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ON THE SEMILOCAL CONVERGENCE OF A TWO-STEP NEWTON-LIKE PROJECTION METHOD FOR ILL-POSED EQUATIONS

Abstract. We present new semilocal convergence conditions for a two-step Newton-like projection method of Lavrentiev regularization for solving ill-posed equations in a Hilbert space setting. The new convergence conditions are weaker than in earlier studies. Examples are presented to show that older convergence conditions are not satisfied but the new conditions are satisfied.

1. Introduction. Let X be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $U(x, R)$ and $\overline{U(x, R)}$, stand respectively, for the open and closed ball in X with center x and radius $R > 0$. Let also $L(X)$ be the space of all bounded linear operators from X into itself.

In this study we are concerned with the problem of approximately solving the ill-posed equation

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to X$ is a nonlinear operator satisfying $\langle F(v) - F(w), v - w \rangle \geq 0$ for all $v, w \in D(F)$, and $y \in X$.

It is assumed that (1.1) has a solution, say $\hat{x}$, and F possesses a locally uniformly bounded Fréchet derivative $F'(x)$ for all $x \in D(F)$ (cf. [18]) i.e.,

$$\|F'(x)\| \leq C_F, \quad x \in D(F),$$

for some constant C_F.

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In applications, usually only noisy data y^δ are available, such that

$$\|y - y^\delta\| \le \delta.$$

Then the problem of recovering $\hat{x}$ from the noisy equation $F(x) = y^\delta$ is ill-posed, in the sense that a small perturbation in the data can cause a large deviation in the solution. To solve (1.1) with monotone operators (see [12, 17, 18, 19]) one usually uses the Lavrentiev regularization method. In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = y^\delta. \tag{1.2}$$

It is known (cf. [19, Theorem 1.1]) that (1.2) has a unique solution x_α^δ for $\alpha > 0$, provided F is Fréchet differentiable and monotone in the ball $U(\hat{x}, r) \subset D(F)$ with radius $r = \|\hat{x} - x_0\| + \delta/\alpha$. However the regularized equation (1.2) remains nonlinear and one may have difficulties in solving it numerically.

In [6], George and Elmahdy considered an iterative regularization method which converges linearly to x_α^δ, and its finite-dimensional realization in [7]. Later in [8] they considered an iterative regularization method which converges quadratically to x_α^δ, and its finite-dimensional realization in [9].

Recall that a sequence (x_n) in X with $\lim x_n = x^*$ is said to be *convergent of order* $p > 1$ if there exist positive reals β, γ such that for all $n \in \mathbb{N}$, $\|x_n - x^*\| \le \beta e^{-\gamma p^n}$. If the sequence (x_n) has the property that $\|x_n - x^*\| \le \beta q^n$ with some $0 < q < 1$ then (x_n) is said to be *linearly convergent*. For an extensive discussion of convergence rates see [13].

Note that the method of [6]–[9] uses a suitably constructed majorizing sequence which heavily depends on the initial guess and hence is not suitable for practical considerations.

Recently, George and Pareth [10] introduced a two-step Newton-like projection method (TSNLPM) of convergence order four to solve (1.2). (TSNLPM) was realized as follows:

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X. Our aim in this section is to obtain an approximation for x_α^δ in the finite-dimensional space $R(P_h)$, the range of P_h. For the results that follow, we impose the following conditions.

Let

$$\varepsilon_h(x) := \|F'(x)(I - P_h)\|, \qquad \forall x \in D(F),$$

and pick $\{b_h : h > 0\}$ such that $\lim_{h\to 0} \|(I - P_h)x_0\|/b_h = 0$ and $\lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h(x) \to 0$ as $h \to 0$ for all $x \in D(F)$. The above assumption is satisfied if $P_h \to I$ pointwise and if $F'(x)$ is a compact operator. Further we assume that $\varepsilon_h(x) \le \varepsilon_0$ for all $x \in D(F)$, $b_h \le b_0$ and $\delta \in (0, \delta_0]$.

1.1. Projection method. We consider the sequences defined iteratively by

$$y_{n,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)], \tag{1.3}$$

$$x_{n+1,\alpha}^{h,\delta} = y_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(y_{n,\alpha}^{h,\delta})P_h[F(y_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n,\alpha}^{h,\delta} - x_0)], \tag{1.4}$$

where $R_\alpha(x) := P_hF'(x)P_h + \alpha P_h$ and $x_{0,\alpha}^{h,\delta} := P_hx_0$, to obtain an approximation for x_α^δ in the finite-dimensional subspace $R(P_h)$ of X. Note that the iterations (1.3) and (1.4) are the finite-dimensional realizations of the iteration (1.3) and (1.4) in [16]. In [10], the parameter $\alpha = \alpha_i$ was chosen from some finite set

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$$

using the adaptive method considered by Perverzev and Schock [17].

The convergence analysis in [10] was carried out using the following assumptions.

Assumption 1 (cf. [18], Assumption 3). There exists a constant $k_0 \geq 0$ such that for every $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x,u,v) \in X$ such that $[F'(x) - F'(u)]v = F'(u)\Phi(x,u,v)$ and $\|\Phi(x,u,v)\| \leq k_0\|v\|\,\|x-u\|$.

Assumption 2. There exists a continuous, strictly increasing function $\varphi : (0,a] \to (0,\infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying:

(i) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$,
(ii) $\sup_{\lambda\geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq c_\varphi\varphi(\alpha)$ for all $\lambda \in (0,a]$,
(iii) there exists $v \in X$ with $\|v\| \leq 1$ such that (cf. [14])

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v.$$

In the present paper we extend the applicability of (TSNLPM) by weakening Assumption 1 which is very difficult to verify (or does not hold) in general. In particular, we replace Assumption 1 by the weaker and easier to verify:

Assumption 3. Let $x_0 \in X$ be fixed. There exists a constant $K_0 \geq 0$ such that for each $x, u \in D(F)$ and $v \in X$ there exists an element $\Phi(x,u,v) \in X$ depending on x_0 such that $[F'(x) - F'(u)]v = F'(u)\Phi(x,u,v)$ and $\|\Phi(x,u,v)\| \leq K_0\|v\|(\|x - P_hx_0\| + \|u - P_hx_0\|)$.

Note that Assumption 1 $\Rightarrow$ Assumption 3 but not necessarily vice versa: at the end of the study we provide examples where Assumption 3 is satisfied but not Assumption 1.

We also replace Assumption 2 by

Assumption 4. There exists a continuous, strictly increasing function $\varphi : (0,a] \to (0,\infty)$ with $a \geq \|F'(x_0)\|$ satisfying:

(i) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$,
(ii) $\sup_{\lambda\geq 0}\frac{\alpha\varphi(\lambda)}{\lambda+\alpha} \leq \varphi(\alpha)$ for all $\lambda \in (0, a]$,
(iii) there exists $v \in X$ with $\|v\| \leq 1$ such that (cf. [14])

$$x_0 - \hat{x} = \varphi(F'(x_0))v.$$

REMARK 1.1. The hypotheses of Assumption 1 may not hold or may be very expensive or impossible to verify in general. In particular, as is the case for well-posed nonlinear equations, the computation of the Lipschitz constant k_0, even if this constant exists, is very difficult. Moreover, there are classes of operators for which Assumption 1 is not satisfied but (TSNLPM) converges.

In this paper, we extend the applicability of (TSNLPM) under smaller computational cost. Let us explain how we achieve this goal.

(1) Assumption 3 is weaker than Assumption 1 (see Examples 5.1 and 5.2).
(2) The computational cost of the constant K_0 is smaller than that of the constant k_0, even when $K_0 = k_0$.
(3) The sufficient convergence criteria are weaker.
(4) The computable error bounds on the distances involved (including K_0) are less costly.
(5) The convergence domain of (TSNLPM) with Assumption 3 can be larger, since K_0/k_0 can be arbitrarily small (see Example 5.3).
(6) The information on the location of the solution is more precise.
(7) Note that Assumption 2 involves the Fréchet derivative at the exact solution $\hat{x}$ which is unknown in practice, while Assumption 4 depends on the Fréchet derivative of F at x_0.

The paper is organized as follows: In Section 2 we present the convergence analysis of (TSNLPM). Section 3 contains the error analysis and parameter choice strategy. The algorithm for implementing (TSNLPM) is given in Section 4. Finally, examples are presented in the concluding Section 5.

2. Semilocal convergence. In order for us to present the semilocal convergence of (TSNLPM) it is convenient to introduce some parameters:

Let

$$e^{h,\delta}_{n,\alpha} := \|y^{h,\delta}_{n,\alpha} - x^{h,\delta}_{n,\alpha}\|, \qquad \forall n \geq 0. \tag{2.1}$$

Suppose that

$$0 < K_0 < \frac{1}{4(1+\varepsilon_0/\alpha_0)}, \tag{2.2}$$

$$\frac{4\delta_0}{\alpha_0}(1+\varepsilon_0/\alpha_0) < 1. \tag{2.3}$$

Define a polynomial P on $(0,\infty)$ by

$$P(t) = (1+\varepsilon_0/\alpha_0)\frac{K_0}{2}t^2 + (1+\varepsilon_0/\alpha_0)t + \frac{\delta_0}{\alpha_0} - \frac{1}{4(1+\varepsilon_0/\alpha_0)}. \tag{2.4}$$

It follows from (2.3) that P has a unique positive root given in closed form by the quadratic formula. Denote this root by p_0.

Let

$$b_0 < p_0, \qquad \|\hat{x} - x_0\| \le \rho, \tag{2.5}$$

where

$$\rho < p_0 - b_0, \tag{2.6}$$

$$\gamma_\rho := (1+\varepsilon_0/\alpha_0)\left[\frac{k_0}{2}(\rho+b_0)^2 + (\rho+b_0)\right] + \frac{\delta_0}{\alpha_0}, \tag{2.7}$$

$$r := \frac{4\gamma_\rho}{1+\sqrt{1+32\gamma_\rho(1+\varepsilon_0/\alpha_0)}}, \tag{2.8}$$

$$b := 4(1+\varepsilon_0/\alpha_0)K_0 r. \tag{2.9}$$

Then by (2.2)–(2.9) we have

$$0 < \gamma_\rho < 1/4, \tag{2.10}$$

$$0 < r < 1, \tag{2.11}$$

$$0 < b < 1. \tag{2.12}$$

Indeed, by (2.4) and (2.12) we have $\gamma_\rho - 1/4 \le P(p_0) = 0$, so $0 < \gamma_\rho < 1/4$, which is (2.10). Estimate (2.11) follows from (2.8) and (2.10). Moreover, estimate (2.12) follows from (2.2) and (2.11). We also have

$$\gamma_\rho < r. \tag{2.13}$$

In view of (2.7) and (2.8), estimate (2.13) reduces to showing that $4\gamma_\rho(1+\varepsilon_0/\alpha_0) < 1$, which is true by the choice of p_0 and (2.4). Finally it follows from (2.13) that

$$0 < \gamma_\rho < 1. \tag{2.14}$$

Lemma 2.1 ([10, Lemma 1]). *Let $x \in D(F)$. Then*

$$\|R_\alpha^{-1}(x)P_h F'(x)\| \le 1 + \varepsilon_0/\alpha_0.$$

Lemma 2.2 ([10, Lemma 2]). *Let $e_0 = e_{0,\alpha}^{h,\delta}$ and γ_ρ be as in (2.7). Then $e_0 \le \gamma_\rho$.*

The proofs below follow along the lines of the corresponding ones in [10]. However, they differ when the weaker Assumption 3 is used in place of Assumption 1.

LEMMA 2.3. *Suppose that* (2.2), (2.3) *and* $\delta \in (0, \delta_0]$ *hold and let Assumption* 3 *be satisfied. Then the following estimates hold for* $(TSNLPM)$:

(a) $$\|x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}\| \le \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)[3\|x_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\| + 5\|y_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\|]e_{n-1,\alpha}^{h,\delta},$$

(b) $$\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\| \le \left\{1 + \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)[3\|x_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\| + 5\|y_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\|]\right\}e_{n-1,\alpha}^{h,\delta}.$$

Proof. Observe that

$$\begin{aligned}
(2.15)\quad & x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta} \\
&= y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta} - R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h[F(y_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n-1,\alpha}^{h,\delta} - x_0)] \\
&\quad + R_\alpha^{-1}(x_{n-1,\alpha}^{h,\delta})P_h[F(x_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta} \\
&\quad - R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h[F(y_{n-1,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}) + \alpha(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})] \\
&\quad + [R_\alpha^{-1}(x_{n-1,\alpha}^{h,\delta}) - R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})]P_h[F(x_{n-1,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n-1,\alpha}^{h,\delta} - x_0)] \\
&= R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h[F'(y_{n-1,\alpha}^{h,\delta})(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}) - (F(y_{n-1,\alpha}^{h,\delta}) - F(x_{n-1,\alpha}^{h,\delta}))] \\
&\quad + R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h(F'(y_{n-1,\alpha}^{h,\delta}) - F'(x_{n-1,\alpha}^{h,\delta}))(x_{n-1,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}) \\
&=: \Gamma_1 + \Gamma_2.
\end{aligned}$$

Note that

$$\|\Gamma_1\| = \Big\|R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h\int_0^1 [F'(y_{n-1,\alpha}^{h,\delta}) - F'(x_{n-1,\alpha}^{h,\delta} + t(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}))] \times (y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})\,dt\Big\|.$$

Using now Assumption 3 for $x = x_{n-1,\alpha}^{h,\delta} + t(y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta})$, $u = y_{n-1,\alpha}^{h,\delta}$, $v = x_{n-1,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}$, $x_0 = x_{0,\alpha}^{h,\delta}$ we get

$$(2.16)\quad \|\Gamma_1\| \le \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)[\|x_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\| + 3\|y_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\|]e_{n-1,\alpha}^{h,\delta}$$

the last step follows from Assumption 3 and Lemma 2.1. Similarly,

$$(2.17)\quad \|\Gamma_2\| \le K_0(1+\varepsilon_0/\alpha_0)[\|y_{n-1,\alpha}^{h,\delta} - x_{0,\alpha}^{h,\delta}\| + \|x_{0,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|]e_{n-1,\alpha}^{h,\delta}.$$

So, (a) follows from (2.15)–(2.17). And (b) follows from (a) and the triangle inequality

$$\|x_{n,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\| \le \|x_{n,\alpha}^{h,\delta} - y_{n-1,\alpha}^{h,\delta}\| + \|y_{n-1,\alpha}^{h,\delta} - x_{n-1,\alpha}^{h,\delta}\|.$$

THEOREM 2.4. *Under the hypotheses of Lemma* 2.3 *the following estimates hold for* (*TSNLPM*):

$$\begin{aligned} e_{n,\alpha}^{h,\delta} &\le \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)[5\|x_{n,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|+3\|y_{n-1,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|]\|y_{n-1,\alpha}^{h,\delta}-x_{n,\alpha}^{h,\delta}\| \\ &\le b^2 e_{n-1,\alpha}^{h,\delta} \le b^{2n} e_{0,\alpha}^{h,\delta} \le b^{2n}\gamma_\rho. \end{aligned}$$

Proof. We have

$$\begin{aligned} &y_{n,\alpha}^{h,\delta}-x_{n,\alpha}^{h,\delta} \\ &\quad = x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}-R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta})-f^\delta+\alpha(x_{n,\alpha}^{h,\delta}-x_0)] \\ &\qquad + R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})P_h[F(y_{n-1,\alpha}^{h,\delta})-f^\delta+\alpha(y_{n-1,\alpha}^{h,\delta}-x_0)] \\ &\quad = x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}-R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F(x_{n,\alpha}^{h,\delta})-F(y_{n-1,\alpha}^{h,\delta})+\alpha(x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta})] \\ &\qquad + [R_\alpha^{-1}(y_{n-1,\alpha}^{h,\delta})-R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})]P_h[F(y_{n-1,\alpha}^{h,\delta})-f^\delta+\alpha(y_{n-1,\alpha}^{h,\delta}-x_0)] \\ &\quad = R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F'(x_{n,\alpha}^{h,\delta})(x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta})-(F(x_{n,\alpha}^{h,\delta})-F(y_{n-1,\alpha}^{h,\delta}))] \\ &\qquad + R_\alpha^{-1}(x_{n,\alpha}^{h,\delta})P_h[F'(x_{n,\alpha}^{h,\delta})-F'(y_{n-1,\alpha}^{h,\delta})](y_{n-1,\alpha}^{h,\delta}-x_{n,\alpha}^{h,\delta}) \\ &\quad =: \Gamma_3+\Gamma_4. \end{aligned}$$

Analogously to the proof of (2.16) and (2.17) one can prove that

$$\begin{aligned} \|\Gamma_3\| &\le \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)[3\|x_{n,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|+\|y_{n-1,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|]\|x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}\|, \\ \|\Gamma_4\| &\le K_0(1+\varepsilon_0/\alpha_0)[\|x_{n,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|+\|y_{n-1,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|]\|x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}\|. \end{aligned}$$

Now

$$\begin{aligned} e_{n,\alpha}^{h,\delta} &\le \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)[5\|x_{n,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|+3\|y_{n-1,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\|]\|x_{n,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}\| \\ &\le \frac{K_0}{2}(1+\varepsilon_0/\alpha_0)(8r)\frac{K_0}{2}(1+\varepsilon_0/\alpha_0)(8r)\|x_{n-1,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}\| \\ &\le b^2\|x_{n-1,\alpha}^{h,\delta}-y_{n-1,\alpha}^{h,\delta}\| \le b^{2n}e_{0,\alpha}^{h,\delta} \le b^{2n}\gamma_\rho. \end{aligned}$$

This completes the proof of the theorem.

THEOREM 2.5. *Suppose that the hypotheses of Theorem* 2.4 *hold. Then the sequences* $\{x_{n,\alpha}^{h,\delta}\}$, $\{y_{n,\alpha}^{h,\delta}\}$ *generated by* (*TSNLPM*) *are well defined and remain in* $U(P_h x_0, r)$ *for all* $n \ge 0$.

Proof. Note that by Lemma 2.3(b) we have

$$\begin{aligned} \|x_{1,\alpha}^{h,\delta}-P_h x_0\| = \|x_{1,\alpha}^{h,\delta}-x_{0,\alpha}^{h,\delta}\| &\le [1+(1+\varepsilon_0/\alpha_0)(K_0/2)(8r)]\gamma_\rho \\ &\le (1+b)\gamma_\rho \le \frac{1-b^2}{1-b}\gamma_\rho < r, \end{aligned} \tag{2.18}$$

i.e., $x_{1,\alpha}^{h,\delta} \in U(P_h x_0, r)$. Again note that from (2) and Theorem 2.4 we get

$$\begin{aligned}\|y_{1,\alpha}^{h,\delta} - P_h x_0\| &\leq \|y_{1,\alpha}^{h,\delta} - x_{1,\alpha}^{h,\delta}\| + \|x_{1,\alpha}^{h,\delta} - P_h x_0\| \\ &\leq [1 + (1+\varepsilon_0/\alpha_0)4K_0 r + ((1+\varepsilon_0/\alpha_0)4K_0 r)^2]\gamma_\rho \\ &\leq (1 + b + b^2)\gamma_\rho \leq \frac{1-b^2}{1-b}\gamma_\rho < r,\end{aligned}$$

i.e., $y_{1,\alpha}^{h,\delta} \in U(P_h x_0, r)$. Further by (2) and Lemma 2.3(b) we have

$$\begin{aligned}\|x_{2,\alpha}^{h,\delta} - P_h x_0\| &\leq \|x_{2,\alpha}^{h,\delta} - x_{1,\alpha}^{h,\delta}\| + \|x_{1,\alpha}^{h,\delta} - P_h x_0\| \\ &\leq (1+b)\gamma_\rho + (1+b)\gamma_\rho = 2(1+b)\gamma_\rho < r\end{aligned}$$

and

$$\begin{aligned}\|y_{2,\alpha}^{h,\delta} - P_h x_0\| &\leq \|y_{2,\alpha}^{h,\delta} - x_{2,\alpha}^{h,\delta}\| + \|x_{2,\alpha}^{h,\delta} - P_h x_0\| \\ &\leq b^4\gamma_\rho + 2(1+b)\gamma_\rho \\ &\leq b^2\gamma_\rho + 2(1+b)\gamma_\rho \\ &\leq \left[\frac{1-b^3}{1-b} + \frac{1-b^2}{1-b}\right]\gamma_\rho \\ \text{(since } b<1\text{)} \quad &< \frac{2\gamma_\rho}{1-b} < r\end{aligned}$$

by the choice of r, i.e., $x_{2,\alpha}^{h,\delta}$, $y_{2,\alpha}^{h,\delta} \in U(P_h x_0, r)$. Continuing this way one can prove that $x_{n,\alpha}^{h,\delta}$, $y_{n,\alpha}^{h,\delta} \in U(P_h x_0, r)$ for all $n \geq 0$. This completes the proof.

Theorem 2.6. *Suppose that the hypotheses of Theorem 2.5 hold. Then:*

(a) $\{x_{n,\alpha}^{h,\delta}\}$ *is a Cauchy sequence in* $U(P_h x_0, r)$ *and converges to* $x_\alpha^{h,\delta} \in \overline{U(P_h x_0, r)}$.
(b) $P_h[F(x_\alpha^{h,\delta}) + \alpha(x_\alpha^{h,\delta} - x_0)] = P_h y^\delta$.
(c) *We have*

$$\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq \frac{(1+b)b^{2n}\gamma_\rho}{1-b^2}$$

where γ_ρ *and* b *are defined by* (2.7) *and* (2.9), *respectively.*

Proof. We have

$$\begin{aligned}\|x_{n+i+1,\alpha}^{h,\delta} - x_{n+i,\alpha}^{h,\delta}\| &\leq (1+b)b^0\|x_{n+i,\alpha}^{h,\delta} - y_{n+i,\alpha}^{h,\delta}\| \\ &\leq (1+b)b\|x_{n+i,\alpha}^{h,\delta} - y_{n+i-1,\alpha}^{h,\delta}\| \\ &\leq (1+b)b^2\|x_{n+i-1,\alpha}^{h,\delta} - y_{n+i,\alpha}^{h,\delta}\| \\ &\leq (1+b)b^{2(n+i)}e_{0,\alpha}^{h,\delta} \leq (1+b)b^{2(n+i)}\gamma_\rho.\end{aligned}$$

So,

$$\|x^{h,\delta}_{n+m,\alpha} - x^{h,\delta}_{n,\alpha}\| \leq \sum_{i=0}^{m-1} \|x^{h,\delta}_{n+i+1,\alpha} - x^{h,\delta}_{n+i,\alpha}\| \leq (1+b)b^{2n} \sum_{i=0}^{m-1} b^{2i}$$
$$= (1+b)b^{2n}\frac{1-b^{2m}}{1-b^2}\gamma_\rho \to \frac{(1+b)b^{2n}}{1-b^2}\gamma_\rho$$

as $m \to \infty$. Thus $x^{h,\delta}_{n,\alpha}$ is a Cauchy sequence in $U(P_h x_0, r)$ and hence it converges, say to $x^{h,\delta}_\alpha \in \overline{U(P_h x_0, r)}$. Observe that

$$\begin{aligned}\|P_h[F(x^{h,\delta}_{n,\alpha}) - f^\delta + \alpha(x^{h,\delta}_{n,\alpha} - x_0)]\| &= \|R_\alpha(x_0)(x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n,\alpha})\| \\ &\leq \|R_\alpha(x_0)\|\, \|x^{h,\delta}_{n,\alpha} - y^{h,\delta}_{n,\alpha}\| \\ &= \|(P_h F'(x^{h,\delta}_{n,\alpha})P_h + \alpha P_h)\| e^{h,\delta}_{n,\alpha} \\ &\leq (C_F + \alpha) e^{h,\delta}_{n,\alpha}.\end{aligned}$$

Now by letting $n \to \infty$ we obtain

$$P_h[F(x^{h,\delta}_\alpha) + \alpha(x^{h,\delta}_\alpha - x_0)] = P_h y^\delta. \tag{2.19}$$

This completes the proof.

Remark 2.7. (a) The convergence order of (TSNLPM) is four [10], under Assumption 1. In Theorem 2.6 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see e.g. [4]) defined by

$$\varrho \approx \ln\left(\frac{\|x_{n+1} - x^\delta_\alpha\|}{\|x_n - x^\delta_\alpha\|}\right) \Big/ \ln\left(\frac{\|x_n - x^\delta_\alpha\|}{\|x_{n-1} - x^\delta_\alpha\|}\right).$$

The (COC) ϱ will then be close to 4, which is the order of convergence of (TSNLPM).

(b) Note that from the proof of Theorem 2.5 a larger r can be obtained from solving the equation

$$[b^4 t + 2(1+bt)]\gamma_\rho - rt = 0.$$

Note that this equation has a minimal root $r^* > r$. Then r^* can replace r in Theorem 2.5. However, we have decided to use r which is given in closed form. Using Mathematica or Maple we found r^* in closed form. But it has a complicated and long form. That is why we decided not to include r in this paper.

3. Error bounds under source conditions. The objective of this section is to obtain an error estimate for $\|x^{h,\delta}_{n,\alpha} - \hat{x}\|$ under a source condition on $x_0 - \hat{x}$.

Proposition 3.1. *Let $F : D(F) \subseteq X \to X$ be a monotone operator in X. Let $x_\alpha^{h,\delta}$ be the solution of* (2.19) *and $x_\alpha^h := x_\alpha^{h,0}$. Then*

$$\|x_\alpha^{h,\delta} - x_\alpha^h\| \le \delta/\alpha.$$

Proof. The result follows from the monotonicity of F and the relation

$$P_h[F(x_\alpha^{h,\delta}) - F(x_\alpha^h) + \alpha(x_\alpha^{h,\delta} - x_\alpha^h)] = P_h(y^\delta - y).$$

Theorem 3.2. *Let $\rho < \frac{2}{K_0(1+\varepsilon_0/\alpha_0)}$ and $\hat{x} \in D(F)$ be a solution of* (1.1)*. Suppose Assumption* 3*, Assumption* 4 *and the assumptions in Proposition* 3.1 *are satisfied. Then*

$$\|x_\alpha^h - \hat{x}\| \le \tilde{C}(\varphi(\alpha) + \varepsilon_h/\alpha)$$

where

$$\tilde{C} := \frac{\max\{1 + (1+\varepsilon_0/\alpha_0)K_0(2b_0+\rho), \rho + \|\hat{x}\|\}}{1 - (1+\varepsilon_0/\alpha_0)\frac{K_0}{2}\rho}.$$

Proof. Let $M := \int_0^1 F'(\hat{x} + t(x_\alpha^h - \hat{x}))\,dt$. Then from the relation

$$P_h[F(x_\alpha^h) - F(\hat{x}) + \alpha(x_\alpha^h - x_0)] = 0$$

we have

$$(P_hMP_h + \alpha P_h)(x_\alpha^h - \hat{x}) = P_h\alpha(x_0 - \hat{x}) + P_hM(I - P_h)\hat{x}.$$

Hence,

$$\begin{aligned}
(3.1)\quad & x_\alpha^h - \hat{x} \\
&= [(P_hMP_h + \alpha P_h)^{-1}P_h - (F'(x_0) + \alpha I)^{-1}]\alpha(x_0 - \hat{x}) \\
&\quad + (F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) + (P_hMP_h + \alpha P_h)^{-1}P_hM(I - P_h)\hat{x} \\
&= (P_hMP_h + \alpha P_h)^{-1}P_h[F'(x_0) - M + M(I - P_h)](F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\
&\quad + [(F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}) + (P_hMP_h + \alpha P_h)^{-1}P_hM(I - P_h)\hat{x}] \\
&=: \zeta_1 + \zeta_2.
\end{aligned}$$

Observe that

$$\begin{aligned}
(3.2)\quad \|\zeta_1\| &\le \Big\|(P_hMP_h + \alpha P_h)^{-1}P_h \\
&\qquad \cdot \int_0^1 [F'(x_0) - F'(\hat{x} + t(x_\alpha^h - \hat{x}))]\,dt\,(F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x})\Big\| \\
&\quad + \|(P_hMP_h + \alpha P_h)^{-1}P_hM(I - P_h)(F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x})\| \\
&\le \Big\|(P_hMP_h + \alpha P_h)^{-1}P_h\int_0^1 \big[F'(\hat{x} + t(x_\alpha^h - \hat{x}))(P_h + I - P_h) \\
&\qquad \cdot \phi\big(x_0, \hat{x} + t(x_\alpha^h - \hat{x}), (F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x})\big)\big]\,dt\Big\| + \frac{\varepsilon_h}{\alpha}\rho;
\end{aligned}$$

here and below $\varepsilon_h := \varepsilon_h(\hat{x} + t(x_\alpha^h - \hat{x}))$. So

$$\begin{aligned}
\|\zeta_1\| &\le (1+\varepsilon_h/\alpha)K_0 \int_0^1 [\|x_0 - P_h x_0\| + \|\hat{x} + t(x_\alpha^h - \hat{x}) - P_h x_0\|] \\
&\qquad \cdot \|F'(x_0) + \alpha I)^{-1}\alpha(x_0 - \hat{x}))\| + \frac{\varepsilon_h}{\alpha}\rho \\
&\le (1+\varepsilon_h/\alpha)K_0[(b_0 + \|\hat{x} - x_0 + x_0 - P_h x_0\|)\varphi(\alpha) + \frac{1}{2}\|x_\alpha^h - \hat{x}\|\rho] + \frac{\varepsilon_h}{\alpha}\rho \\
&\le (1+\varepsilon_h/\alpha)K_0[(2b_0 + \rho)\varphi(\alpha) + \frac{1}{2}\|x_\alpha^h - \hat{x}\|\rho] + \frac{\varepsilon_h}{\alpha}\rho
\end{aligned}$$

and

$$\|\zeta_2\| \le \varphi(\alpha) + \frac{\varepsilon_h}{\alpha}\|\hat{x}\|. \tag{3.3}$$

The result now follows from (3.1)–(3.3).

THEOREM 3.3. *Let $x_{n,\alpha}^{h,\delta}$ be as in* (1.4), *and suppose the assumptions in Theorems* 2.6 *and* 3.2 *hold. Then*

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \le \frac{1+b}{1-b^2}\gamma_\rho b^{2n} + \max\{1, \tilde{C}\}\left(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right).$$

Proof. Observe that

$$\|x_{n,\alpha}^{h,\delta} - \hat{x}\| \le \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| + \|x_\alpha^{h,\delta} - x_\alpha^h\| + \|x_\alpha^h - \hat{x}\|$$

so, by Proposition 3.1, Theorem 2.6 and Theorem 3.2 we obtain

$$\begin{aligned}
\|x_{n,\alpha}^{h,\delta} - \hat{x}\| &\le \frac{1+b}{1-b^2}\gamma_\rho b^{2n} + \frac{\delta}{\alpha} + \tilde{C}(\varphi(\alpha) + \varepsilon_h/\alpha) \\
&\le \frac{1+b}{1-b^2}\gamma_\rho b^{2n} + \max\{1, \tilde{C}\}\left(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right).
\end{aligned}$$

Let

$$n_\delta := \min\left\{n : b^{2n} \le \frac{\delta + \varepsilon_h}{\alpha}\right\}, \tag{3.4}$$

$$C_0 := \frac{1+b}{1-b^2}\gamma_\rho + \max\{1, \tilde{C}\}. \tag{3.5}$$

THEOREM 3.4. *Let n_δ and C_0 be as in* (3.4) *and* (3.5) *respectively. Moreover, let $x_{n_\delta,\alpha}^{h,\delta}$ be as in* (1.4) *and suppose the assumptions in Theorem* 3.3 *are satisfied. Then*

$$\|x_{n_\delta,\alpha}^{h,\delta} - \hat{x}\| \le C_0\left(\varphi(\alpha) + \frac{\delta + \varepsilon_h}{\alpha}\right). \tag{3.6}$$

3.1. A priori choice of the parameter. Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$, $0 < \lambda \le a$. Then the choice

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$$

gives the optimal order error estimate (see [10]) for $\varphi(\alpha) + (\delta + \varepsilon_h)/\alpha$. So the relation (3.6) leads to the following.

THEOREM 3.5. *Let* $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ *for* $0 < \lambda \le a$, *and suppose the assumptions in Theorem* 3.4 *hold. For* $\delta > 0$, *let* $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta + \varepsilon_h))$ *and let* n_δ *be as in* (3.4). *Then*

$$\|x^{h,\delta}_{n_\delta,\alpha} - \hat{x}\| = O(\psi^{-1}(\delta + \varepsilon_h)).$$

3.2. An adaptive choice of the parameter. As in [10], the parameter α is chosen according to the balancing principle studied in [15], [17], i.e., it is selected from some finite set

$$D_N(\alpha) := \{\alpha_i = \mu^i \alpha_0 : i = 0, 1, \dots, N\}$$

where $\mu > 1$, $\alpha_0 > 0$. Moreover, let

$$n_i := \min\left\{n : b^{2n} \le \frac{\delta + \varepsilon_h}{\alpha_i}\right\}.$$

Then for $i = 0, 1, \dots, N$, we have

$$\|x^{h,\delta}_{n_i,\alpha_i} - x^{h,\delta}_{\alpha_i}\| \le C\frac{\delta + \varepsilon_h}{\alpha_i}, \qquad \forall i = 0, 1, \dots, N.$$

Let $x_i := x^{h,\delta}_{n_i,\alpha_i}$. In this paper we select $\alpha = \alpha_i$ from $D_N(\alpha)$ for computing x_i, for each $i = 0, 1, \dots, N$.

THEOREM 3.6 (cf. [18, Theorem 3.1]). *Assume that there exists* $i \in \{0, 1, \dots, N\}$ *such that* $\varphi(\alpha_i) \le (\delta + \varepsilon_h)/\alpha_i$. *Let the assumptions of Theorems* 3.4 *and* 3.5 *hold and let*

$$l := \max\left\{i : \varphi(\alpha_i) \le \frac{\delta + \varepsilon_h}{\alpha_i}\right\} < N,$$

$$k := \max\left\{i : \|x_i - x_j\| \le 4C_0\frac{\delta + \varepsilon_h}{\alpha_j},\ j = 0, 1, \dots, i\right\}.$$

Then $l \le k$ *and* $\|\hat{x} - x_k\| \le c\psi^{-1}(\delta + \varepsilon_h)$ *where* $c = 6C_0\mu$.

4. Implementation of the adaptive choice rule. The balancing algorithm associated with the choice of the parameter specified in Theorem 3.6 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0$, $i = 0, 1, \dots, N$.

4.1. Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min\{n : b^{2n} \le (\delta + \varepsilon_h)/\alpha_i\}$.
3. Solve $x_i := x^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (1.3) and (1.4).

4. If $\|x_i - x_j\| > 4C_0(\delta + \varepsilon_h)/\alpha_j$, $j < i$, then take $k = i-1$ and return x_k.
5. Else set $i = i + 1$ and go to 2.

5. Example

EXAMPLE 5.1. Let $X = Y = \mathbb{R}$, $D = [0, \infty)$, $x_0 = 1$ and define a function F on D by

$$F(x) = \frac{x^{1+1/i}}{1 + 1/i} + c_1 x + c_2, \tag{5.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. However Assumption 3 holds for $K_0 = 1$.

Using the identity

$$\mu^i - \mu_0^i = (\mu - \mu_0)(\mu^{i-1} + \mu^{i-2}\mu_0 + \cdots + \mu_0^{i-1}) \tag{5.2}$$

for $\mu = x^{1/i}$, $\mu_0 = x_0^{1/i}$ and (5.1) we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| = |x^{1/i} - x_0^{1/i}| &= \frac{|x - x_0|}{x_0^{(i-1)/i} + \cdots + x^{(i-1)/i}} \\ &\le |x - x_0| \quad \text{for } x \in [0, \infty) \text{ and } x_0 = 1. \end{aligned}$$

Hence, we get $\|F'(x) - F'(x_0)\| \le K_0 |x - x_0|$.

EXAMPLE 5.2. We consider the integral equations

$$u(s) = f(s) + \lambda \int_a^b G(s,t) u(t)^{1+1/n}\, dt, \qquad n \in \mathbb{N}. \tag{5.3}$$

Here, f is a given continuous function satisfying $f(s) > 0, s \in [a, b]$, λ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s, t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$\begin{gathered} u'' = \lambda u^{1+1/n}, \\ u(a) = f(a), \quad u(b) = f(b). \end{gathered}$$

Problems of this type have been considered in [1]–[4].

Equations of the form (5.3) generalize equations of the form

$$u(s) = \int_a^b G(s,t) u(t)^n\, dt \tag{5.4}$$

studied in [1]–[4]. Instead of (5.3) we can try to solve the equation $F(u) = 0$ where

$$F : \varOmega \to C[a, b], \qquad \varOmega = \{u \in C[a, b] : u(s) \ge 0,\ s \in [a, b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t) u(t)^{1+1/n}\, dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda(1+1/n) \int_a^b G(s,t) u(t)^{1/n} v(t)\, dt, \qquad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1]$, $G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\lambda|(1+1/n) \int_a^b x(t)^{1/n}\, dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \le L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}\, dt \le L_2 \max_{s\in[0,1]} x(s), \tag{5.5}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = t/j, \qquad j \ge 1,\ t \in [0,1].$$

If these are substituted into (5.5) then

$$\frac{1}{j^{1/n}(1+1/n)} \le \frac{L_2}{j}, \quad \text{i.e.} \quad j^{1-1/n} \le L_2(1+1/n), \qquad \forall j \ge 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (5.5) is not satisfied in this case. However, Assumption 3 holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s\in[a,b]} f(s)$, $\alpha > 0$. Then for $v \in \Omega$,

$$\begin{aligned}\|[F'(x) - F'(x_0)]v\| &\le |\lambda|(1+1/n) \max_{s\in[a,b]} \left| \int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n}) v(t)\, dt \right| \\ &\le |\lambda|(1+1/n) \max_{s\in[a,b]} \left| \int_a^b G_n(s,t)\, dt \right|\end{aligned}$$

where

$$G_n(s,t) = \frac{G(s,t)|x(t) - f(t)|}{|x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}|} \|v\|$$

and we used (5.2) for $i = n$, $\mu = x(t)^{1/n}$ and $\mu_0 = x_0(t)^{1/n}$. Hence,

$$\begin{aligned}\|[F'(x) - F'(x_0)]v\| &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)\,dt\,\|x - x_0\| \\ &\le K_0\|x - x_0\|,\end{aligned}$$

where

$$K_0 = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} N \quad \text{and} \quad N = \max_{s\in[a,b]} \int_a^b G(s,t)\,dt.$$

Thus Assumption 3 holds for sufficiently small λ.

EXAMPLE 5.3. Let $X = D(F) = \mathbb{R}$, $x_0 = 0$, and define a function F on $D(F)$ by

$$F(x) = d_0 x + d_1 + d_2 \sin e^{d_3 x},$$

where d_0, d_1, d_2 and d_3 are given parameters. Then it can easily be seen that for d_3 sufficiently large and d_1 sufficiently small, K_0/k_0 can be arbitrarily small.

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