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SOME REMARKS ON THE CONTROL OF FALSE DISCOVERY RATE UNDER DEPENDENCE

Abstract. We investigate controlling false discovery rate (FDR) under dependence. Our main result is a generalization of the results obtained by Genovese and Wasserman (2004) and Farcomeni (2007).

1. Introduction. We consider a multiple testing procedure in which m tests are being performed simultaneously. Suppose that m_0 of the null hypotheses are true and $m-m_0$ are false. The *false discovery proportion* (FDP) is defined to be the proportion of erroneously rejected null hypotheses:

$$FDP = \begin{cases} V/R & \text{if } R > 0, \\ 0 & \text{if } R = 0, \end{cases}$$

where V is the number of erroneously rejected null hypotheses and R it the total number of rejected hypotheses in the multiple testing procedure. Benjamini and Hochberg (1995) defined the *False Discovery Rate* (FDR) to be the expectation value of the FDP:

$$FDR = \mathbb{E}(FDP).$$

Multiple testing procedures which control FDR have good power even when thousands of hypotheses are tested simultaneously, especially in modern biology applications. Benjamini and Hochberg (1995) introduced the BH procedure which guarantees control of FDR for independent test statistics. Genovese and Wasserman (2004) showed that, asymptotically, the BH procedure corresponds to a fixed threshold method that rejects all p-values less than a threshold u^*. Recently there have been many new results extending the BH procedure to classes of dependent test statistics (see Benjamini and Yekutieli (2001), Sarkar (2002), Storey (2002), Farcomeni (2007)). Under

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the assumption that the p-values are independent Genovese and Wasserman (2004) formulated asymptotic results on controlling *FDR* by using methods from the theory of stochastic processes. Some progress has been achieved by Farcomeni (2007) in the case when the p-values satisfy some dependent models including mixing and associated dependence. Wu (2008) extended these results when the null hypotheses (H_i) are 0-1 valued stationary processes, and, given (H_i), the p-values are independent. In our study the null hypotheses (H_i) are i.i.d. Bernoulli random variables. Additionally, we assume that the p-values satisfy some weak dependence model (see (4.10)) which is not covered by Farcomeni (2007).

The paper is organized as follows. In Section 2 we present the mixture model of simultaneous testing of hypotheses. In Section 3 we present an overview of known asymptotic results on controlling *FDR* and we give a generalization of those results in the mixture model. In Section 4 we give a new dependence model for test statistics (p-values).

2. Model. Suppose that data $\mathbb{X}$ come from some probability distribution $\mathbb{P} \in \Omega$, where Ω is the set of all available probability distributions. On the base of $\mathbb{X}$ we are testing m hypotheses simultaneously $H_i : \mathbb{P} \in \omega_i$ or $H_i' : \mathbb{P} \notin \omega_i$ for $i = 1, \ldots, m$. We assume that we test the hypothesis H_i versus H_i' based on a statistic T_i. Let $(K_\alpha)_\alpha$ be a given family of rejection sets for H_i such that:

(i) $K_\alpha \subseteq K_\beta$ for $\alpha \le \beta$,
(ii) $\mathbb{P}(T_i \in K_\alpha) = \alpha$ for all $\mathbb{P} \in \omega_i$, $i = 1, \ldots, m$.

Then *p-value* for H_i is defined by

$$p_i(T) = \inf\{\alpha : T_i \in K_\alpha\}.$$

The multitesting procedure *controls FDR at level* α if

$$\mathbb{E}_{\mathbb{P}}(FDP) \le \alpha \quad \text{for all } \mathbb{P} \in \Omega.$$

A most popular framework for *FDR* is the mixture model (Storey (2002)).

2.1. Mixture model. We assume that the null hypotheses H_i, $1 \le i \le m$, are i.i.d. Bernoulli random variables and

$$\mathbb{P}(H_i = 0) = 1 - \pi \tag{2.1}$$

for some $0 < \pi < 1$. We write $H_i = 0$ if the null hypothesis H_i is true and $H_i = 1$ if it is false. In particular, we assume that the 2-dimensional random vectors (p_i, H_i) for $i = 1, \ldots, m$ are i.i.d. and such that

$$\mathbb{P}(p_i \le t \mid H_i = 0) = t, \tag{2.2}$$

$$\mathbb{P}(p_i \le t \mid H_i = 1) = F(t), \tag{2.3}$$

for $t \in [0,1]$, where F is the distribution function of the p-value under the alternative hypothesis. Then the marginal distribution function of p_i has the form

$$G(t) = \pi t + (1-\pi)F(t) \quad \text{for } t \in [0,1]. \tag{2.4}$$

3. Asymptotic control of ***FDR*****.** We define the following stochastic process:

$$\Gamma_m(t) = \frac{\sum_{i=1}^m \mathbf{1}\{p_i \le t\}(1-H_i)}{\sum_{i=1}^m \mathbf{1}\{p_i \le t\} + \mathbf{1}\{p_{(1)} > t\}}$$

for $t \in [0,1]$, where $p_{(1)} = \min\{p_1, \ldots, p_m\}$.

Storey (2002) showed that in the mixture model for any $t > 0$,

$$\mathbb{E}(FDP) = \mathbb{E}(\Gamma_m(t)) = Q(t)(1-(1-G(t))^m),$$

where

$$Q(t) = (1-\pi)\frac{t}{G(t)}.$$

First, we assume that π is known. Let

$$T_{PI} = \sup\{0 \le t \le 1 : Q_m(t) \le \alpha\},$$

where

$$Q_m(t) = (1-\pi)\frac{t}{G_m(t)}, \qquad G_m(t) = \frac{1}{m}\sum_{i=1}^m \mathbf{1}\{p_i \le t\}.$$

In the mixture model Genovese and Wasserman (2004) obtained

$$\mathbb{E}(\Gamma_m(T_{PI})) = \alpha + o(1) \quad \text{as } m \to \infty.$$

In the case where π is unknown, we use an estimator $\hat{\pi}$ and

$$\hat{Q}_m(t) = (1-\hat{\pi})\frac{t}{G_m(t)}, \qquad \hat{T} = \sup\{0 \le t \le 1 : \hat{Q}_m(t) \le \alpha\}.$$

Genovese and Wasserman (2004) showed that if G is concave and

$$\hat{\pi} \xrightarrow{\mathbb{P}} \pi_0 < \pi,$$

then

$$\mathbb{E}(\Gamma_m(\hat{T})) \le \alpha + o(1) \quad \text{as } m \to \infty.$$

A discussion of identifiability of the parameter π appears in Genovese and Wasserman (2004). Various methods of estimating π can be found in Langaas and Lindqvist (2005). We mention the following estimator:

$$\hat{\pi} = \left(\frac{G_m(s) - s}{1-s}\right)_+$$

for some $s \in (0,1)$, where $a_+ = \max(a,0)$. Storey (2002) proved that if $G(s) > s$, then

$$\hat{\pi} \xrightarrow{\mathbb{P}} \frac{G(s)-s}{1-s}$$

and

$$\sqrt{m}\biggl(\hat{\pi} - \frac{G(s)-s}{1-s}\biggr) \xrightarrow{d} N\biggl(0, \frac{G(s)(G(s)-s)}{(1-s)^2}\biggr)$$

as $m \to \infty$ (see also Genovese and Wasserman (2004, Proposition 3.2)).

3.1. Some generalization of the mixture model. We assume that the (H_i) are i.i.d. Bernoulli random variables satisfying (2.1) and the p-values (p_j) satisfy (2.2)–(2.3), and come from some weak dependence model. Let

$$\Lambda_{0,m}(t) = \frac{1}{m}\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{p_i \le t\},$$
$$\Lambda_{1,m}(t) = \frac{1}{m}\sum_{i=1}^{m}H_i\mathbf{1}\{p_i \le t\}.$$

We consider the space $L^\infty([0,1])$ of all uniformly bounded, real functions z on $[0,1]$ with uniform norm

$$\|z\|_\infty = \sup_{t\in[0,1]} |z(t)|.$$

Our basic assumption is

(D) $$\sqrt{m}(\Lambda_{0,m}(t) - (1-\pi)t, \Lambda_{1,m}(t) - \pi F(t)) \rightsquigarrow (Z_1(t), Z_2(t))$$

in $L^\infty([0,1]) \times L^\infty([0,1])$ as $m \to \infty$, where (Z_1, Z_2) is a mean zero Gaussian process with bounded covariance kernel

(3.1) $$K_{ij}(s,t) = \mathrm{Cov}(Z_i(s), Z_j(t)) \quad \text{for } i,j = 0,1.$$

Lemma 1. *Under condition* (D), *we have*

$$W_m(t) := \sqrt{m}(\Gamma_m(t) - Q(t)) \rightsquigarrow Z(t)$$

as $m \to \infty$ for $t \in [\delta, 1]$ for some $\delta > 0$, where Z is a mean zero Gaussian process with covariance kernel

$$K(s,t) = \frac{\pi^2 F(s)F(t)}{G^2(s)G^2(t)}K_{11}(s,t) - \frac{\pi(1-\pi)F(t)s}{G^2(s)G^2(t)}K_{12}(s,t)$$
$$- \frac{\pi(1-\pi)F(s)t}{G^2(s)G^2(t)}K_{21}(s,t) + \frac{(1-\pi)^2 st}{G^2(s)G^2(t)}K_{22}(s,t),$$

where K_{ij} are defined by (3.1).

Proof. This follows immediately by a similar reasoning to the one in Genovese and Wasserman (2004, proof of Theorem 4.2). ∎

REMARK 1. Condition (D) has been obtained by Genovese and Wasserman (2004) in the case where the p-values (p_i) are independent, and by Farcomeni (2007) for various dependent models for (p_i).

The theorem below can be obtained as in Genovese and Wasserman (2004, proof of Theorem 5.1).

THEOREM 1. *In the case where π is known, condition* (D) *implies*

$$\mathbb{E}(\Gamma_m(T_{PI})) = \alpha + o(1) \quad \text{as } m \to \infty. \tag{3.2}$$

4. Dependence model of p-values. Let the p-values be of the form

$$p_j = G(\ldots, \eta_{j-1}, \eta_j, \eta_{j+1}, \ldots), \tag{4.1}$$

where (η_j) are i.i.d. and $G : \mathbb{R}^\infty \to [0,1]$ is a measurable function. Let $\mathcal{F}_i := (\ldots, \eta_{i-1}, \eta_i)$

$$\mathcal{P}_k(\xi_i^{s,t}) := \mathbb{E}(\xi_i^{s,t} \mid \mathcal{F}_k) - \mathbb{E}(\xi_i^{s,t} \mid \mathcal{F}_{k-1}),$$
$$\|\mathcal{P}_0(\xi_i)\| := \sqrt{\mathbb{E}(\mathcal{P}_0(\xi_i))^2},$$

where

$$\xi_i = \xi_i^{0,t}, \qquad \xi_i^{s,t} = \mathbf{1}\{s \le p_i \le t\},$$

for $s, t \in [0,1]$. Now, we give conditions which imply (D).

LEMMA 2. *If the hypotheses (H_i) are i.i.d. Bernoulli random variables, the p-values (p_i) have the form* (4.1), *and*

$$\sum_{i=1}^{\infty} \|\mathcal{P}_0(\xi_i^{s,t})\| \le C d(s,t) \tag{a}$$

for all $s, t \in (\delta, 1]$ for some $\delta > 0$ and some constant $C > 0$, where $d(s,t)$ is a pseudo-metric on $[0,1]$ such that the space $((\delta, 1], d)$ is totally bounded, then (D) *holds.*

Proof. Let

$$A_m(t) := \sqrt{m}(\Lambda_{0,m}(t) - (1-\pi)t),$$
$$B_m(t) := \sqrt{m}(\Lambda_{1,m}(t) - \pi F(t)).$$

By weak convergence theory (see Van der Vaart and Wellner (1996, p. 42)) it is sufficient to check asymptotic tightness of the processes $A_m(t)$ and $B_m(t)$ and finite-dimensional convergence: for all $l \in \mathbb{N}$ and all $t_1, \ldots, t_l \in [0,1]$,

$$\begin{aligned}(A_m(t_1), B_m(t_1), A_m(t_2), B_m(t_2), \ldots, A_m(t_l), B_m(t_l)) \\ \rightsquigarrow (Z_1(t_1), Z_2(t_1), Z_1(t_2), Z_2(t_2), \ldots, Z_1(t_l), Z_2(t_l)).\end{aligned} \tag{4.2}$$

Since the space $((\delta, 1], d)$ is totally bounded, the processes A_m and B_m are asymptotically tight if $A_m(t)$ and $B_m(t)$ are tight in $\mathbb{R}$ and the processes A_m and B_m are asymptotically uniformly d-equicontinuous in probability (see Van der Vaart and Wellner (1996, Th. 1.5.7, p. 37)). Asymptotic tightness in $\mathbb{R}$ of $A_m(t)$ and $B_m(t)$ is trivial from (4.2). Asymptotic uniform d-equicontinuity in probability of A_m and B_m will follow once we prove the following conditions:

(i) for all m and for all $s, t \in (\delta, 1]$ for some $\delta > 0$,
$$\|A_m(t) - A_m(s)\| \le Cd(s,t)$$
for some constant $C > 0$,

(ii) for all m and for all $s, t \in (\delta, 1]$ for some $\delta > 0$,
$$\|B_m(t) - B_m(s)\| \le Cd(s,t)$$
for some constant $C > 0$

(see Furmańczyk (2008, Lemma 3.1 for $Q = 2$, p. 135)).

We will deduce those conditions from condition (a). Indeed, we may assume that $s < t$. Obviously
$$\mathcal{P}_k\Big(\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{s \le p_i \le t\}\Big) = \sum_{i=1}^{m}\mathcal{P}_k((1-H_i)\mathbf{1}\{s \le p_i \le t\}).$$
From the triangle inequality, we have
$$\Big\|\mathcal{P}_k\Big(\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{s \le p_i \le t\}\Big)\Big\| \le \sum_{i=1}^{m}\|\mathcal{P}_k((1-H_i)\mathbf{1}\{s \le p_i \le t\})\|.$$
Observe that
$$\begin{aligned}
&\mathbb{E}(\mathcal{P}_k((1-H_i)\mathbf{1}\{s \le p_i \le t\}))^2 \\
&\quad = \mathbb{E}\big((\mathcal{P}_k((1-H_i)\mathbf{1}\{s \le p_i \le t\}))^2 \bigm| H_i = 0\big)\mathbb{P}(H_i = 0) \\
&\qquad + \mathbb{E}\big((\mathcal{P}_k((1-H_i)\mathbf{1}\{s \le p_i \le t\}))^2 \bigm| H_i = 1\big)\mathbb{P}(H_i = 1) \\
&\quad = \mathbb{E}\big((\mathcal{P}_k(\mathbf{1}\{s \le p_i \le t\}))^2 \bigm| H_i = 0\big)\mathbb{P}(H_i = 0) \\
&\quad \le \mathbb{E}\big((\mathcal{P}_k(\mathbf{1}\{s \le p_i \le t\}))^2 \bigm| H_i = 0\big)\mathbb{P}(H_i = 0) \\
&\qquad + \mathbb{E}\big((\mathcal{P}_k(\mathbf{1}\{s \le p_i \le t\}))^2 \bigm| H_i = 1\big)\mathbb{P}(H_i = 1) \\
&\quad = \mathbb{E}(\mathcal{P}_k(\mathbf{1}\{s \le p_i \le t\}))^2.
\end{aligned}$$
From stationarity of (p_i), we have
$$\|\mathcal{P}_k(\xi_i^{s,t})\| = \|\mathcal{P}_0(\xi_{i-k}^{s,t})\|.$$
Therefore from (a) we get
$$\Big\|\mathcal{P}_k\Big(\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{s \le p_i \le t\}\Big)\Big\| \le \sum_{i=1}^{m}\|\mathcal{P}_k(\xi_i^{s,t})\| = \sum_{i=1}^{m}\|\mathcal{P}_0(\xi_{i-k}^{s,t})\| \le Cd(s,t) \tag{4.3}$$

for some constant $C > 0$. Since $(\mathcal{P}_k)$ are orthogonal, from (4.3) we have

$$\begin{aligned}
&\Big\|\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{s \le p_i \le t\} - m(1-\pi)(t-s)\Big\|^2 \\
&\qquad = \Big\|\sum_{k=-\infty}^{\infty}\mathcal{P}_k\Big(\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{s \le p_i \le t\} - m(1-\pi)(t-s)\Big)\Big\|^2 \\
&\qquad = \sum_{k=-\infty}^{\infty}\Big\|\mathcal{P}_k\Big(\sum_{i=1}^{m}(1-H_i)\mathbf{1}\{s \le p_i \le t\}\Big)\Big\|^2 \\
&\qquad \le Cd(s,t)\sum_{k=-\infty}^{\infty}\sum_{i=1}^{m}\|\mathcal{P}_0(\xi_{i-k}^{s,t})\| \le C^2 m d^2(s,t).
\end{aligned}$$

Hence we have (i). Similarly we obtain (ii).

Now, we show (4.2). By the Cramer–Wald theorem the finite-dimensional convergence (4.2) holds if for any $a_i, b_i \in \mathbb{R}$ and for fixed $t_i \in [0,1]$ for $i = 1, \dots, l$ the random variable

$$L_m := \sum_{i=1}^{l}(a_i A_m(t_i) + b_i B_m(t_i))$$

is convergent to a normal distribution $N(0, \sigma^2)$, where

$$\begin{aligned}
\text{(4.4)}\qquad \sigma^2 = &\sum_{i,j=1}^{l} a_i a_j K_{11}(t_i, t_j) + \sum_{i,j=1}^{l} b_i b_j K_{22}(t_i, t_j) \\
&+ \sum_{i,j=1}^{l} a_i b_j K_{12}(t_i, t_j) + \sum_{i,j=1}^{l} a_j b_i K_{21}(t_i, t_j),
\end{aligned}$$

and (K_{ij}) are defined in (3.1). Therefore (4.2) holds if

$$\text{(4.5)}\qquad \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\sum_{j=1}^{l}(\tilde{\xi}_{i,j} - E(\tilde{\xi}_{i,j})) \xrightarrow{d} N(0, \sigma^2) \quad \text{as } m \to \infty,$$

where

$$\text{(4.6)}\qquad \tilde{\xi}_{i,j} := (a_j + (b_j - a_j)H_i)\mathbf{1}\{p_i \le t_j\}.$$

Let

$$\tilde{\pi}_{1,j} := E(\tilde{\xi}_{i,j}) = (a_j + (b_j - a_j)\pi)G(t).$$

Since

$$\sum_{i=-\infty}^{\infty}\Big\|\mathcal{P}_0\Big(\sum_{j=1}^{l}\tilde{\xi}_{i,j}\Big)\Big\| \le \sum_{j=1}^{l}\sum_{i=-\infty}^{\infty}\Big\|\mathcal{P}_0\Big(\sum_{j=1}^{l}\tilde{\xi}_{i,j}\Big)\Big\|,$$

reasoning as in Lemma 1 (see Wu (2008)) we find that the condition

$$\sum_{i=-\infty}^{\infty} \|\mathcal{P}_0(\tilde{\xi}_{i,j})\| < \infty \tag{4.7}$$

for $j = 1, \ldots, l$ implies

$$\Big\|\sum_{i=1}^{m} \tilde{\xi}_i - m\tilde{\pi}_1 - M_m\Big\|^2 = o(m)$$

as $m \to \infty$, where $\tilde{\xi}_i = \sum_{j=1}^{l} \tilde{\xi}_{i,j}$, $\tilde{\pi}_1 = \sum_{j=1}^{l} \tilde{\pi}_{1,j}$, and $M_m = \sum_{k=1}^{m} D_k$ is a martingale with respect to $(\mathcal{F}_k)$, because the processes

$$D_k = \sum_{i=-\infty}^{\infty} \mathcal{P}_k(\tilde{\xi}_i) \tag{4.8}$$

are martingale differences with respect to $(\mathcal{F}_k)$. By the central limit theorem for martingales we have (4.5) for $\sigma = \|D_k\|$. On the other hand, (4.6)–(4.8) imply that σ has the form (4.4). Similarly to Wu (2008) we show that conditions (4.12)–(4.13) imply (4.7). Let $\xi_i := \xi_i^{0,t} = \mathbf{1}\{p_i \le t\}$. Then

$$\begin{aligned}
\mathbb{E}(\mathcal{P}_0(\tilde{\xi}_i))^2 &= \mathbb{E}\big((\mathcal{P}_0(\tilde{\xi}_i))^2 \mid H_i = 0\big)\mathbb{P}(H_i = 0)\\
&\quad + \mathbb{E}\big((\mathcal{P}_0(\tilde{\xi}_i))^2 \mid H_i = 1\big)\mathbb{P}(H_i = 1)\\
&= \mathbb{E}\big((a^2\mathcal{P}_0(\xi_i))^2 \mid H_i = 0\big)\mathbb{P}(H_i = 0)\\
&\quad + \mathbb{E}\big((b^2\mathcal{P}_0(\xi_i))^2 \mid H_i = 1\big)\mathbb{P}(H_i = 1)\\
&\le \max(a^2, b^2)\mathbb{E}(\mathcal{P}_0(\xi_i))^2,
\end{aligned}$$

and consequently

$$\|\mathcal{P}_0(\tilde{\xi}_i)\| \le \max(|a|, |b|)\|\mathcal{P}_0(\xi_i)\|. \tag{4.9}$$

From (a) for $s = 0$ we obtain

$$\sum_{i=-\infty}^{\infty} \|\mathcal{P}_0(\xi_i)\| < \infty,$$

which implies (4.7) and (4.2). ■

4.1. Linear process. We consider a special model of (4.1), where the p-value p_j is a function of a linear process,

$$p_j = g\Big(\sum_{r=-\infty}^{\infty} a_r \eta_{j-r}\Big), \tag{4.10}$$

where $g : \mathbb{R} \to [0, 1]$ is measurable such that $g \in C^1(\mathbb{R})$, $g'(x) \neq 0$ and

$$\int_s^t |(g^{-1}(u))'|\, du < Cd(s, t) \quad \text{for all } s, t \in (\delta, 1] \text{ for some } \delta > 0, \tag{4.11}$$

and for some constant $C > 0$, (η_i) are i.i.d. with bounded and Lipschitz marginal density f_η, and

$$E(\eta_1)^2 < \infty. \tag{4.12}$$

We assume additionally that the sequence of coefficients of the linear process satisfies

$$\sum_{r=-\infty}^{\infty} |a_r| < \infty. \tag{4.13}$$

LEMMA 3. *Under the mixture model, if the hypotheses* (H_i) *are i.i.d. Bernoulli random variables, and* p_j *is of the form* (4.10) *satisfying conditions* (4.11)–(4.13), *then* (4.5) *holds.*

Proof. From Lemma 2 it is sufficient to show condition (a). We may assume $s < t$. Let $\psi_i := \sum_{r=-\infty}^{\infty} a_r\eta_{i-r} - a_i\eta_0$ and η_0' be an independent copy of η_0. Let $p_i' := g(\psi_i + a_i\eta_0')$.

Observe that

$$\mathcal{P}_0(\xi_i^{s,t}) = \mathbb{E}\big(\mathbf{1}\{s \le p_i \le t\} - \mathbf{1}\{s \le p_i' \le t\} \mid \mathcal{F}_0\big). \tag{4.14}$$

Let $\Lambda_i := \sum_{r=i+1}^{\infty} a_r\eta_{i-r}$, $X := \sum_{r=-\infty}^{i-1} a_r\eta_{i-r}$, $Y := a_i\eta_0'$. Then

$$\mathcal{P}_0(\xi_i^{s,t}) = \mathbb{P}_X(s \le g(X + a_i\eta_0 + \Lambda_i) \le t) - \mathbb{P}_{X+Y}(s \le g(X+Y+\Lambda_i) \le t),$$

where $\mathbb{P}_X$ denotes the probability measure of the random variable X. Under the regularity conditions on g, we have

$$\mathbb{P}_X(s \le g(X + a_i\eta_0 + \Lambda_i) \le t) = \int_s^t f_{h_i(X)}(u)\,du,$$

where $f_{h_i(X)}$ is the density of the random variable $h_i(X) := g(X + a_i\eta_0 + \Lambda_i)$ with respect to $\mathbb{P}_X$ for given $a_i\eta_0 + \Lambda_i$, and

$$\mathbb{P}_{X+Y}(s \le g(X+Y+\Lambda_i) \le t) = \int_s^t f_{r_i(X+Y)}(u)\,du,$$

where $f_{r_i(X+Y)}$ is the density of $r_i(X+Y) := g(X+Y+\Lambda_i)$ with respect to $\mathbb{P}_{X+Y}$ for given Λ_i. Moreover

$$f_{h_i(X)}(u) = f_X(g^{-1}(u) - a_i\eta_0 - \Lambda_i)|(g^{-1}(u))'|,$$

where f_X is the density of X, and from the independence of X and Y we get

$$f_{r_i(X+Y)}(u) = f_X * f_Y(g^{-1}(u) - \Lambda_i)|(g^{-1}(u))'|,$$

where $*$ stands for convolution. Therefore

$$|\mathcal{P}_0(\xi_i^{s,t})| = \Big|\int_{-\infty}^{\infty}\Big(\int_s^t (f_X(u_i - a_i\eta_0) - f_X(u_i - y))|(g^{-1}(u))'|\,du\Big) f_Y(y)\,dy\Big|,$$

where $u_i := g^{-1}(u) - \Lambda_i$. Since f_η is bounded and Lipschitz, so is f_X, and

$$|\mathcal{P}_0(\xi_i^{s,t})| \le Cd(s,t)\mathrm{Lip}(f_X) \int_{-\infty}^{\infty} (|a_i\eta_0| + |y|) f_Y(y)\, dy$$
$$\le Cd(s,t)\mathrm{Lip}(f_X)(|a_i\eta_0| + E|Y|).$$

Then

$$\|\mathcal{P}_0(\xi_i^{s,t})\| \le C'd(s,t)\mathrm{Lip}(f_X)|a_i|$$

for some constant $C' > 0$. Therefore from (4.13) we obtain (a), which ends the proof. ∎

Example 1. The condition (4.11) holds for the logistic transformation

$$g(x) = \frac{\exp(x)}{1 + \exp(x)}.$$

In this case

$$\int_s^t |(g^{-1}(u))'|\, du = \int_s^t \frac{1}{u(1-u)}\, du = \ln(t) - \ln(s) + \ln(1-t) - \ln(1-s)$$
$$\le \big|\ln(t/s)\big|$$

and the metric has the form $d(s,t) = |\ln(t/s)|$.

When π is known, from Theorem 1 and from Lemmas 3 and 4 we obtain

Corollary 1. *Under the mixture model, if the hypotheses* (H_i) *are i.i.d. Bernoulli random variables, and* p_j *is of the form* (4.10) *satisfying conditions* (4.11)–(4.13), *then* (3.2) *holds.*

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