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ON THE OPTIMAL REINSURANCE PROBLEM

Abstract. In this paper we consider the optimal reinsurance problem in endogenous form with respect to general convex risk measures ϱ and pricing rules π. By means of a subdifferential formula for compositions in Banach spaces we first characterize optimal reinsurance contracts in the case of one insurance taker and one insurer. In the second step we generalize the characterization to the case of several insurance takers. As a consequence we obtain a result saying that cooperation brings less risk compared to insurance takers acting individually. Our results extend previously known results from the literature.

1. Introduction. (Re)insurance problems are classical problems in mathematical economics and insurance. They have been studied in the context of expected utilities in extenso, to name but a few papers: Borch (1962), Arrow (1963), Raviv (1979), Deprez and Gerber (1985), Zagrodny (2003), Kałuszka (2004), Aase (2006), Dana and Scarsini (2007), Kałuszka and Okolewski (2008), and Kuciński (2011). Since the upcoming of risk measure theory in the late 90's there have been several papers which carried over insurance problems to risk measures. Here we refer to Gajek and Zagrodny (2004), Barrieu and El Karoui (2005), Jouini et al. (2007) Balbás et al. (2009), [KR] (2008)([1]), [KR] (2010), Balbás et al. (2011), and Cheung et al. (2011).

In the context of risk measures the authors mostly studied insurance problems for specific (classes of) risk measures ϱ and pricing rules π and derived explicit solutions of the infimal convolution problem which in the case of one insurer takes the form

$$\operatorname*{argmin}_R\{\varrho(X-R)+\pi(R)\}. \tag{1.1}$$

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([1]) Kiesel and Rüschendorf is abbreviated within this paper to [KR].

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In this paper we allow general risk and pricing functionals ϱ, π and assume that the premium the insurer charges has a direct endogenous impact on the insurance takers' decision. Thus the problem under consideration has the general form

$$\operatorname*{argmin}_{R} \varrho(X - R + \pi(R)). \tag{1.2}$$

In the expected utility framework this problem was already considered by Deprez and Gerber (1985). They studied the maximization problem

$$\operatorname*{argmax}_{R} \mathbf{E}[u(-X - H(R) + R)],$$

where $u : \mathbb{R} \to \mathbb{R}$ is a risk averse utility function and H is a convex Gâteaux differentiable (pricing) principle.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\varrho_i : L^p(\mathbf{P}) \to [0, \infty]$ for $i \in \{1, \dots, n\}$ be convex, proper, normed, monotone with respect to the almost sure order, lower semicontinuous, and subdifferentiable mappings, called in the following *risk functionals*. The value $\varrho_i(X_i)$ is called the *risk* of the loss $X_i \in L^p$ and describes the risk evaluation of individual i regarding X_i. A natural property of the risk functionals is their monotonicity with respect to the almost sure order, i.e. if $X_i \geq Y_i$ almost surely, then $\varrho_i(X_i) \geq \varrho_i(Y_i)$. We focus on unbounded losses and assume that $1 < p < \infty$.

In the models we analyze we have either one or $n > 1$ individuals, in the following called *insurance taker(s)*, who want to insure their initial loss $X \in L^p_+$ with a suitable insurance coverage $R \in L^p$ such that the residual loss minimizes their risk. The insurance coverages are provided by one insurance company, called the *insurer*, who charges the insurance taker(s) a premium according to a pricing rule π.

Depending on the model we analyze, each individual is provided with a capital endowment $c \in \mathbb{R}_+$ which represents the maximal amount the insurance taker is willing to spend for the premium of an insurance coverage R. This results in the side constraint $\pi(R) \leq c$. The pricing rules $\pi : L^p \to [0, \infty]$ are exogenously given normed, non-negative, convex, L^p-continuous, thus subdifferentiable, functions defined on the space of p-integrable random variables.

For the infimal convolution problem (1.1) a general characterization of solutions is known (see Jouini et al. (2007), Acciaio (2007), and [KR] (2008)). Under certain assumptions the optimal coverages R^* are characterized by the condition

$$\partial\varrho(X - R^*) \cap \partial\pi(R^*) \neq \emptyset. \tag{1.3}$$

Reformulated, this means that there exist $V \in \partial\varrho(X - R^*)$ and $W \in \partial\pi(R^*)$ such that

$$0 = W - V \quad \text{a.s.}$$

In the present paper we show that the solutions to problem (1.2) have a similar characterization. In fact R^* is an optimal coverage if and only if there exist $V \in \partial\varrho(X - R^* + \pi(R^*))$ and $W \in \partial\pi(R^*)$ such that

$$0 = \mathbf{E}[V]W - V \quad \text{a.s.} \tag{1.4}$$

For translation equivariant risk functionals with $\varrho(X + c) = \varrho(X) + c$, $c \in \mathbb{R}$, we show

$$V \in \partial\varrho(X) \;\Rightarrow\; \mathbf{E}[V] = 1.$$

Therefore, the characterization (1.4) reduces in this case to the known condition (1.3) for the infimal convolution problem.

The structure of the paper is the following. First we adapt and specify a chain rule for subdifferentials of the composition $\Psi = \varrho \circ g$ in general Banach spaces to the optimal insurance problem where $g(R) := X - R + \pi(R)$. Based on this rule we are able to characterize explicitly optimal insurance coverages in the framework of subdifferentiable risk functionals.

Then in Section 2 we analyze the insurance model where one insurance taker insures his initial loss with one insurance company, and in Section 3 we deal with the case where n insurance takers pool (aggregate) their initial losses and seek to insure them with one insurance company.

Each of these two sections is divided into two subsections. The first part of each covers the case where the insurance coverage is chosen arbitrarily, and the second handles the case where only specific insurance coverages are allowed; in particular, the side condition that the premium cannot exceed the capital endowment cannot be violated.

In the final section we deduce that cooperation between insurance takers brings less risk compared to their acting individually. The results of this paper are mainly based on the thesis of Kiesel (2013).

2. One insurance taker and one insurer.

In this section we deal with the endogenous insurance problem (1.2) in the case of one insurance taker and one insurer. In the first subsection we consider the case of unrestricted insurance coverage R.

2.1. Unrestricted contracts.

First we consider the case where one insurance company is willing to cover the initial loss of one insurance taker to any extent $R \in L^p$. Thus the insurance coverage problem can be formulated as follows:

$$\operatorname*{argmin}_{R \in L^p} \varrho(X - R + \pi(R)). \tag{2.1}$$

For a given loss $X \in L^p$ we define the mapping $g(R) := X - R + \pi(R)$. Since the underlying measure $\mathbf{P}$ is a probability measure, the real numbers $\mathbb{R}$ can be regarded as p-integrable constant functions, and $g : L^p \to L^p$.

Obviously the composite function $\Psi := \varrho \circ g$ is proper and convex, thus Fermat's rule applies and gives the following optimality condition:

$$R_0 \text{ is a minimizer of } h \text{ if and only if } 0 \in \partial\Psi(R_0). \tag{2.2}$$

Therefore, it is crucial to describe the subdifferential of the composition Ψ. Since g maps into a Banach space, we need some basic notions of convex analysis in Banach lattices. Some of these notions, like subdifferentials of Banach lattice valued mappings and the necessary definitions and statements including a general chain rule for subdifferentials, are collected in Appendix A.

As noted there, L^p-spaces, $1 < p < \infty$, are conditionally complete Banach lattices with σ-order continuous norm. Thus Theorem A.7 is applicable to the composition function Ψ and we have

$$\partial\Psi(R_0) = \{X^* = A^*[\mu] \mid \mu \in \partial\varrho(g(R_0)),\ A \in \partial g(R_0)\}, \tag{2.3}$$

where $A^*[\mu]$ is the value of the adjoint operator A^* on μ.

Lemma 2.1. *The subdifferential of g at R_0 is given by*

$$\partial g(R_0) = \{A = Y^* - \mathrm{id}_{L^p} \in L(L^p, L^p) \mid Y^* \in \partial\pi(R_0)\}. \tag{2.4}$$

Proof. Consider the right directional derivative of g at R_0, given by

$$\begin{aligned}\mathcal{D}(g, R_0)(R) &= \lim_{\lambda\searrow 0} \frac{X - (R_0 + \lambda R) + \pi(R_0 + \lambda R) - X + R_0 - \pi(R_0)}{\lambda} \\ &= \lim_{\lambda\searrow 0} \frac{-\lambda R + \pi(R_0 + \lambda R) - \pi(R_0)}{\lambda} \\ &= -R + \mathcal{D}(\pi, R_0)(R) = (\mathcal{D}(\pi, R_0) - \mathrm{id}_{L^p})(R).\end{aligned}$$

Then, by Proposition A.4,

$$\begin{aligned}\partial g(R_0) &= \{A \in \mathbf{L}(L^p, L^p) \mid A[R] \le (\mathcal{D}(\pi, R_0) - \mathrm{id}_{L^p})(R),\ \forall R \in L^p\} \\ &= \{A \in \mathbf{L}(L^p, L^p) \mid (A + \mathrm{id}_{L^p})[R] \le \mathcal{D}(\pi, R_0)(R),\ \forall R \in L^p\}.\end{aligned}$$

Hence $A \in \partial g(R_0)$ if and only if $A + \mathrm{id}_{L^p} \in \partial\pi(R_0)$. ∎

Next we determine the adjoint operator A^* of $A \in \partial g(R_0)$.

Lemma 2.2. *Let $Z \in L^q$ and $A \in \partial g(R_0)$. Then the adjoint operator A^* is given by*

$$A^*[Z] = \mathbf{E}[Z] \cdot Y^* - Z, \tag{2.5}$$

where $Y^* \in \partial\pi(R_0)$ is such that $A = Y^* - \mathrm{id}_{L^p}$.

Proof. Note first that $\partial\pi(R_0) \subset (L^p)^* = \mathbf{L}(L^p, \mathbb{R}) \subseteq \mathbf{L}(L^p, L^p)$, because $\mathbb{R}$ can be seen as a subset of L^p. Thus for $Y^* \in \partial\pi(R_0)$ the value $Y^*[Z]$ can be identified with $\langle Z \mid Y^*\rangle$ for the dual pairing of $(L^p, L^q, \langle\cdot \mid \cdot\rangle)$. For $X \in L^p$

we derive

$$\begin{aligned}\langle A[X] \mid Z\rangle &= \langle (Y^* - \mathrm{id}_{L^p})[X] \mid Z\rangle = \langle Y^*[X] - X \mid Z\rangle \\ &= \langle\langle X \mid Y^*\rangle \mid Z\rangle - \langle X \mid Z\rangle = \langle X \mid Y^*\rangle \cdot \langle 1 \mid Z\rangle - \langle X \mid Z\rangle \\ &= \langle X \mid \mathbf{E}[Z] \cdot Y^* - Z\rangle,\end{aligned}$$

which proves the claim. ∎

As a consequence we obtain the following description of the subdifferential of Ψ.

THEOREM 2.3. *The composition function* $\Psi := \varrho \circ g$ *with* $g(R) := X - R + \pi(R)$ *is subdifferentiable and its subdifferential is given by*

$$\partial(\varrho \circ g)(R) = \{X^* \in L^q \mid \exists Z \in \partial\varrho(g(R)),\, Y \in \partial\pi(R) : X^* = \mathbf{E}[Z] \cdot Y - Z\}.$$

This theorem enables us to extend the known characterizations of optimal insurance coverages. As a consequence of Theorem 2.3 and Fermat's rule we obtain the following characterization.

COROLLARY 2.4. $R_0 \in L^p$ *is an optimal insurance coverage of problem* (2.1) *if and only if there exist* $Z \in \partial\varrho(g(R_0))$ *and* $Y \in \partial\pi(R_0)$ *such that*

$$0 = \mathbf{E}[Z] \cdot Y - Z \quad a.s. \tag{2.6}$$

REMARKS 2.5. (a) A sufficient condition for the validity of condition (2.6) is the following. If there exists an insurance coverage R_0 such that

$$0 \in \partial\varrho(g(R_0)), \tag{2.7}$$

then (2.6) holds for every $Y \in \partial\pi(R_0)$. Hence R_0 is then an optimal insurance coverage of (2.1).

For lower semicontinuous convex risk functionals, (2.7) is equivalent to

$$g(R_0) \in \partial\varrho^*(0). \tag{2.8}$$

Under this condition the solutions of the optimization problem

$$\underset{R:\, g(R) \in \partial\varrho^*(0)}{\operatorname{argmin}} \quad \pi(R) \tag{2.9}$$

are optimal insurance coverages which additionally minimize the premium.

If there is no $R_0 \in L^p$ such that (2.7) holds, then we get at least a necessary condition for $Y \in \partial\pi(R)$. Taking the expectation in (2.6) we see that for any optimal insurance coverage R_0 the following has to hold:

$$\mathbf{E}[Y] = 1.$$

(b) For $\varrho(X) := -\mathbf{E}[u(-X)]$ and $\pi(X) := H(X)$, where $u : \mathbb{R} \to \mathbb{R}$ is a risk averse utility function and H is a convex Gâteaux differentiable pricing principle, Corollary 2.4 yields the characterization of optimal insurance contracts R_0 by

$$\nabla H(R_0) = \frac{u'(-X + R - H(R_0))}{\mathbf{E}[u'(-X + R - H(R_0))]}, \tag{2.10}$$

which corresponds to Deprez and Gerber (1985, Theorem 9).

We next consider condition (2.6) for the special class of cash invariant risk functionals as in classical monetary risk measure theory.

PROPOSITION 2.6. *Let $f : L^p \to [0,\infty]$ be a lower semicontinuous, convex, and cash invariant function with $f(0) < \infty$. Then for any $X \in L^p$ the following implication holds:*

$$Y \in \partial f(X) \;\Rightarrow\; \mathbf{E}[Y] = 1.$$

Proof. From the definition of the convex conjugate and the properties of f we derive

$$\begin{aligned} f^*(Y) &= \sup_{X\in L^p} \{\mathbf{E}[XY] - f(X)\} \geq \sup_{c\in\mathbb{R}}\{c\,\mathbf{E}[Y] - f(c)\} \\ &= \sup_{c\in\mathbb{R}}\{c\,\mathbf{E}[Y] - c\} - f(0) = \sup_{c\in\mathbb{R}}\{c(\mathbf{E}[Y] - 1)\} - f(0). \end{aligned}$$

Thus $Y \notin \operatorname{dom}(f^*)$ if $\mathbf{E}[Y] \neq 1$.

Since

$$\{\partial f(X) \mid X \in L^p\} =: \operatorname{range}(\partial f) \subseteq \operatorname{dom}(f^*),$$

the proof is complete. ∎

With this result, for cash invariant risk functionals Corollary 2.4 reads as follows.

COROLLARY 2.7. *If the underlying risk functional ϱ is additionally cash invariant, then R_0 is an optimal insurance coverage of* (2.1) *if and only if there exist $Z \in \partial\varrho(g(R_0))$ and $Y \in \partial\pi(R_0)$ such that*

$$Y = Z \quad a.s. \tag{2.11}$$

As mentioned in the introduction, this statement corresponds to the characterization of an optimal allocation of the minimal total risk problem with respect to ϱ and π in (1.3).

2.2. Restricted contracts. Classical insurance contracts only cover part of the risk and do not allow overinsurance $R > X$ or negative risk increasing parts $R < 0$. The coverage R taken by the insurer is determined by a function I of the initial loss $X \in L^p_+$. The coverage R of the initial loss covered by the insurer is described by $R = I(X)$, and I is called an *insurance contract.*

The set of all admissible insurance contracts is given by

$$\mathcal{I} := \{I : \mathbb{R}_+ \to \mathbb{R}_+ \mid 0 \leq I(x) \leq x,\ \forall x \in \mathbb{R}_+\}. \tag{2.12}$$

For a loss $X \in L^p_+$ the set of its admissible coverages is thus given by

$$\mathcal{R}_{\mathcal{I}}(X) = \mathcal{R} := \{R \in L^p_+ \mid \exists I \in \mathcal{I} : R = I(X)\}. \tag{2.13}$$

With the cost constraint $\pi(R) \le c$, where $c > 0$ represents the maximal amount of money the insurance taker is willing to pay for an insurance, the minimization problem of interest is

$$\underset{R\in\mathcal{R},\, \pi(R)\le c}{\operatorname{argmin}} \ \varrho(X - R + \pi(R)). \tag{2.14}$$

In the classical papers like Deprez and Gerber (1985) this problem is considered for the linear pricing rule

$$\pi(R) := (1+\theta)\,\mathbf{E}[R].$$

In several of the papers mentioned in the introduction it is shown that for law invariant risk measures the stop–loss reinsurance and related contracts are optimal.

In order to apply the Kuhn–Tucker Theorem (see Theorem B.1) to characterize solutions of the optimal insurance problem (2.14) we next establish the closedness of the class $\mathcal{R}$ of insurance claims.

PROPOSITION 2.8. *$\mathcal{R}$ is a convex, closed and bounded subset of L^p.*

Proof. The convexity and boundedness of $\mathcal{R}$ are obvious. To prove its closedness, let $(R_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{R}$ which converges in L^p to some $R \in L^p$. For each $R_k \in \mathcal{R}$ let $I_k \in \mathcal{I}$ be an insurance contract with $I_k(X) = R_k$. By the modified Komlos Lemma as in Delbaen and Schachermayer (1994) there exists a sequence $\widetilde{I}_k \in \operatorname{conv}(I_j : j \ge k)$, $k \in \mathbb{N}$, such that $\widetilde{I}_k \to I$ a.s. We show that $I \in \mathcal{I}$ and $I(X) = R$. Let $(\beta^k_j)_{j\ge0}$ be the corresponding weights with

$$\widetilde{I}_k = \sum_{j\ge0} \beta^k_j I_{k+j}.$$

As $\mathcal{I}$ is convex, we have $\widetilde{I}_k \in \mathcal{I}$ for all $k \in \mathbb{N}$. From the Komlos Lemma we get the non-negativity of I. Further, since $\widetilde{I} \in \mathcal{I}$ it follows that

$$I(X) = \lim_{k\to\infty} \widetilde{I}_k(X) \le X.$$

Thus $I \in \mathcal{I}$. It remains to show that $I(X) = \lim_{k\to\infty} R_k = R \in \mathcal{R}$. This follows from

$$I(X) = \lim_{k\to\infty} \widetilde{I}_k = \lim_{k\to\infty} \sum_{j\ge0} \beta^k_j I_{k+j}(X) = \lim_{k\to\infty} \sum_{j\ge0} \beta^k_j R_{k+j} = R.$$

Hence there exists an insurance contract $I \in \mathcal{I}$ with $I(X) = R$ and thus $R \in \mathcal{R}$. ∎

The Kuhn–Tucker Theorem provides a characterization for solutions of restricted minimization problems with functional side conditions.

Defining $f_1(R) := \pi(R) - c$, we see that for $0 \in \mathcal{R}$ we have $f_1(0) = -c < 0$. Thus the Slater condition (B.15) is fulfilled and by using Theorem B.1 we conclude that R_0 is a minimizer of (2.14) if and only if there exists a Lagrange multiplier $\lambda_1 \in \mathbb{R}_+$ such that

$$0 \in \partial((\varrho \circ g) + \lambda_1 f_1 + \mathbb{1}_{\mathcal{R}})(R_0) \quad \text{with} \quad \lambda_1 f_1(R_0) = 0. \tag{2.15}$$

Here $\mathbb{1}_A(x)$ denotes as usual the convex indicator function (see Appendix B).

If this problem is well-posed, i.e. if

$$\operatorname{domc}(\varrho \circ g) \cap \operatorname{domc}(f_1) \cap \mathcal{R} \neq \emptyset, \tag{2.16}$$

where domc(f) stands for the domain of continuity of the function f,

$$\operatorname{domc}(f) := \{x \mid f \text{ is finite and continuous at } x\},$$

then the subdifferential sum formula (cf. Barbu and Precupanu (1986, Section 3, Theorem 2.6)) is applicable to (2.15) and yields

$$0 \in \partial(\varrho \circ g)(R_0) + \lambda_1 \partial f_1(R_0) + \partial \mathbb{1}_{\mathcal{R}}(R_0). \tag{2.17}$$

Due to Theorem 2.3 we obtain the following Kuhn–Tucker type characterization of optimal insurance coverages.

Theorem 2.9 (Kuhn–Tucker characterization of optimal insurances). *If problem* (2.14) *is well-posed, then R_0 is an optimal insurance coverage of the insurance problem in* (2.14) *if and only if there exist $Z \in \partial\varrho(g(R_0))$, $Y \in \partial\pi(R_0)$, $W \in \partial\mathbb{1}_{\mathcal{R}}(R_0)$ and a Lagrange multiplier $\lambda_1 \geq 0$ such that*

$$0 = -Z + \mathbf{E}[Z]Y + \lambda_1 Y + W, \quad \lambda_1 f_1(R_0) = 0. \tag{2.18}$$

In order to get a better understanding of the preceding statement we describe the subdifferential $\partial\mathbb{1}_{\mathcal{R}}$.

Lemma 2.10. *For $W \in L^q$ we have $W \in \partial\mathbb{1}_{\mathcal{R}}(R_0)$ if and only if*

$$\begin{array}{ll} W \leq 0 & \text{on } A := \{R_0 = 0 \wedge X \neq 0\}, \\ W = 0 & \text{on } B := \{0 < R_0 < X\}, \\ W \geq 0 & \text{on } C := \{R_0 = X \wedge X \neq 0\}, \\ W \text{ is arbitrary} & \text{on } D := \{R_0 = X = 0\}. \end{array} \tag{2.19}$$

Proof. Obviously $\mathbf{P}(A \cup B \cup C \cup D) = 1$. As $\partial\mathbb{1}_{\mathcal{R}}(R_0)$ is defined by

$$\partial\mathbb{1}_{\mathcal{R}}(R_0) = \{W \in L^q \mid \langle W, R - R_0\rangle \leq 0, \forall R \in \mathcal{R}\}$$

the sufficiency of these conditions is clear. For the converse we have to discuss every condition separately. Let $W \in \partial\mathbb{1}_{\mathcal{R}}(R_0)$.

(a) We assume that $W > 0$ on A. Then for $R := X\mathbb{1}_A + R_0\mathbb{1}_{A^c} \in \mathcal{R}$ we conclude

$$\langle W, R - R_0\rangle = \langle W, (X - R_0)\mathbb{1}_A\rangle + \langle W, (R_0 - R_0)\mathbb{1}_{A^c}\rangle = \langle W, X\mathbb{1}_A\rangle > 0,$$

which contradicts $W \in \partial\mathbb{1}_{\mathcal{R}}(R_0)$.

By the same arguments the following insurance coverages produce contradictions to the respective sets:

(b) $R := R_0 \mathbb{1}_{B^c} \in \mathcal{R}$ for $W < 0$ and $R := X\mathbb{1}_B + R_0\mathbb{1}_{B^c} \in \mathcal{R}$ for $W > 0$.

(c) $R := R_0\mathbb{1}_{C^c} \in \mathcal{R}$.

(d) On the set D it is not possible to specify the form of a subgradient W. If $X = 0$ on a certain set U, then every insurance coverage $R \in \mathcal{R}$ has to be zero itself on U. Thus $(R - R_0)\mathbb{1}_D = 0$, which implies $\langle W, R - R_0\rangle = 0$ on D for all $W \in L^q$. ∎

Remark. The undeterminedness on D can be overcome by considering only risks $X \in L^p_+$ with $\mathbf{P}(X > 0) = 1$, which yields $\mathbf{P}(D) = 0$.

Based on Lemma 2.10 we next describe the optimality condition of Theorem 2.9 in a more precise form.

Theorem 2.11. *Let* $\mathbf{P}(X > 0) = 1$. *If* (2.14) *is well-posed, then* R_0 *is an optimal insurance coverage of* (2.14) *if and only if there exist* $Z \in \partial\varrho(g(R_0))$, $Y \in \partial\pi(R_0)$, $W \in \partial\mathbb{1}_{\mathcal{R}}(R_0)$ *and a Lagrange multiplier* $\lambda_1 \geq 0$ *such that*

$$0 \leq -Z + Y(\lambda_1 + \mathbf{E}[Z]) \quad \text{on } A, \tag{2.20}$$

$$0 = -Z + Y(\lambda_1 + \mathbf{E}[Z]) \quad \text{on } B, \tag{2.21}$$

$$0 \geq -Z + Y(\lambda_1 + \mathbf{E}[Z]) \quad \text{on } C, \tag{2.22}$$

$$\lambda_1 f_1(R_0) = 0. \tag{2.23}$$

In order to establish the existence of solutions of (2.14) we next prove the lower semicontinuity of $\varrho \circ g$.

Lemma 2.12. *For* ϱ *lower semicontinuous and a pricing rule* π *the composite function* $\varrho \circ g$ *is lower semicontinuous on* $\mathcal{R}$.

Proof. Let $(R_n)_{n\in\mathbb{N}} \subset \mathcal{R}$ converge in L^p to $R \in \mathcal{R}$. Then from the lower semicontinuity of ϱ and the L^p-continuity of π we get

$$\begin{aligned}\liminf_{n\to\infty} (\varrho \circ g)(R_n) &= \liminf_{n\to\infty} \varrho(g(R_n)) \\ &\geq \varrho\Big(\lim_{n\to\infty} g(R_n)\Big) = \varrho\Big(X - R + \lim_{n\to\infty} \pi(R_n)\Big) \\ &= \varrho(X - R + \pi(R)) = (\varrho \circ g)(R).\end{aligned}$$ ∎

We define the admissible contract set $F := \{R \in L^p_+ \mid \pi(R) \leq c\}$. From the continuity and convexity of π we see that F is closed and convex. We reformulate problem (2.14) as

$$\operatorname*{argmin}_{R\in\mathcal{R}\cap F} (\varrho \circ g)(R). \tag{2.24}$$

Lemma 2.13. *$\mathcal{R} \cap F$ is a closed, bounded and convex subset of L^p_+.*

Proof. Due to Proposition 2.8 and the previous considerations these properties are immediate. ∎

Classical results in functional analysis imply that in reflexive Banach spaces bounded sets are relatively weakly compact. Moreover, the closure of a convex set coincides with its weak closure. Thus convex, closed, and bounded sets in $L^p\,(1 < p < \infty)$ are weakly compact. On the other hand, proper functions defined on a linear normed space are lower semicontinuous if and only if they are weakly lower semicontinuous (cf. Barbu and Precupanu (1986, Chapter 2, Proposition 1.5)).

Thus Lemma 2.13 and the classical Weierstrass Theorem yield the existence of solutions of (2.14).

Theorem 2.14. *Let $1 < p < \infty$. For a lower semicontinuous convex risk functional $\varrho : L^p \to \mathbb{R}_+$, a pricing rule $\pi : L^p \to \mathbb{R}_+$ and an initial loss $X \in L^p_+$ there exists a solution of the optimal insurance problem*

$$\operatorname*{argmin}_{R \in \mathcal{R} \cap F} \varrho(X - R + \pi(R)).$$

3. Several insurance takers and one insurer. In this section we assume that there are n insurance takers with initial losses X_i, $i \in \{1, \dots, n\}$, and risk functionals ϱ_i respectively, and one insurance company with one pricing rule π. We are now interested in optimal insurance coverages occurring under cooperation. The individuals cooperate by forming a coalition and hence pool their initial losses X_i to the total loss $\bar{X} = \sum_{i=1}^n X_i$. Then they buy an insurance contract from the insurance company and redistribute the residual loss back. In this way we view the coalition itself as one individual who intends to insure one initial loss $\bar{X}$. The appropriate mutual risk functional this new individual uses has to reflect the procedure of redistributing the residual losses back to the individuals. This, however, depends on the individual risk functionals and suggests using the infimal convolution

$$\widehat{\varrho}(S) := \inf_{S_i \in L^p :\, \sum_{i=1}^n S_i = S} \sum_{i=1}^n \varrho_i(S_i) \tag{3.1}$$

as a joint risk functional. In the following we assume *exactness* of the infimal convolution $\widehat{\varrho}$, i.e. for any $S \in L^p$ there exist $(S_1, \dots, S_n)$ with $\sum_{i=1}^n S_i = S$ such that $\widehat{\varrho}(S) = \sum_{i=1}^n \varrho_i(S_i)$. This implies its subdifferentiability as well as its lower semicontinuity (see [KR] (2010)). Interior point conditions are known (see [KR] (2010)) which are sufficient for the validity of the epigraph condition. This epigraph condition in turn is equivalent to the exactness of $\widehat{\varrho}$.

3.1. Unrestricted contracts. We define the unrestricted *coalitionary insurance problem* to be

$$\operatorname*{argmin}_{R\in L^p} \widehat{\varrho}(\bar{X} - R + \pi(R)). \tag{3.2}$$

Setting $\bar{g}(R) := \bar{X} - R + \pi(R)$ we define

DEFINITION 3.1. A tuple $(R_0, S_1, \ldots, S_n) \in L^p_{n+1}$ with $\sum_{i=1}^n S_i = \bar{g}(R_0)$ is called a *coalitional solution* of the unrestricted coalitionary insurance problem (3.2) if

- R_0 solves (3.2),
- $(S_1, \ldots, S_n)$ minimizes $\widehat{\varrho}(\bar{g}(R_0))$.

An immediate consequence of the characterization of optimal allocations (see [KR] (2010), Theorem 3.1) and Corollary 2.4 is:

COROLLARY 3.2. *If $\widehat{\varrho}$ is exact and well-posed, then the tuple $(R_0, S_1, \ldots, S_n) \in L^p_{n+1}$ with $\sum_{i=1}^n S_i = \bar{g}(R_0)$ is a coalitional solution of problem* (3.2) *if and only if there exist $Z \in \bigcap_{i=1}^n \partial\varrho_i(S_i)$ and $Y \in \partial\pi(R_0)$ such that*

$$0 = \mathbf{E}[Z] \cdot Y - Z. \tag{3.3}$$

As a consequence of Corollary 3.2 we recover the known characterizations of coalitional solutions for cash invariant risk functionals ϱ and Gâteaux differentiable price functionals π.

REMARKS 3.3. (a) If there exist $k \in \{1, \ldots, n\}$ such that ϱ_k is cash invariant, then R_0 solves (3.2) if and only if

$$\partial\widehat{\varrho}(\bar{g}(R_0)) \cap \partial\pi(R_0) \neq \emptyset.$$

Similarly $(R_0, S_1, \ldots, S_n)$ with $\sum_{i=1}^n S_i = \bar{g}(R_0)$ is a coalitional solution of (3.2) if and only if

$$\bigcap_{i=1}^n \partial\varrho_i(S_i) \cap \partial\pi(R_0) \neq \emptyset. \tag{3.4}$$

(b) If additionally to (a) the pricing rule π is Gâteaux differentiable, then R_0 solves (3.2) if and only if

$$\nabla\pi(R_0) \in \partial\widehat{\varrho}(\bar{g}(R_0)).$$

Similarly $(R_0, S_1, \ldots, S_n)$ with $\sum_{i=1}^n S_i = \bar{g}(R_0)$ is a coalitional solution of (3.2) if and only if

$$\nabla\pi(R_0) \in \bigcap_{i=1}^n \partial\varrho_i(S_i). \tag{3.5}$$

For lower semicontinuous risk functionals ϱ_i the latter is equivalent to

$$S_i \in \partial\varrho_i^*(\nabla\pi(R_0)), \quad \forall i \in \{1, \ldots, n\}. \tag{3.6}$$

3.2. Restricted contracts. As in Section 2.2, we restrict the minimization problem (3.2) to the admissible insurance coverages

$$\bar{\mathcal{R}} = \mathcal{R}(\bar{X}) := \{R \in L^p_+ \mid \exists I \in \mathcal{I} : R = I(\bar{X})\}. \tag{3.7}$$

Using similar arguments to those for Theorem 2.14 we obtain the corresponding existence result for the restricted coalitional insurance problem. With $\bar{X} = \sum_{i=1}^n X_i$ and $\bar{c} = \sum_{i=1}^n c_i$ we obtain

COROLLARY 3.4. *For lower semicontinuous convex risk functionals $\varrho_i : L^p \to \mathbb{R}_+$, $1 < p < \infty$, such that $\widehat{\varrho}$ is exact, a pricing rule $\pi : L^p \to \mathbb{R}_+$ and initial losses $X_i \in L^p$, there exists an optimal insurance coverage of the problem*

$$\operatorname*{argmin}_{R \in \bar{\mathcal{R}}, \pi(R) \le \bar{c}} \widehat{\varrho}(\bar{X} - R + \pi(R)). \tag{3.8}$$

Proof. Due to the exactness of $\widehat{\varrho}$ and its lower semicontinuity, this follows as in the proof of Theorem 2.14. ■

Further by the arguments in Subsection 2.2 we obtain

COROLLARY 3.5. *Let* $\mathbf{P}(X > 0) = 1$. *If* (3.8) *is well-posed, then R_0 is an optimal insurance coverage of* (2.14) *if and only if there exist $Z \in \partial\widehat{\varrho}(g(R_0))$, $Y \in \partial\pi(R_0)$ and a Lagrange multiplier $\lambda_1 \ge 0$ such that*

$$\begin{aligned}
&0 \le -Z + Y(\lambda_1 + \mathbf{E}[Z]) && \text{on } A,\\
&0 = -Z + Y(\lambda_1 + \mathbf{E}[Z]) && \text{on } B,\\
&0 \ge -Z + Y(\lambda_1 + \mathbf{E}[Z]) && \text{on } C,\\
&\lambda_1 f_1(R_0) = 0.
\end{aligned}$$

Combining this with Corollary 3.2 we get the following Kuhn–Tucker type characterization of restricted coalitional solutions.

COROLLARY 3.6 (Kuhn–Tucker characterization of coalitional solutions). *If in the situation of Corollary* 3.5 *additionally $\widehat{\varrho}$ is exact and well-posed, then the tuple $(R_0, S_1, \dots, S_n) \in \mathcal{R} \times L^p_n$ with $\sum_{i=1}^n S_i = g(R_0)$ is a coalitional solution of the restricted problem* (3.8) *if and only if there exist $Z \in \bigcap_{i=1}^n \partial\varrho_i(S_i)$, $Y \in \partial\pi(R_0)$ and a Lagrange multiplier $\lambda_1 \ge 0$ such that the inequalities in Corollary* 3.5 *hold.*

4. Whether to act individually or cooperatively. In the context of a group of n insurance takers and one insurance company the natural question arises whether individual or cooperative insurance contracts bring a lower minimal total risk in restricted models. We will see in the following that cooperation brings less risk and therefore is preferable.

The total minimization problem where every individual acts alone is given by

$$\sum_{i=1}^{n} \underset{\substack{R_i \in \mathcal{R}(X_i) \\ \pi(R_i)-c_k \le 0}}{\operatorname{argmin}} \ \varrho_i(X_i - R_i + \pi(R_i)). \tag{4.1}$$

The objective function is the sum of the objective functions of the corresponding individual insurance problems in (2.14). Aiming at comparing (4.1) with the coalitional insurance problem in (3.8) we introduce the following notation. This notation aims to include the side conditions of the corresponding minimization problems into the minimization sets.

For each individual $i \in \{1, \dots, n\}$ the set of *extended insurance contracts* for individual insurance problem is defined by

$$\widetilde{\mathcal{I}}_i := \{I : \mathbb{R}_+ \to \mathbb{R}_+ \mid 0 \le I(x) \le x,\ \forall x \in \mathbb{R}_+,\ \pi(I(x)) \le c_i\},\ i \in \{1, \dots, n\}.$$

The set of *extended contracts* for insurance coverages is given by

$$\widetilde{\mathcal{R}}_i = \widetilde{\mathcal{R}}_i(X_i) := \{R \in L_+^p \mid \exists I \in \widetilde{\mathcal{I}}_i : R = I(X_i)\},\ i \in \{1, \dots, n\}.$$

Additionally we denote the corresponding sets of *residual losses after insurance* by

$$\mathcal{L}_i = \mathcal{L}_i(X_i) := \{L \mid \exists R \in \widetilde{\mathcal{R}}_i : L = X_i - R + \pi(R)\},\ i \in \{1, \dots, n\}.$$

The sets of *extended coalitional contracts and losses* corresponding to the cooperative insurance problem are defined by

$$\begin{aligned}
\widetilde{\mathcal{I}} &:= \{I : \mathbb{R}_+ \to \mathbb{R}_+ \mid 0 \le I(x) \le x,\ \forall x \in \mathbb{R}_+,\ \pi(I(x)) \le \bar{c}\},\\
\widetilde{\mathcal{R}} &= \widetilde{\mathcal{R}}(\bar{X}) := \{R \in L_+^p \mid \exists I \in \widetilde{\mathcal{I}} : R = I(\bar{X})\},\\
\mathcal{L} &= \mathcal{L}(\bar{X}) := \{L \mid \exists R \in \widetilde{\mathcal{R}} : L = \bar{X} - R + \pi(R)\},
\end{aligned}$$

with $\bar{X} = \sum_{i=1}^n X_i$ and $\bar{c} = \sum_{i=1}^n c_i$. Furthermore,

$$\mathcal{A}_{\mathcal{L}} = \mathcal{A}_{\mathcal{L}(\bar{X})} := \left\{(L_1, \dots, L_n) \in (L_+^p)^n \,\middle|\, \exists L \in \mathcal{L} : \sum_{i=1}^n L_i = L\right\}$$

denotes the set of all admissible redistributions of the cooperative insurance problem. For $(L_1, \dots, L_n) \in \mathcal{A}_{\mathcal{L}}$ the component L_i reflects the part of $L \in \mathcal{L}(\bar{X})$ that is reassigned to individual i. And

$$\mathcal{A}(M) := \left\{(M_1, \dots, M_n) \in (L_+^p)^n \,\middle|\, \sum_{i=1}^n M_i = M\right\}$$

is called the set of all admissible allocations of $M \in L_+^p$.

Proposition 4.1. *The value of the individual insurance problem* (4.1) *is identical to*

$$\sum_{i=1}^{n} \inf_{K_i \in \mathcal{L}_i(X_i)} \varrho_i(K_i).$$

The value of the coalitional insurance problem (3.8) *is identical to*

$$\inf_{(M_i)_i \in \mathcal{A}_{\mathcal{L}}} \sum_{i=1}^{n} \varrho_i(M_i).$$

Proof. We only show the second equality. The first one follows similarly. For the value of problem (3.8) we have

$$\inf_{\substack{R \in \overline{\mathcal{R}} \\ \pi(R) \le \bar{c}}} \widehat{\varrho}(\bar{g}(R)) = \inf_{R \in \widetilde{\mathcal{R}}} \widehat{\varrho}(\bar{g}(R)) = \inf_{R \in \widetilde{\mathcal{R}}} \inf \Big\{ \sum_{i=1}^{n} \varrho_i(M_i) \,\Big|\, (M_i)_i \in \mathcal{A}(\bar{g}(R)) \Big\}$$

$$= \inf_{(M_i)_i \in \mathcal{A}_{\mathcal{L}}} \sum_{i=1}^{n} \varrho_i(M_i). \blacksquare$$

For subadditive pricing rules we have the following relation between individual and cooperative residual losses.

Proposition 4.2. *Let π be a subadditive pricing rule. Then for every $K = (K_1, \dots, K_n) \in \times_{i=1}^{n} \mathcal{L}_i$ there exists an $L \in \mathcal{L}(\bar{X})$ such that*

$$\sum_{i=1}^{n} K_i \ge L \quad \textit{a.s.}$$

Proof. Let $K_i \in \mathcal{L}_i(X_i)$. Then there exists an $R_i \in \widetilde{\mathcal{R}}_i(X_i)$ such that $K_i = X_i - R_i + \pi(R_i)$ with $\pi(R_i) \le c_i$. From the subadditivity of π we conclude that

$$\sum_{i=1}^{n} K_i \ge \bar{X} - \sum_{i=1}^{n} R_i + \pi\Big(\sum_{i=1}^{n} R_i \Big) =: L.$$

Obviously $R_0 := \sum_{i=1}^{n} R_i \in \widetilde{\mathcal{R}}(\bar{X})$ and thus $L \in \mathcal{L}(\bar{X})$. $\blacksquare$

Now we are ready to state the main result of this section.

Theorem 4.3. *The value of the individual insurance problem dominates the value of the coalitional insurance problem, i.e.*

$$\sum_{i=1}^{n} \inf_{R_i \in \widetilde{\mathcal{R}}_i} \varrho_i(X_i - R_i + \pi(R_i)) \ge \inf_{R \in \widetilde{\mathcal{R}}} \widehat{\varrho}(\bar{X} - R + \pi(R)). \tag{4.2}$$

Proof. The infimal convolution $\widehat{\varrho}$ inherits the monotonicity with respect to the almost sure order from the risk functionals ϱ_i (cf. Acciaio (2007)). Let $K = (K_1, \dots, K_n) \in \times_{i=1}^{n} \mathcal{L}_i$. Then we know from Proposition 4.2 that there exists an $L \in \mathcal{L}(\bar{X})$ such that $\sum_{i=1}^{n} K_i \ge L$ a.s. From the inclusion

$\mathcal{A}(L) \subseteq \mathcal{A}_{\mathcal{L}}$, we conclude that

$$\sum_{i=1}^{n} \varrho_i(K_i) \geq \inf_{(L_i)_i \in \mathcal{A}(\sum K_i)} \sum_{i=1}^{n} \varrho_i(L_i) \geq \inf_{(L_i)_i \in \mathcal{A}(L)} \sum_{i=1}^{n} \varrho_i(L_i)$$
$$\geq \inf_{(L_i)_i \in \mathcal{A}_{\mathcal{L}}} \sum_{i=1}^{n} \varrho_i(L_i).$$

Since this holds for all $(K_1, \ldots, K_n) \in \bigtimes_{i=1}^{n} \mathcal{L}_i(X_i)$ we get

$$\sum_{i=1}^{n} \inf_{K_i \in \mathcal{L}_i(X_i)} \varrho_i(K_i) \geq \inf_{(L_i)_i \in \mathcal{A}_{\mathcal{L}}} \sum_{i=1}^{n} \varrho_i(L_i),$$

and the claim follows from Proposition 4.1. ∎

Appendix A. Subdifferentiability of Banach lattice valued mappings. In this section we collect some notions and results on subdifferentiability of Banach lattice valued mappings as used in Sections 2–4 of this paper. Let $(Y, \leq)$ be a Banach lattice. We assume throughout that Y is conditionally (or Dedekind) complete, i.e. every subset $A \subset Y$ which is bounded above has a least upper bound $y_0 = \sup A$. In particular reflexive Banach lattices, like $L^p, 1 < p < \infty$, are conditionally complete.

Let $F : X \to Y$ be a convex mapping from the Banach space X to Y and denote the directional derivative in x_0 in direction x by

$$\mathcal{D}(F, x_0)(x) := \lim_{h \searrow 0} \frac{F(x_0 + hx) - F(x_0)}{h}. \tag{A.1}$$

Then conditional completeness of Y implies

Proposition A.1.

$$\mathcal{D}(F, x_0)(x) \in Y \quad \textit{for all } x_0, x \in X. \tag{A.2}$$

Proof. For $h > 0$ the difference quotient

$$g(x_0, x, h) := \frac{F(x_0 + hx) - F(x_0)}{h}$$

lies in Y for any $x_0, x \in X$. Moreover it is increasing in h. Let $h_1 < h_2$. Then the convexity of F gives

$$\begin{aligned} F(x_0 + h_1 x) - F(x_0) &= F\left(\frac{h_1}{h_2} x_0 + \left(1 - \frac{h_1}{h_2}\right) x_0 + \frac{h_1}{h_2} h_2 x\right) - F(x_0) \\ &= F\left(\frac{h_1}{h_2}(x_0 + h_2 x) + \left(1 - \frac{h_1}{h_2}\right) x_0\right) - F(x_0) \\ &\leq \frac{h_1}{h_2} F(x_0 + h_2 x) + \left(1 - \frac{h_1}{h_2}\right) F(x_0) - F(x_0) \\ &= \frac{h_1}{h_2} F(x_0 + h_2 x) + \frac{h_1}{h_2} F(x_0). \end{aligned}$$

This is equivalent to

$$\frac{F(x_0 + h_1 x) - F(x_0)}{h_1} \leq \frac{F(x_0 + h_2 x) - F(x_0)}{h_2}.$$

Thus $g(x_0, x, h)$ decreases as $h \searrow 0$, and

$$\mathcal{D}(F, x_0)(x) = \inf_{h>0} g(x_0, x, h). \tag{A.3}$$

Set $x_0 := \frac{1}{1+h}(x_0 + hx) + \frac{h}{1+h}(x_0 - x)$. Then the convexity of F yields

$$F(x_0) \leq \frac{1}{1+h} F(x_0 + hx) + \frac{h}{1+h} F(x_0 - x),$$

which implies that for all $h > 0$,

$$F(x_0) - F(x_0 - x) \leq \frac{F(x_0 + hx) - F(x_0)}{h}.$$

Thus $g(x_0, x, h)$, $h > 0$, are bounded from below by $-g(x_0, x, -1)$, and conditional completeness of Y implies the existence of the element $\mathcal{D}(F, x_0)(x)$ in Y. ∎

The subdifferential of a Banach lattice valued mapping is defined analogously to the real case.

Definition A.2. The subdifferential of the convex mapping $F : X \to Y$ at $x_0 \in X$ is defined by

$$\partial F(x_0) := \{A \in \mathbf{L}(X, Y) \mid A(x - x_0) \leq F(x) - F(x_0) \ \forall x \in X\}. \tag{A.4}$$

Here $\mathbf{L}(X, Y)$ stands for the set of all continuous linear operators on X with values in Y.

We next collect some useful results stated in Ioffe and Levin (1972) concerning right directional derivatives and subdifferentials.

Proposition A.3 (Continuity of right directional derivative). *The right directional derivative $x \mapsto \mathcal{D}(F, x_0)(x)$ at $x_0 \in X$ is sublinear. If $F : X \to Y$ is additionally continuous at x_0 then $\mathcal{D}(F, x_0)(\cdot)$ is a continuous mapping from X to Y.*

Proposition A.4. *$\partial F(x_0) = \partial(\mathcal{D}(F, x_0))(0)$.*

In lattices the concept of order convergence can be introduced in a natural way. Therefore, when speaking of an increasing sequence $(y_n)_{n\in\mathbb{N}}$ we understand that $y_1 \leq y_2 \leq \cdots$.

Definition A.5. A sequence $(y_n)_{n\in\mathbb{N}}$ in a Banach lattice Y is called *order convergent* to $y_0 \in Y$ ($y_n \xrightarrow{(o)} y_0$ or $y_0 = (o)\text{-}\lim_{n\to\infty} y_n$) if there exist two monotonic sequences in Y: one decreasing, (x_n), and the other increasing, (z_n), such that

- $\sup(z_n) = y_0 = \inf(x_n)$,
- $z_n \le y_n \le x_n$ for all $n \in \mathbb{N}$.

Following Ioffe and Levin (1972) and Vulikh (1967) a Banach lattice Y is said to have *property* (**A**) if:

(**A**) Every decreasing sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ with $y_n \xrightarrow{(o)} 0$ converges in norm, $\|y_n\|_Y \to 0$.

Proposition A.6 (Compactness of subdifferentials). *Let Y have property* (**A**). *Further let $G \subset X$ and $U \subset Y$ be open convex sets and $F : G \to U$ be a continuous convex mapping. Then for $x_0 \in G$ the subdifferential $\partial F(x_0)$ is a non-empty convex set that is compact in the weak operator topology of $\mathbf{L}(X, Y)$.*

The following theorem is a subdifferential chain rule for the composition of a real valued function and a Banach lattice valued mapping.

Theorem A.7 (Chain rule for subdifferentials). *Let $F : G \to U$ be a continuous convex mapping, $G \subset X$ and $U \subset Y$ be open convex sets, where X is a Banach space and Y is a conditionally complete Banach lattice with property* (**A**) *and let ϱ be a monotonic convex real valued function on U. Then for the composition $\Psi := \varrho \circ F$ and $x_0 \in G$ we have*

$$\partial\Psi(x_0) = \{x^* = A^*[\mu] \mid \mu \in \partial\varrho(F(x_0)),\ A \in \partial F(x_0)\}, \tag{A.5}$$

where A^* denotes the adjoint operator of A.

To study the subdifferential sum formula for Banach lattice valued mappings we rely on the following results in Kusraev and Kutateladze (1995). These authors introduce a concept of *general position* which guarantees the subdifferential sum formula, similarly to the interior point conditions in the real case (see [KR] (2010)).

Let X and Y be two topological vector spaces and Φ be a subset of the product $X \times Y$. Then Φ is called a *correspondence* from X to Y. We define its *domain* $\mathrm{dom}(\Phi)$ and *image* $\mathrm{im}(\Phi)$ by

$$\mathrm{dom}(\Phi) := \{x \in X \mid \exists y \in Y : (x, y) \in \Phi\},$$
$$\mathrm{im}(\Phi) := \{y \in Y \mid \exists x \in X : (x, y) \in \Phi\}.$$

For $U \subset X$ the correspondence $\Phi \cap (U \times Y)$ is called the restriction of Φ to U and is denoted by $\Phi\restriction U$. The set $\Phi(U) := \mathrm{im}(\Phi\restriction U)$ is called the image of U under Φ and we have

$$\Phi(x) := \Phi(\{x\}) = \{y \in Y \mid (x, y) \in \Phi\},$$
$$\mathrm{dom}(\Phi) = \{x \in X \mid \Phi(x) \ne \emptyset\},$$
$$\Phi(U) = \{\Phi(x) \mid x \in U\} = \{y \in Y \mid \exists x \in U : y \in \Phi(x)\}.$$

Definition A.8. A correspondence $\Phi \subset X \times Y$ is called

- *convex* if Φ is a convex subset of $X \times Y$,
- *conic* if Φ is a cone in $X \times Y$,
- *open* at $(x_0, y_0) \in \Phi$ if for every neighborhood U of x_0 the set $\Phi(U) - y_0$ is a neighborhood of the origin in Y. For $x_0 = 0$ and $y_0 = 0$ we speak of openness at the origin.

Definition A.9. Consider two cones K_1 and K_2 in the topological space X and put $\kappa := (K_1, K_2)$. We say κ is a *non-oblate pair* if the conic correspondence $\Phi_\kappa \subset X^2 \times X$ defined by

$$\Phi_\kappa := \{(k_1, k_2, x) \in X^2 \times X \mid x = k_1 - k_2,\ k_i \in K_i,\ i = 1, 2\} \tag{A.6}$$

is open at the origin.

Thus openness of the correspondence Φ_κ in the definition above (or non-oblateness of the pair K_1, K_2) means that for every neighborhood $V \subset X$ of the origin in X the set

$$\Phi_\kappa(V^2) = V \cap K_1 - V \cap K_2$$

is a neighborhood of the origin in X.

The following is a useful characterization of non-oblateness. Let $\Delta_n : x \mapsto (x, \ldots, x)$ denote the embedding of X into the diagonal $\Delta_n(X)$ of the space X^n.

Lemma A.10 (Characterization of non-oblate pairs). *A pair of cones $\kappa := (K_1, K_2)$ is non-oblate if and only if the pair $\lambda := (K_1 \times K_2, \Delta_2(X))$ is non-oblate in X^2.*

Definition A.11. We say that the cones K_1 and K_2 are *in general position* if the following three conditions are satisfied:

- K_1 and K_2 reproduce (algebraically) some subspace $X_0 \subseteq X$, i.e. $X_0 = K_1 - K_2 = K_2 - K_1$.
- The subspace X_0 is complemented, i.e. there exists a continuous projection $\pi : X \to X$ such that $\pi(X) = X_0$.
- (K_1, K_2) is a non-oblate pair in X.

Let $\sigma_n : (X \times Y)^n \to X^n \times Y^n$ denote the natural isomorphism between $(X \times Y)^n$ and $X^n \times Y^n$ defined by the rearrangement of the coordinates

$$\sigma^n : ((x_1, y_1), \ldots, (x_n, y_n)) \mapsto ((x_1, \ldots x_n), (y_1, \ldots, y_n)).$$

Definition A.12. We say that sublinear operators $\pi_1, \ldots, \pi_n : X \to Y$, with $\mathrm{dom}(\pi_i) \subseteq X$, are *in general position* if the sets $\Delta_n(X) \times Y^n$ and $\sigma_n(\mathrm{epi}(\pi_1) \times \cdots \times \mathrm{epi}(\pi_n))$ are in general position.

Theorem A.13 (Subdifferential sum formula). *Let X be a Banach space and let Y be a conditionally complete Banach lattice. If sublinear operators $\pi_1, \ldots, \pi_n : X \to Y$ are in general position, then the following subdifferential sum formula holds at zero:*

$$\partial\Big(\sum_{i=1}^{n} \pi_i\Big)(0) = \sum_{i=1}^{n} \partial\pi_i(0).$$

Appendix B. Minimization of convex functions. Let $f : E \to \overline{\mathbb{R}}$ be a proper convex function on a locally convex topological vector space E. Convex analysis then gives important tools for minimization problems. For general background we refer to Barbu and Precupanu (1986). We give a collection of results related to Fermat's rule which are used throughout the text.

For $x \in \mathrm{dom}(\partial f)$ we have

$$x^* \in \partial f(x) \;\Leftrightarrow\; \langle x^* \mid x\rangle - f(x) = \sup_{y\in E}(\langle x^* \mid y\rangle - f(y)). \tag{B.1}$$

In case $x^* = 0$ this equivalence becomes

$$0 \in \partial f(x) \;\Leftrightarrow\; f(x) = \inf_{y\in E} f(y). \tag{B.2}$$

Thus x is a (global) minimizer of f if and only if *Fermat's rule*

$$0 \in \partial f(x) \tag{B.3}$$

is valid. If f is furthermore lower semicontinuous, then by the Fenchel–Moreau theorem we get the equivalence

$$0 \in \partial f(x) \;\Leftrightarrow\; x \in \partial f^*(0). \tag{B.4}$$

Thus in this case $\partial f^*(0)$ represents the set of all minimizers of f.

Fermat's rule for restricted minimization problems. Minimization problems are seldom globally defined. Thus the question arises what Fermat's rule looks like in the case of restricted minimization problems

$$\inf_{x\in A} f(x), \tag{B.5}$$

where $A \subseteq E$ is a closed convex subset and f is a proper function on E. For such a set A, $\mathbb{1}_A$ denotes the *convex indicator* function

$$\mathbb{1}_A(x) := \begin{cases} 0, & x \in A,\\ \infty, & x \notin A.\end{cases} \tag{B.6}$$

With this notation, (B.5) can be equivalently expressed by

$$\inf_{x\in E}(f(x) + \mathbb{1}_A(x)), \tag{B.7}$$

and Fermat's condition reads now

$$0 \in \partial(f(x) + \mathbb{1}_A(x)). \tag{B.8}$$

In the context of restricted minimization problems we generally assume that there exists at least one $x \in A$ where f is continuous and finite. The *domain of continuity* of f is defined by

$$\operatorname{domc}(f) := \{x \in E \mid f \text{ is finite and continuous in } x\}. \tag{B.9}$$

Then the minimization problem (B.5) is called *well-posed* for $A \subseteq E$ if

$$\operatorname{domc}(f) \cap A \neq \emptyset. \tag{B.10}$$

By the subdifferential sum formula as in Barbu and Precupanu (1986, Chapter 3) the right hand side of (B.8) yields

$$0 \in \partial f(x) + \partial \mathbb{1}_A(x). \tag{B.11}$$

Thus $x \in A$ is a minimizer of (B.5) if and only if there exists $v \in \partial \mathbb{1}_A(x)$ such that $-v \in \partial f(x)$. For the indicator function $\mathbb{1}_A$ the definition of the subdifferential yields

$$\partial \mathbb{1}_A(x) = \{x^* \in E^* \mid \langle x^* \mid x - y\rangle \geq 0 \text{ for all } y \in A\}. \tag{B.12}$$

This is the *normal cone* $N_A(x)$ to the set A at a point $x \in A$; it consists of all vectors which are perpendicular to half-spaces that support A at x. It is a closed convex cone with vertex at the origin and we get the following two properties:

- $\operatorname{dom}(\partial \mathbb{1}_A) = A$,
- $\partial \mathbb{1}_A(x) = \{0\}$ for $x \in \operatorname{int} A$.

Fermat's rule under a functional side condition. Here we consider a functional form of the preceding restricted minimization problem. Let again E be a Banach space paired with its dual space E^* by $(E, E^*, \langle \cdot \mid \cdot \rangle)$. Let $f_i : E \to \overline{\mathbb{R}}$, $i \in \{0, \dots, n\}$, be convex functions and $A \subset E$ be a closed convex subset. Then we consider the minimization problem

$$\inf\{f_0(x) \mid x \in A,\ f_i(x) \leq 0,\ i \in \{1, \dots, n\}\}. \tag{B.13}$$

Such problems can be solved by using the Lagrangian function

$$\mathcal{L}(x, \lambda_1, \dots, \lambda_n) := \sum_{i=0}^{n} \lambda_i f_i(x) + \mathbb{1}_A(x). \tag{B.14}$$

Then the Kuhn–Tucker Theorem, stated in the version of Ioffe and Tikhomirov (1979, Chapter 1.1.2), provides necessary conditions for $x \in A$ to be a solution to problem (B.13). If the *Slater condition*

$$\exists\, x \in A \text{ such that } f_i(x) < 0 \text{ for all } i \in \{1, \dots, n\} \tag{B.15}$$

is fulfilled, the above mentioned necessary conditions are sufficient as well.

THEOREM B.1 (Kuhn–Tucker Theorem). *Let* $f_i : E \to \overline{\mathbb{R}}$, $i \in \{0, \dots, n\}$, *be convex functions and* $A \subset E$ *be a convex set. If there is a* $y \in A$ *which solves problem* (B.13), *then there exist Lagrangian multipliers* $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ *such that*

$$\mathcal{L}(y, \lambda_0, \dots, \lambda_n) = \min_{x \in A} \mathcal{L}(x, \lambda_0, \dots, \lambda_n) \tag{B.16}$$

and

$$\lambda_i f_i(y) = 0 \quad \text{for } i \in \{1, \dots, n\}. \tag{B.17}$$

If the Slater condition (B.15) *holds true, then* $\lambda_0 \neq 0$ *and we can assume* $\lambda_0 = 1$. *In the latter case conditions* (B.16) *and* (B.17) *are sufficient for* y *to minimize* (B.13).

Again we assume that the minimization problem is well-posed. In this context this means

$$\bigcap_{i=0}^{n} \operatorname{domc}(\lambda_i f_i) \cap A \neq \emptyset.$$

Thus Fermat's rule and the subdifferential sum formula yield under the assumption of the Slater condition that $x \in A$ solves problem (B.13) if and only if there exists a weight vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \setminus \{0\}$ such that

$$0 \in \partial f_0(x) + \sum_{i=1}^{n} \lambda_i \partial f_i(x) + \partial \mathbb{1}_A(x),$$
$$\lambda_i f_i(x) = 0 \quad \text{for } i \in \{1, \dots, n\}.$$

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