta > 0$ corresponding to the pseudoinverses $h_\theta^{(7)}$.

For $\theta = 1$ we have

$$g_1^{(7)}(t) = \frac{1-t}{t}\, e^{1-t}.$$

The form of the function $v_\theta^{(6)}$ from Example 3 is closely connected with normal copulas. This will be discussed in a forthcoming paper on adjoint copulas.

The considerations leading to the definition of $\boldsymbol{\mathcal{V}}^-$ and Theorem 1 show that there is a one-to-one correspondence $v : \boldsymbol{\mathcal{H}}^- \to \boldsymbol{\mathcal{V}}^-$ given by $v(H) = v_H$.

The function v_H will be called the *vector generator* of the Archimedean copula H. Obtaining an analogue of Theorem 1 for the sets $\overset{*}{\boldsymbol{\mathcal{H}}}{}^0$ and $\boldsymbol{\mathcal{V}}^0$ does not seem easy. For consistency we will call the elements of $\boldsymbol{\mathcal{V}}^0$ *vector pseudogenerators* of Archimedean copulas.

The vector generator v_H generates a one-parameter transformation semigroup of the form (2). It is easy to show that the semigroup can be obtained by solving the differential equation $\dot{y} = v_H(y)$ with initial condition $y(0) = u$. Then $y(t) = f_t(u)$. The one-parameter semigroup (2) corresponding to the Archimedean copula $H \in \boldsymbol{\mathcal{H}}^-$ will be called the transformation semigroup *induced by* H and will be denoted by $\mathcal{F}_H$.

Solving the differential equation $\dot{y} = v_H(y)$ for

$$v(u) = \frac{1}{2} u(2-u) \ln\left[\frac{u}{2-u}\right]$$

we obtain

$$f_t(u) = \frac{2u^{e^t}}{(2-u)^{e^t} + u^{e^t}}.$$

EXAMPLE 4. The Archimedean copulas in items 1, 3 and 5 of Example 1 induce the following transformation semigroups:

$$\mathcal{F}_{H_\theta^{(1)}} = \{f_t^\theta(u) = [u^{-\theta} e^t + 1 - e^t]^{-1/\theta} : t \in \mathbb{T}\} \qquad \text{for } \theta > 0,$$
$$\mathcal{F}_{H_\theta^{(3)}} = \{f_t^\theta(u) = [u^{1/\theta} e^t + 1 - e^t]^{\theta} : t \in \mathbb{T}\} \qquad \text{for } \theta \geq 2,$$

where the support of the function f_t^θ is $[\alpha_t, 1]$ with $\alpha_t = (1 - e^{-t})^\theta$, and

$$\mathcal{F}_{H^{(5)}} = \left\{ f_t(u) = \frac{u[u^2 - 4ue^t + 4e^t]^{1/2} - u^2}{2(1-u)e^t} : t \in \mathbb{T} \right\}.$$

To end this section we give a characterization of the Archimedean copulas of the family $\boldsymbol{\mathcal{H}}^-$. To this end, we define on $\boldsymbol{\mathcal{H}}$ an operator $\widehat{\ }$ as follows:

$$\widehat{H}(u) = \iint_{\{(u_1,u_2)\,:\,H(u_1,u_2)\leq u\}} dH(u_1, u_2).$$

If $\boldsymbol{U} = (U_1, U_2)$ is a random vector with distribution H, then $\widehat{H}$ is the distribution function of the random variable $H \circ \boldsymbol{U}$. Genest and Rivest (1993) showed that

$$\widehat{H}(u) = u - v_H(u), \tag{6}$$

and called $\widehat{H}$ the *projection* of H.

LEMMA 2. *If $H \in \boldsymbol{\mathcal{H}}^-$, then $\widehat{H}$ has the following properties*:

(1) *$\widehat{H}$ is an increasing function of $\mathbb{I}$ onto $\mathbb{I}$ having continuous derivative on $(0, 1]$.*
(2) $(d\widehat{H}/du)(1) = 0$.
(3) *$\widehat{H}(u) > u$ for all $u \in (0, 1)$.*

Proof. All the properties are evident. ∎

Let $\widehat{\boldsymbol{\mathcal{H}}}$ be the set of all functions with the properties listed in Lemma 2.

THEOREM 2. *The following conditions are equivalent*:

(A) *H is a copula in the family $\boldsymbol{\mathcal{H}}^-$.*
(B) $\widehat{H} \in \widehat{\boldsymbol{\mathcal{H}}}$.

Proof. (B)⇒(A). Assume that $\widehat{H} \in \widehat{\boldsymbol{\mathcal{H}}}$. Then the function $\widetilde{v}(u) = u - \widehat{H}(u)$ is in $\boldsymbol{\mathcal{V}}^-$. Thus Theorem 1 indicates that to $\widetilde{v}$ there corresponds a copula $\widetilde{H} \in \boldsymbol{\mathcal{H}}^-$. As $\widetilde{H}$ has vector generator $u - \widehat{H}(u)$, it follows that $H = \widetilde{H}$. ∎

EXAMPLE 5. The function $f(u) = \sqrt{1-(1-u)^2}$ is in $\widehat{\boldsymbol{\mathcal{H}}}$. Hence $v(u) = u - f(u)$ is the vector generator of an Archimedean copula. Solving the differential equation (5) we obtain the additive generator

$$g(t) = \{1 - [1-(1-t)^2]^{1/2}\}^{1/2} e^{\frac{1}{2}\arcsin(1-t)}.$$

Two characterizations of Archimedean copulas are known: an "algebraic" characterization due to Ling (1965) and a "differential" one given in Genest–MacKay (1986a) [1].

The next section gives applications of the vector generators of Archimedean copulas.

3. Weak convergence of Archimedean copulas. In this section we show that convergence in $\boldsymbol{\mathcal{H}}$ can be characterized in terms of convergence of vector generators and pseudogenerators.

Let us recall the notion of convergence in distribution of a sequence of random vectors (see e.g. Billingsley (1979)). Let $\boldsymbol{X}$ and $\boldsymbol{X}_n$ for $n = 1, 2, \ldots$ be two-dimensional random vectors on a common probability space

[1] More precisely, this is a sufficient criterion for a copula to be Archimedean. The converse appears in Nelsen's book (1997).

$(\Omega, \mathcal{A}, P)$. Let moreover G and G_n for $n = 1, 2, \ldots$ be the distribution functions of $\boldsymbol{X}$ and $\boldsymbol{X}_n$, respectively.

We say that the sequence $(\boldsymbol{X}_n)$ is *convergent in distribution* to $\boldsymbol{X}$ if for every bounded continuous function $f : \mathbb{R}^2 \to \mathbb{R}^1$ we have the convergence $\int_{\mathbb{R}^2} f(\boldsymbol{x})\, dG_n(\boldsymbol{x}) \to \int_{\mathbb{R}^2} f(\boldsymbol{x})\, dG(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$.

It is known that the convergence in distribution $\boldsymbol{X}_n \to \boldsymbol{X}$ is equivalent to the pointwise convergence of $G_n \to G$ at each continuity point of G. If G is continuous, then the latter convergence is uniform. We recall another well known fact.

LEMMA 3. *Assume that the sequence $(\boldsymbol{Z}_n)$ of two-dimensional random vectors converges in distribution to a random vector $\boldsymbol{Z}$, and the sequence of continuous functions $f_n : \mathbb{R}^2 \to \mathbb{R}^1$ is uniformly convergent to a function f. Then the sequence $(f_n \circ \boldsymbol{Z}_n)$ of random variables converges in distribution to the random variable $f \circ \boldsymbol{Z}$.*

THEOREM 3. *Let H and H_n for $n = 1, 2, \ldots$ be Archimedean copulas in the family $\boldsymbol{\mathcal{H}}^-$, corresponding to the uniform representations $\boldsymbol{U}$ and $\boldsymbol{U}_n$ of random vectors $\boldsymbol{X}$ and $\boldsymbol{X}_n$, respectively. Assume that the second derivative of the pseudoinverse h_n of the additive generator g_n of H_n is continuous at $x = g_n(0)$ for $n = 1, 2, \ldots$. Then the following conditions are equivalent*:

(A_1) *The sequence $(\boldsymbol{U}_n)$ converges in distribution to $\boldsymbol{U}$.*
(A_2) *The sequence (H_n) converges uniformly to H.*
(A_3) *The sequence (v_{H_n}) converges uniformly to v_H.*

Proof. It suffices to prove the implications $(\mathrm{A}_2)\Rightarrow(\mathrm{A}_3)$ and $(\mathrm{A}_3)\Rightarrow(\mathrm{A}_2)$.

$(\mathrm{A}_2)\Rightarrow(\mathrm{A}_3)$. Let $\widehat{H}$ and $\widehat{H}_n$ for $n = 1, 2, \ldots$ be the projections of H and H_n defined by (6). If $\boldsymbol{Z} = \boldsymbol{U}$, $f = H$ and $\boldsymbol{Z}_n = \boldsymbol{U}_n$, $f_n = H_n$ for $n = 1, 2, \ldots$, then by Lemma 3 the convergence $H_n \to H$ implies the uniform convergence

$$\widehat{H}_n \to \widehat{H}, \tag{7}$$

which in turn yields the uniform convergence $v_{H_n} \to v_H$.

$(\mathrm{A}_3)\Rightarrow(\mathrm{A}_2)$. By Theorem 1, to the generators v_{H_n}, v_H there correspond the pseudoinverses h_n, h. The functions $v_{H_n} \circ h_n$ are equicontinuous. To see this, it suffices to show that they are all Lipschitz with constant 1. To this end, write (5) in the form

$$v_{H_n} \circ h_n(x) = x\, \frac{dh_n(x)}{dx}.$$

Differentiating both sides we obtain

$$\left[\frac{dv_{h_n}}{du} \circ h_n(x)\right] \frac{dh_n(x)}{dx} = \frac{dh_n(x)}{dx} + x\, \frac{d^2 h_n(x)}{dx^2} = \alpha(x).$$

Note that for each positive integer n there exists $x_{0n} > 0$ such that $\alpha(x_{0n}) = 0$. The function $\alpha(x)$, continuous in the interval $[0, x_{0n}]$, takes all values from

$[-1, 0]$. It turns out that if $g_n(0) = \infty$, then

$$\lim_{x\to\infty} x\,\frac{d^2h_n(x)}{dx^2} = \lim_{x\to\infty} x\left[\frac{\frac{dh_n}{dx}(2x) - \frac{dh_n}{dx}(x)}{x}\right] = \lim_{x\to\infty}\left[\frac{dh_n}{dx}(2x) - \frac{dh_n}{dx}(x)\right] = 0.$$

Since $dh_n(x)/dx$ is negative and increases to 0, it follows that the values of the positive function $xd^2h_n(x)/dx^2$ are such that $\alpha(x)$ is less than 1 for $x \in [x_{0n}, \infty]$. If $g_n(0) < \infty$, then $\alpha(x)$ also has values < 1 in $[x_{0n}, g_n(0)]$, since then $d^2h_n(x)/dx^2 = 0$ and $\alpha(x) = 0$ for $x \geq g_n(0)$. This argument leads to

$$\sup_{x\geq 0}\left|\left[\frac{dv_{H_n}}{du}\circ h_n(x)\right]\frac{dh_n(x)}{dx}\right| = 1,$$

implying that all the functions $v_{H_n}\circ h_n(x)$ are Lipschitz with constant 1. They are also uniformly bounded. Hence by the Ascoli theorem (see e.g. Schwartz (1967)) there exists a subsequence $v_{H_{n_k}}\circ h_{n_k}(x)$ almost uniformly convergent (that is, uniformly convergent on each compact set) to a continuous function f. Similarly, all the h_n are uniformly bounded and equicontinuous. Hence in this case there also exists a subsequence h_{n_k} almost uniformly convergent to a continuous function $\widetilde{f}$. By (A_3) we deduce that $\widetilde{f} = h$, and the subsequence $v_{H_{n_k}}\circ h_{n_k}(x)$ is almost uniformly convergent to $f(x) = v_H\circ h(x)$. Since $v_{H_n}(u)$ is convergent, it has a unique accumulation point. Hence $v_{H_n}\circ h_n(x)$ must converge almost uniformly to $v_H\circ h(x)$. This, however, means that $h_n(x)$ converges almost uniformly to $h(x)$, which in turn implies the almost uniform convergence of $g_n(t)$ to $g(t)$. The last two convergences ensure the uniform convergence of the Archimedean copulas H_n to H. ∎

The convergence (7) yields

COROLLARY 1. *The operator* $\widehat{\ }$ *is continuous in the topology of uniform covergence.*

Theorem 3 implies

COROLLARY 2. $v : \boldsymbol{\mathcal{H}}^- \to \boldsymbol{\mathcal{V}}^-$ *is a homeomorphism in the topology of uniform convergence.*

EXAMPLE 6. Let H_θ be the Archimedean copula corresponding to the generator $g_\theta^{(1)}$ (for $\theta \geq 1$) of Example 3,

$$H_1(u_1, u_2) = \frac{u_1u_2}{u_1 + u_2 - u_1u_2} \quad \text{for all } (u_1, u_2) \in \mathbb{I}^2.$$

Suppose that $\theta_n \to 1$, $\theta_n > 1$. Then $\Phi\left(\theta_n\Phi^{-1}(u)\right)$ converges uniformly to the identity map for all $t \in \mathbb{I}$. By Lebesgue's theorem $\int_0^u \Phi(\theta_n\Phi^{-1}(t))\,dt$ converges to $\frac{1}{2}u^2$ for all $u \in \mathbb{I}$. This yields the uniform convergence $v_{\theta_n}^{(1)}(u) \to v_1^{(1)}(u) = u^2 - u$. Theorem 3 now shows that H_{θ_n} tends uniformly to H_1.

The convergence of Archimedean copulas was considered by Genest and MacKay (1986a). Their condition for convergence $H_n \to H$, although equivalent to $v_{H_n} \to v_H$, is unnatural and may be difficult to check.

As the last result we prove a theorem stating convergence of Archimedean copulas in $\boldsymbol{\mathcal{H}}$ to a copula that is not Archimedean. It is a modification of the result of Genest and MacKay (1986a). Let H^+ be the copula defined by $H^+(u_1, u_2) = \min(u_1, u_2)$ for all $(u_1, u_2) \in \mathbb{I}^2$. Then H^+ is not Archimedean. It is called the *Fréchet upper bound.*

THEOREM 4. *Let H_n for $n = 1, 2, \ldots$ be Archimedean copulas of the family $\boldsymbol{\mathcal{H}}$, corresponding to uniform representations $\boldsymbol{U}_n$ of random vectors $\boldsymbol{X}_n$. Then the following conditions are equivalent*:

(B_1) *The sequence $(\boldsymbol{U}_n)$ converges in distribution to $\boldsymbol{U}^+$.*
(B_2) *The sequence (H_n) converges uniformly to H^+.*
(B_3) *The sequence (v_{H_n}) converges uniformly on $\mathbb{I}$ to the identically zero function.*

The proof of (B_2)$\Rightarrow$(B_3) is similar to that of (A_2)$\Rightarrow$(A_3) in the proof of Theorem 3. In this case we have $\widehat{H}^+(u) = u$. Thus (7) yields (B_3).

To prove (B_3)$\Rightarrow$(B_2) we apply the form of Kendall's familiar index of stochastic dependence τ, given by Genest and MacKay (1986a):

$$\tau(H) = 4\int_0^1 v_H(t)\,dt + 1. \tag{8}$$

By Lebesgue's theorem $\int_0^1 v_{H_n}(t)\,dt \to 0$, and hence (8) implies $\tau(H_n) \to 1$. As is well known, $\tau(H) = 1$ if and only if $H = H^+$, and so $\tau(H_n) \to \tau(H^+)$. The copulas H_n for $n = 1, 2, \ldots$ are Lipschitz with constant 1 in the metric $d(\boldsymbol{u}, \boldsymbol{u}') = |u_1 - u_1'| + |u_2 - u_2'|$, where $\boldsymbol{u} = (u_1, u_2)$. Therefore the H_n are equicontinuous and uniformly bounded. By the Ascoli theorem there exists a subsequence H_{n_k} uniformly convergent to a copula $\widetilde{H}$. By the continuity of τ we have $\tau(H_{n_k}) \to \tau(\widetilde{H}) = 1$. This means that $\widetilde{H} = H^+$. Since $\tau(H_n) \to \tau(H^+)$, we finally conclude that the sequence H_n of Archimedean copulas uniformly converges to H^+. ∎

EXAMPLE 7. Let $\theta_n \to \infty$. The sequence of additive generators (g_{θ_n}) of the family $\{g_\theta^{(4)} : \theta > 1\}$ corresponds to the sequence of vector pseudogenerators

$$v_{H_{\theta_n}}(u) = -\frac{1}{\pi\theta_n}\sin(\pi u).$$

Thus $H_{\theta_n} \to H^+$.

EXAMPLE 8. Let H_n for $n = 1, 2, \ldots$ be Archimedean copulas from $\boldsymbol{\mathcal{H}}^-$ defined by

$$H_n(u_1, u_2) = \frac{u_1 u_2}{[u_1^n + u_2^n - u_1^n u_2^n]^{1/n}} \qquad \text{for all } (u_1, u_2) \in \mathbb{I}^2$$

(cf. the family $g_\theta^{(1)}$ of Example 1). Their additive and vector generators are respectively

$$g_n(t) = \frac{1}{n}\,(t^{-n} - 1), \qquad v_n(u) = -\frac{1}{n}\,t(1 - t^{-n}).$$

The sequence $v_n(u)$ tends uniformly to the constantly zero function. Hence Theorem 4 yields the uniform convergence of H_n to H^+.

Example 8 implies

COROLLARY 3. *$\boldsymbol{\mathcal{H}}^-$ is not closed under uniform convergence.*

The study of the closure of $\boldsymbol{\mathcal{H}}^-$ and of further applications of vector generators (e.g. to characterize stochastic domination of Archimedean copulas) will be the object of a forthcoming publication.

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