APPLICATIONES MATHEMATICAE
39,4 (2012), pp. 413–424

Marek Andrzej Kociński (Warszawa)

THE MARTINGALE METHOD OF SHORTFALL RISK MINIMIZATION IN A DISCRETE TIME MARKET

Abstract. The shortfall risk minimization problem for the investor who hedges a contingent claim is studied. It is shown that in case the nonnegativity of the final wealth is not imposed, the optimal strategy in a finite market model is obtained by super-hedging a contingent claim connected with a martingale measure which is a solution of an auxiliary maximization problem.

1. Introduction. In the paper we consider the problem of minimization of the risk of loss of the investor who hedges a contingent claim and starts with an initial capital which is smaller than the initial cost of the super-hedging strategy. This is an important problem, especially in incomplete markets where in general super-hedging requires a large initial capital. One of the most natural measures of risk is the expected shortfall which is defined as the expected value, with respect to an objective probability, of the positive part of the difference between the value of the contingent claim and the final wealth of the investor. The shortfall risk minimization in a continuous time model was studied e.g. in [4]. In this paper a discrete time market is considered. One can distinguish two cases: when the final wealth of the investor is nonnegative and when this condition is not imposed. In the constrained case the dynamic problem can be reduced to the static one and solved via an auxiliary dual problem. This approach was presented in [5], where the convex duality method of [1] was used. In the unconstrained case the problem of shortfall risk minimization can be solved by dynamic programming (see [2], [3], [8] and [9]).

2010 *Mathematics Subject Classification*: Primary 91G20; Secondary 91B30.
Key words and phrases: contingent claim, self-financing strategy, shortfall risk minimization, super-hedging.

DOI: 10.4064/am39-4-2

In this paper it is shown that for a finite market model the static problem of minimization can be solved via an auxiliary problem of maximization and the maximum is searched in the set of all martingale measures.

This paper generalizes some results of [3], [8] and [11].

In Section 2 we present the basic definitions and facts in a discrete time market model. In Section 3 the results in the constrained case obtained in [5] are briefly recalled. Then, these results are used to obtain the proper auxiliary maximization problem and to solve the shortfall risk minimization problem in the unconstrained case.

2. The model. Let $(\Omega, \mathcal{F}, P)$ be a probability space, T a positive natural number and $\mathcal{F} = \{\mathcal{F}_t, t = 0, \ldots, T\}$ a family of σ-algebras such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ for $t = 0, \ldots, T-1$ and $\mathcal{F}_T = \mathcal{F}$.

Throughout this paper, equalities and inequalities depending on $\omega \in \Omega$ hold P-almost surely unless stated otherwise. Moreover, we assume that if Ω is finite then $\mathcal{F}$ consists of all subsets of Ω and $P(\{\omega\}) > 0$ for all $\omega \in \Omega$.

We consider a market in which one can trade in one stock and in the bank account with interest rate for simplicity equal to 0. The generalization to the case of an arbitrary nonnegative interest rate is straightforward. We assume that all assets are infinitely divisible.

The stock price movement is modeled by a process $\{S_t, t = 0, \ldots, T\}$ where S_t denotes the price of the stock at time t, for $t = 0, \ldots, T$. We assume that S_t is $\mathcal{F}_t$-measurable and positive P-almost surely for $t = 0, \ldots, T$.

A *contingent claim* is a nonnegative, random variable $H \in L^1(P)$.

A *trading strategy* $\beta = \{\beta_t\}_{t=0,\ldots,T-1}$ is a process adapted to the filtration $\{\mathcal{F}_t, t = 0, \ldots, T-1\}$ where the random variable β_t represents the number of shares of the stock held by the seller of the contingent claim at time t after a possible transaction at that time instance.

For a trading strategy β and an initial capital $x > 0$ we define the *wealth process* $V(x, \beta)$ as follows:

$$V_0(x, \beta) = x,$$

$$V_{t+1}(x, \beta) = x + \sum_{n=0}^{t} \beta_n (S_{n+1} - S_n) \quad \text{for } t = 0, \ldots, T-1.$$

This means that at every trading date, sales must finance purchases, that is, the strategy is self-financing.

Denote by B the set of all trading strategies.

A probability measure Q such that the discounted stock price process is a martingale under Q is called a *martingale measure*. We assume that in our market model there exists at least one equivalent martingale measure, which means that in this model there is no arbitrage opportunity (see [10]).

3. The shortfall risk minimization. We define the *super-replication cost* V^* of a contingent claim H by

$$V^* = \inf\{x \in \mathbb{R} : V_T(x, \beta) \geq H \text{ for some } \beta \in B\}.$$

Denote $L^\infty(\Omega, \mathcal{F}, P)$ by $L^\infty(P)$ and let $\mathbf{Q}$ be the set of probability measures defined as follows:

$$\mathbf{Q} = \left\{ Q \ll P : \frac{dQ}{dP} \in L^\infty(P) \text{ and } Q \text{ is a martingale measure} \right\}.$$

By Theorem 3.1 in [6] the super-replication cost V^* of the contingent claim is given by the formula

$$V^* = \sup_{Q \in \mathbf{Q}} E^Q(H).$$

REMARK 3.1. It is easily seen that if Ω is finite then for all $\beta \in B$ the wealth process $V(x, \beta)$ is a martingale under every $Q \in \mathbf{Q}$.

Let $x \in \mathbb{R}$ denote an initial capital of the investor. We say that a strategy $\beta \in B$ is a *super-hedging strategy* for a contingent claim H at the initial capital $x \in \mathbb{R}$ if $V_T(x, \beta) \geq H$.

For the initial capital $x \in \mathbb{R}$ and a strategy $\beta \in B$ the *expected shortfall* is defined as the expected value under P of $(H - V_T(x, \beta))^+$.

From now on we assume that $0 < x < V^*$.

Our aim is to minimize the shortfall risk, that is, to minimize the expected shortfall.

3.1. Minimization of shortfall risk with the nonnegativity condition. Let $B(x) = \{\beta \in B : V_T(x, \beta) \geq 0\}$. We consider the following optimization problem:

$$\inf_{\beta \in B(x)} E^P((H - V_T(x, \beta))^+). \tag{3.1}$$

Let $\mathcal{X}(x) = \{X \in L^1(P) : 0 \leq X \leq H, \sup_{Q \in \mathbf{Q}} E^Q(X) \leq x\}$. The next proposition is a consequence of Proposition 4.2 in [5] and reduces the dynamic problem (3.1) to a static one.

PROPOSITION 3.2. *Suppose that $\widehat{X} \in X(x)$ is a solution to the static problem*

$$\inf_{X \in \mathcal{X}(x)} E^P(H - X). \tag{3.2}$$

Then there exists a super-hedging strategy $\widehat{\beta} \in B(x)$ for X that solves the dynamic problem (3.1). *Moreover, $E^P((H - V_T(x, \beta))^+) = E^P(H - \widehat{X})$.*

Set

$$\mathcal{Z}=\left\{Z\in L^{\infty}(P):\ \begin{array}{l} Z\ge 0,\ E^P(Z)\le 1,\\ E^P(ZX)\le E^P(X)\ \text{for all}\ X\in L^1_+(P)\end{array}\right\},$$

$$\mathcal{G}=\left\{G\in L^1_+(P):\ \begin{array}{l} E^P(G)\le 1,\ E^P(GH)\le V^*,\\ E^P(GX)\le x\ \text{for all}\ x>0\ \text{and}\ X\in\mathcal{X}(x)\end{array}\right\}$$

(cf. [5] where the sets $\mathcal{G}$ and $\mathcal{Z}$ are defined for a more general problem).

Lemma 3.3. *We have* $\mathcal{Z}=\{Z\in L^{\infty}(P):0\le Z\le 1\}$.

Proof. It is easily seen that $\{Z\in L^{\infty}(P):0\le Z\le 1\}\subseteq\mathcal{Z}$. Conversely, let $Z\in\mathcal{Z}$ and $A=\{Z>1\}$. It is clear that $\mathbf{1}_A\in L^1_+(P)$ and consequently $E^P(Z\mathbf{1}_A)\le E^P(\mathbf{1}_A)$. Moreover, by the definition of A we have $E^P(Z\mathbf{1}_A)\ge E^P(\mathbf{1}_A)$ and thus $E^P(Z\mathbf{1}_A)=E^P(\mathbf{1}_A)$. Consequently, by the definition of A we have $P(A)=0$. Therefore $\mathcal{Z}\subseteq\{Z\in L^{\infty}(P):0\le Z\le 1\}$. ∎

Consider the following auxiliary problems:

$$\sup\{E^P(H(Z\wedge yG))-xy:Z\in\mathcal{Z},\,G\in\mathcal{G},\,y\ge 0\},\tag{3.3}$$

$$\sup\{E^P(H(1\wedge yG))-xy:G\in\mathcal{G},\,y\ge 0\}.\tag{3.4}$$

By the inequality $H\ge 0$ and Lemma 3.3 if the triple $(\widehat{Z},\widehat{G},\widehat{y})$ is a solution of the problem (3.3) then so is $(1,\widehat{G},\widehat{y})$. Therefore using Theorem 4.11 in [5] we have the following fact:

Theorem 3.4. *Let* $(\widehat{G},\widehat{y})$ *be an optimal pair for problem* (3.4).

- (i) *There exists a* $[0,1]$*-valued random variable* C *such that the random variable* $\widehat{X}=H\mathbf{1}_{\{\widehat{y}\widehat{G}<1\}}+HC\mathbf{1}_{\{\widehat{y}\widehat{G}=1\}}$ *is a solution to problem* (3.2). *Moreover, the supremum in* (3.4) *equals the infimum in* (3.2).
- (ii) *If* β *is a super-hedging strategy for* $\widehat{X}$ *at the initial capital* x *then* β *is a solution to* (3.1).

A consequence of Remark 4.6 in [5] is the following lemma:

Lemma 3.5. *Theorem* 3.4 *holds if we replace* $\mathcal{G}$ *by any set* $\mathcal{G}'$ *satisfying the following conditions:*

- (i) $\mathcal{G}'$ *is convex, closed under* P*-a.s. convergence, bounded in* $L^1(P)$ *and it includes the convex hull of the densities with respect to* P *of all elements of* $\mathbf{Q}$, *that is,* $\mathrm{conv}\{dQ/dP:Q\in\mathbf{Q}\}\subseteq\mathcal{G}'$.
- (ii) *If a sequence* $\{G_n\}_{n=1}^{\infty}$ *from* $\mathcal{G}'$ *converges to some random variable* G,P*-a.s. on* $\{H>0\}$ *and if* $G=0$ *on the set* $\{H=0\}$, *then* $G\in\mathcal{G}'$.

By Lemma 3.5 we have the following fact:

LEMMA 3.6. *Assume that Ω is finite. Then Theorem* 3.4 *holds if we replace $\mathcal{G}$ by the set $\{dQ/dP : Q \in \mathbf{Q}\}$.*

Proof. Set

$$\mathcal{G}' = \operatorname{conv}\left(\left\{\frac{dQ}{dP} : Q \in \mathbf{Q}\right\} \cup \left\{\frac{dQ}{dP}\mathbf{1}_{\{H>0\}} : Q \in \mathbf{Q}\right\}\right).$$

It is clear that $\mathcal{G}'$ is convex and $\operatorname{conv}\{dQ/dP : Q \in \mathbf{Q}\} \subseteq \mathcal{G}'$. Since Ω is finite, $\mathcal{G}'$ is bounded in $L^1(P)$. Moreover, the same argument and Theorem 17.2 in [7] imply that $\mathcal{G}'$ is closed under P-a.s. convergence. It is easily seen that the set $\{dQ/dP : Q \in \mathbf{Q}\}$ is convex. Hence for each $G \in \mathcal{G}'$ there exists $Q(G) \in \mathbf{Q}$ such that $G = dQ(G)/dP$ on $\{H > 0\}$. Moreover, since Ω is finite, each sequence in $\{dQ/dP : Q \in \mathbf{Q}\}$ has a subsequence converging P-a.s. to an element of $\{dQ/dP : Q \in \mathbf{Q}\}$. Therefore, $\mathcal{G}'$ satisfies conditions (i) and (ii) in Lemma 3.5, and consequently Theorem 3.4 holds with $\mathcal{G}$ replaced by $\mathcal{G}'$. Then, since for each $G \in \mathcal{G}'$ there exists $Q(G) \in \mathbf{Q}$ such that $G = dQ(G)/dP$ on $\{H > 0\}$, it is clear that Theorem 3.4 holds if we replace $\mathcal{G}'$ by $\{dQ/dP : Q \in \mathbf{Q}\}$. ∎

3.2. Minimization of shortfall risk without the nonnegativity condition. Now we consider the following optimization problem:

$$\inf_{\beta \in B} E^P((H - V_T(x,\beta))^+). \tag{3.5}$$

We assume throughout this subsection that Ω is finite.

REMARK 3.7. It is easily seen that a strategy $\widehat{\beta}$ solves problem (3.5) if and only if for all $l \in \mathbb{R}$ it solves problem (3.5) with the contingent claim $H + l$ and the initial capital $x + l$, which means that

$$E^P((H + l - V_T(x + l, \widehat{\beta}))^+) = \inf_{\beta \in B} E^P((H + l - V_T(x + l, \beta))^+).$$

Remark 3.7 easily implies the following fact:

LEMMA 3.8. *If a strategy $\widehat{\beta} \in B(x)$ is a solution to problems* (3.1) *and* (3.5) *then for all $l \in \mathbb{R}_+$ it solves problem* (3.1) *with the contingent claim $H + l$ and the initial capital $x + l$, which means that*

$$\begin{aligned} &E^P((H + l - V_T(x + l, \widehat{\beta}))^+) \\ &\qquad = \inf_{\beta \in B(x+l)} E^P((H + l - V_T(x + l, \beta))^+) \quad \textit{for all } l \in \mathbb{R}_+. \end{aligned}$$

Proof. Let $l \in \mathbb{R}_+$. It is clear that $\widehat{\beta} \in B(x+l)$. Moreover, by Remark 3.7 it solves the problem $\inf_{\beta\in B} E^P((H + l - V_T(x + l, \beta))^+)$. Consequently, the inclusion $B(x + l) \subseteq B$ implies that $\widehat{\beta}$ solves the problem

$$\inf_{\beta \in B(x+l)} E^P((H + l - V_T(x + l, \beta))^+). \ \blacksquare$$

Let $\mathbf{X}(x,H)$ denote the set of all random variables satisfying the inequalities $X \le H$ and $\sup_{Q\in\mathbf{Q}} E^Q(X) \le x$.

As in the case when the nonnegativity of the final wealth is imposed, we can reduce the dynamic problem (3.5) to a static one (see also Lemma 4.3 in [3] and Proposition 4 in [11] for the case of the complete market model).

LEMMA 3.9. *Suppose that $\widehat{X} \in \mathbf{X}(x,H)$ is a solution to the static problem*

$$\inf_{X\in\mathbf{X}(x,H)} E^P(H-X). \tag{3.6}$$

Then there exists a super-hedging strategy for $\widehat{X}$ at the initial capital x. Moreover, if $\widehat{\beta} \in B$ is such a super-hedging strategy then $\widehat{\beta}$ solves the dynamic problem (3.5) *and we have $E^P((H-V_T(x,\widehat{\beta}))^+) = E^P(H-\widehat{X})$.*

Proof. Let $\beta \in B$. By Remark 3.1, $\min\{V_T(x,\beta),H\} \in \mathbf{X}(x,H)$, and since $\widehat{X}$ is a solution to the static problem (3.6) we obtain

$$E^P((H-V_T(x,\beta))^+) \ge E^P(H-\widehat{X}). \tag{3.7}$$

Since Ω is finite, by the inequality $\inf_{Q\in\mathbf{Q}} E^P(\widehat{X}) \le x$ and Theorem 3.1 in [6] it is easily seen that there exists a super-hedging strategy $\widehat{\beta} \in B$ for $\widehat{X}$ at the initial capital x. It is clear that $E^P((H-V_T(x,\widehat{\beta}))^+) \le E(H-\widehat{X})$. Thus, by (3.7) the strategy $\widehat{\beta}$ solves the dynamic problem (3.5). Moreover, it is clear that $E^P((H-V_T(x,\widehat{\beta}))^+) = E(H-\widehat{X})$. ∎

It is easily seen that the following lemma holds:

LEMMA 3.10. *Let $l \in \mathbb{R}$. A random variable $\widehat{X}$ is a solution to problem* (3.6) *if and only if the random variable $\widehat{X}+l$ is a solution to the problem $\inf_{X\in\mathbf{X}(x+l,H+l)} E^P(H+l-X)$. Moreover,*

$$\inf_{X\in\mathbf{X}(x,H)} E^P(H-X) = \inf_{X\in\mathbf{X}(x+l,H+l)} E^P(H+l-X).$$

For every $Q \in \mathbf{Q}$ let $G_Q = dQ/dP$, that is, $G_Q(\omega) = Q(\{\omega\})/P(\{\omega\})$ for each $\omega \in \Omega$. Moreover, for all $Q \in \mathbf{Q}$ let $y_Q = \min\{1/G_Q(\omega) : \omega \in \Omega, Q(\{\omega\}) > 0\}$.

Consider the following optimization problem:

$$\sup\{E^P(H(1\wedge yG_Q)) - xy : Q \in \mathbf{Q},\ y \ge 0\}. \tag{3.8}$$

REMARK 3.11. Let $(\widetilde{Q},\widetilde{y})$ be an optimal pair for problem (3.8). By Lemma 3.6 we can replace the pair $(\widehat{G},\widehat{y})$ and problem (3.4) in Theorem 3.4 respectively by the pair $(G_{\widetilde{Q}},\widetilde{y})$ and problem (3.8).

Consider the following maximization problem:

$$\sup_{Q\in\mathbf{Q}} y_Q(E^Q(H)-x). \tag{3.9}$$

LEMMA 3.12. *Let $X \in \mathbf{X}(x,H)$ and $Q \in \mathbf{Q}$. Then $E^P(H-X) \geq y_Q(E^Q(H)-x)$.*

Proof. By the inequality $X \leq H$ and the definitions of G_Q and y_Q we have

$$\begin{aligned} E^P(H-X) &\geq E^P((H-X)\mathbf{1}_{\{\omega\in\Omega:\, Q(\{\omega\})>0\}}) \\ &= E^Q\biggl(\frac{1}{G_Q}(H-X)\mathbf{1}_{\{\omega\in\Omega:\, Q(\{\omega\})>0\}}\biggr) \\ &\geq E^Q(y_Q(H-X)1_{\{\omega\in\Omega:\, Q(\{\omega\})>0\}}) = y_Q E^Q(H-X). \end{aligned}$$

Consequently, by the definition of $\mathbf{X}(x,H)$ we obtain

$$E^P(H-X) \geq y_Q(E^Q(H)-x). \quad \blacksquare$$

Since in the proof of the next theorem the problem of shortfall risk minimization will be considered when some nonnegative capital l is added both to x and H, a generalization of the set $\mathcal{X}(x)$ is needed.

For all $l \in \mathbb{R}_+$ let

$$\mathcal{X}_l(x) = \Bigl\{X \in L^1(P) : 0 \leq X \leq H + l,\ \sup_{Q\in\mathbf{Q}} E^Q(X) \leq x + l\Bigr\}.$$

THEOREM 3.13. *There exists a measure $\widehat{Q} \in \mathbf{Q}$ which is a solution to problem* (3.9) *such that:*

(i) *there exists a random variable C such that $C \leq 1$ and the random variable*

$$\widehat{X} = H\mathbf{1}_{\{y_{\widehat{Q}}\frac{d\widehat{Q}}{dP}<1\}} + HC\mathbf{1}_{\{y_{\widehat{Q}}\frac{d\widehat{Q}}{dP}=1\}}$$

is a solution to problem (3.6). *Moreover, the supremum in* (3.9) *equals the infimum in* (3.6).

(ii) *If $\widehat{\beta}$ is a super-hedging strategy for $\widehat{X}$ at the initial capital x then $\widehat{\beta}$ is a solution to* (3.5).

Proof. Let $x > 0$ be the initial capital and let $\widehat{\mathbf{X}}(x,H)$ denote the set of all random variables which are solutions of problem (3.6). Define

$$v = \inf_{X\in\widehat{\mathbf{X}}(x,H)} \min_{\omega\in\Omega} X(\omega).$$

Since Ω is finite we have $v > -\infty$. Consider a sequence $\{l_n\}_{n=1}^\infty$ of real numbers such that $l_n > -\min\{v,0\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} l_n = \infty$. Since Ω is finite it is easily seen that there exist sequences $\{\widetilde{Q}_n\}_{n=1}^\infty$ and $\{\widetilde{y}_n\}_{n=1}^\infty$ such that $\widetilde{Q}_n \in \mathbf{Q}$, $\widetilde{y}_n \geq 0$ and the pair $(G_{\widetilde{Q}_n}, \widetilde{y}_n)$ is a solution to the problem

$$\sup\{E^P((H+l_n)(1\wedge yG_Q)) - (x+l_n)y : Q \in \mathbf{Q},\ y \geq 0\}.$$

Consequently, since $l_n \geq 0$ for all $n \in \mathbb{N}$, Theorem 3.4 and Remark 3.11 imply that there exists a sequence $\{C_n\}_{n=1}^\infty$ of $[0,1]$-valued random variables such that for all $n \in \mathbb{N}$ the random variable

$$\widetilde{X}_n = (H + l_n)\mathbf{1}_{\{\widetilde{y}_n G_{\widetilde{Q}_n} < 1\}} + (H + l_n) C_n \mathbf{1}_{\{\widetilde{y}_n G_{\widetilde{Q}_n} = 1\}}$$

is a solution to problem (3.2) with the contingent claim $H + l_n$ and the initial capital $x + l_n$, that is, $\widetilde{X}_n \in X_{l_n}(x)$ and

$$E^P(H + l_n - \widetilde{X}_n) = \inf_{X \in \mathcal{X}_{l_n}(x)} E^P(H + l_n - X). \tag{3.10}$$

Let $n \in \mathbb{N}$. It is clear that $\widetilde{X}_n \in \mathbf{X}(x + l_n, H + l_n)$. From the inequality $l_n > -\min\{v, 0\}$ and Lemma 3.10 it follows that every random variable $X \in \mathbf{X}(x + l_n, H + l_n)$ which is a solution to the static problem

$$\inf_{X \in \mathbf{X}(x + l_n, H + l_n)} E^P(H + l_n - X)$$

is strictly positive and therefore it is also a solution to problem (3.2) with the contingent claim $H + l_n$ and the initial capital $x + l_n$. This implies that

$$\begin{aligned} E^P(H + l_n - \widetilde{X}_n) \\ = \inf_{X \in \mathbf{X}(x + l_n, H + l_n)} E^P(H + l_n - X_n) \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.11}$$

For all $n \in \mathbb{N}$ define the random variable $\widehat{X}_n$ by $\widehat{X}_n = \widetilde{X}_n - l_n$. It is clear that $\widehat{X}_n \in \mathbf{X}(x, H)$ for all $n \in \mathbb{N}$. Moreover, by Lemma 3.10 and (3.11),

$$E^P(H - \widehat{X}_n) = \inf_{X \in \mathbf{X}(x, H)} E^P(H - X) \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

By Theorem 3.4 and Remark 3.11 it follows that

$$E^P((H + l_n)(1 \wedge \widetilde{y}_n G_{\widetilde{Q}_n})) - (x + l_n)\widetilde{y}_n = \inf_{X \in \mathcal{X}_{l_n}(x)} E^P(H + l_n - X).$$

Thus by (3.10) and (3.11) we have

$$E^P((H + l_n)(1 \wedge \widetilde{y}_n G_{\widetilde{Q}_n})) - (x + l_n)\widetilde{y}_n = \inf_{X \in \mathbf{X}(x + l_n, H + l_n)} E^P(H + l_n - X).$$

The last equality, Lemma 3.10 and the definition of $\mathbf{X}(x, H)$ imply that

$$\begin{aligned} E^P((H + l_n)(1 \wedge \widetilde{y}_n G_{\widetilde{Q}_n})) - (x + l_n)\widetilde{y}_n \\ = \inf_{X \in \mathbf{X}(x, H)} E^P(H - X) \geq 0 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.13}$$

By (3.13) and since $l_n > 0$ for each $n \in \mathbb{N}$, we have

$$0 \leq E^P(H) - x\widetilde{y}_n + l_n(1 - \widetilde{y}_n) \quad \text{for each } n \in \mathbb{N},$$

and since $x > 0$ and $l_n > 0$ for all $n \in \mathbb{N}$, it follows that there exists a number y^* such that $0 \leq \widetilde{y}_n < y^*$ for all $n \in \mathbb{N}$.

By the definition of $\mathbf{X}(x, H)$ and (3.12) there exists $x^* \in \mathbb{R}$ such that $x^* < \widehat{X}_n \leq H$ for all $n \in \mathbb{N}$. Moreover, it is easily seen that $0 \leq G_{\widetilde{Q}_n} \leq$

$1/\min_{\omega\in\Omega} P(\omega)$ for all $n \in \mathbb{N}$. Therefore, since Ω is finite, there exist subsequences $\{G_{\widetilde{Q}_{n_k}}\}_{k=1}^{\infty}, \{\widehat{X}_{n_k}\}_{k=1}^{\infty}, \{\widetilde{y}_{n_k}\}_{k=1}^{\infty}$ of $\{G_{\widetilde{Q}_n}\}_{n=1}^{\infty}, \{\widehat{X}_n\}_{n=1}^{\infty}, \{\widetilde{y}_n\}_{n=1}^{\infty}$ respectively, such that $\lim_{k\to\infty} G_{\widetilde{Q}_{n_k}} = \widehat{G}$, $\lim_{k\to\infty} \widetilde{y}_{n_k} = \widehat{y}$, $\lim_{k\to\infty} \widehat{X}_{n_k} = \widehat{X}$ for some random variable $\widehat{G}$, $\widehat{y} \in \mathbb{R}_+$ and $\widehat{X} \in \mathbf{X}(x,H)$.

It is clear that

$$\begin{aligned}(3.14)\qquad & E^P((H+l_n)(1\wedge \widetilde{y}_n G_{\widetilde{Q}_n})) - (x+l_n)\widetilde{y}_n \\ &= E^P(H(1\wedge \widetilde{y}_n G_{\widetilde{Q}_n})) - x\widetilde{y}_n + l_n(E^P(1\wedge \widetilde{y}_n G_{\widetilde{Q}_n}) - \widetilde{y}_n) \text{ for all } n\in\mathbb{N}.\end{aligned}$$

Thus, by (3.13), (3.14) and since $\lim_{n\to\infty} l_n = \infty$, we get

$$\lim_{k\to\infty} (E^P(1\wedge \widetilde{y}_{n_k} \widetilde{G}_{\widetilde{Q}_{n_k}}) - \widetilde{y}_{n_k}) = 0.$$

Consequently,

$$\widehat{y}\widehat{G} \le 1. \tag{3.15}$$

It is easily seen that there exists $\widehat{Q} \in \mathbf{Q}$ such that $\widehat{G} = G_{\widehat{Q}}$. Moreover, we have $E^P(1\wedge \widetilde{y}_n \widetilde{G}_{\widetilde{Q}_n}) - \widetilde{y}_n \le 0$ for each $n \in \mathbb{N}$. Thus, by (3.13) and (3.14),

$$\widehat{y}(E^{\widehat{Q}}(H) - x) \ge \inf_{X\in\mathbf{X}(x,H)} E^P(H-X). \tag{3.16}$$

From the inequality $x < V^*$ it follows easily that $\inf_{X\in\mathbf{X}(x,H)} E^P(H-X) > 0$, and thus (3.16) implies that $E^{\widehat{Q}}(H) - x > 0$. Thus, by (3.15), (3.16) and the definition of $y_{\widehat{Q}}$ we get

$$y_{\widehat{Q}}(E^{\widehat{Q}}(H) - x) \ge \widehat{y}(E^{\widehat{Q}}(H) - x) \ge \inf_{X\in\mathbf{X}(x,H)} E^P(H-X). \tag{3.17}$$

Consequently, by (3.17), Lemma 3.12 and the inequality $E^{\widehat{Q}}(H) - x > 0$ we obtain $\widehat{y} = y_{\widehat{Q}}$ and

$$y_{\widehat{Q}}(E^{\widehat{Q}}(H) - x) = \sup_{Q\in\mathbf{Q}} y_Q(E^Q(H) - x) = \inf_{X\in\mathbf{X}(x,H)} E^P(H-X).$$

Thus, the measure $\widehat{Q} \in \mathbf{Q}$ is a solution to problem (3.9) and the supremum in (3.9) equals the infimum in (3.6). Moreover, from the definition of $\widehat{X}$ it follows easily that there exists a random variable C such that $C \le 1$ and $\widehat{X} = H\mathbf{1}_{\{y_{\widehat{Q}} G_{\widehat{Q}} < 1\}} + HC\mathbf{1}_{\{y_{\widehat{Q}} G_{\widehat{Q}} = 1\}}$, and (3.12) implies that $\widehat{X}$ is a solution to the static problem (3.6). This finishes the proof of (i).

Item (ii) is a consequence of Lemma 3.9. ∎

The next proposition easily implies some of the results obtained in a complete market model (see [3], [11]).

PROPOSITION 3.14. *Let* $\widehat{Q} \in \mathbf{Q}$ *and let* $X = H\mathbf{1}_{\{y_{\widehat{Q}} G_{\widehat{Q}} < 1\}} + HC\mathbf{1}_{\{y_{\widehat{Q}} G_{\widehat{Q}} = 1\}}$ *where* C *is a random variable such that* $E^{\widehat{Q}}(\widehat{X}) = \sup_{Q \in \mathbf{Q}} E^Q(\widehat{X}) = x$ *and* $C \le H$*. Then* $\widehat{X} \in \mathbf{X}(x, H)$ *and* $\widehat{X}$ *is a solution to problem* (3.6)*. Moreover, if* $\widehat{\beta} \in B$ *is a super-hedging strategy for the contingent claim* $\widehat{X}$ *then* $\widehat{\beta}$ *is a solution to problem* (3.5)*.*

Proof. It is clear that $\widehat{X} \in \mathbf{X}(x, H)$. To prove that $\widehat{X}$ is a solution to problem (3.6) it is sufficient to show that $E^P(\widehat{X} - X) \ge 0$ for all $X \in \mathbf{X}(x, H)$. Let $X \in \mathbf{X}(x, H)$. We have

$$\begin{aligned} E^P(\widehat{X} - X) &= E^P((H - X)\mathbf{1}_{\{y_{\widehat{Q}} G_{\widehat{Q}} < 1\}}) + E^P((\widehat{X} - X)\mathbf{1}_{\{y_{\widehat{Q}} G_{\widehat{Q}} = 1\}}) \\ &\ge y_{\widehat{Q}} E^{\widehat{Q}}(\widehat{X} - X) = y_{\widehat{Q}}(x - E^{\widehat{Q}}(X)) \\ &\ge y_{\widehat{Q}}\Big(x - \sup_{Q \in \mathbf{Q}} E^Q(X)\Big) = 0. \end{aligned}$$

The rest of the assertion is a consequence of Lemma 3.9. ∎

REMARK 3.15. Since the super-replication cost of the contingent claim H is equal to $\sup_{Q \in \mathbf{Q}} E^Q(H)$, from Theorem 3.13 it follows immediately that for an initial capital $x \in \mathbb{R}$ we have

$$\inf_{\beta \in B} E^P((H - V_T(x, \beta))^+) = \sup_{Q \in \mathbf{Q}} y_Q((E^Q(H) - x)^+).$$

3.3. The binomial model. Now we will consider the problem of minimizing the shortfall risk in a binomial market. This model was considered in [2], [3], [8] and these papers use dynamic programming. It can be noticed that in the case of complete information about the distribution of the stock price at each time, the solution of the shortfall risk minimization problem in the binomial model follows easily from Theorem 3.13.

We assume that the stock price satisfies the recursive formula

$$S_{t+1} = (1 + \gamma_{t+1})S_t, \quad t = 0, \ldots, T - 1, \quad S_0 > 0,$$

where $\{\gamma_t\}_{t=1}^T$ is a sequence of i.i.d. random variables such that $P(\gamma_t = b) = p = 1 - P(\gamma_t = a)$ for each $t = 1, \ldots, T$ where $p \in (0, 1)$ and $0 < a < 1 < b$ are given constants.

In this subsection we set $\mathcal{F}_t = \sigma(\gamma_u, 1 \le u \le t)$ for $t = 0, \ldots, T$ and assume that H is a function of S_T.

It is easily seen that in this model there exists a unique martingale measure Q^* such that $q^* = Q^*(\gamma_t = b) = \frac{-a}{b-a} = 1 - Q^*(\gamma_t = a)$ for $t = 1, \ldots, T$.

Let $V_n^* = E^{Q^*}(H(S_T) \mid \mathcal{F}_n)$ for $n = 0, \ldots, T$. It is easily seen that V_n^* is a function of S_n.

THEOREM 3.16. *Consider a contingent claim* H *in the binomial market model.*

If $p \geq q^*$ *then*

$$\inf_{\beta \in B} E^P((H - V_T(x,\beta))^+) = \left(\frac{1-p}{1-q^*}\right)^T (V^* - x)$$

and an optimal strategy is given by

$$\widehat{\beta}_t = \frac{V^*_{t+1}(S_t(1+b)) - V_t(x,\widehat{\beta})}{S_t b} \quad \text{for } t = 0,\dots,T-1.$$

If $p < q^*$ *then*

$$\inf_{\beta \in B} E^P((H - V_T(x,\beta))^+) = \left(\frac{p}{q^*}\right)^T (V^* - x)$$

and an optimal strategy is given by

$$\widehat{\beta}_t = \frac{V^*_{t+1}(S_t(1+a)) - V_t(x,\widehat{\beta})}{S_t a} \quad \text{for } t = 0,\dots,T-1.$$

This theorem is an easy consequence of Theorem 3.13. For the proof using dynamic programming see [8].

Acknowledgements. This research was supported by MNiSW grant no. 1 P03A 013 28.

References

[1] J. Cvitanić and I. Karatzas, *Generalized Neyman–Pearson lemma via convex duality*, Bernoulli 7 (2001), 79–97.

[2] G. Favero, *Shortfall risk minimization under model uncertainty in the binomial case: adaptive and robust approaches*, Math. Methods Oper. Res. 53 (2001), 493–503.

[3] G. Favero and T. Vargiolu, *Shortfall risk minimising strategies in the binomial model: characterisation and convergence*, Math. Methods Oper. Res. 64 (2006), 237–253.

[4] H. Föllmer and P. Leukert, *Efficient hedging: Cost versus shortfall risk*, Finance Stoch. 4 (2000), 117–146.

[5] Y. Nakano, *Minimizing coherent risk measures of shortfall in discrete-time models with cone constraints*, Appl. Math. Finance 10 (2003), 163–181.

[6] H. Pham, *Dynamic L^p-hedging in discrete time under cone constraints*, SIAM J. Control Optim. 38 (2000), 665–682.

[7] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, NJ, 1997.

[8] W. J. Runggaldier, B. Trivellato and T. Vargiolu, *A Bayesian adaptive approach to risk management in a binomial model*, in: Seminar on Stochastic Analysis, Random Fields and Applications, III (Ascona 1999), R. C. Dalang et al. (eds), Progr. Probab. 52, Birkhäuser, Basel, 2002, 243–258.

[9] C. Scagnellato and T. Vargiolu, *Explicit solutions for shortfall risk minimization in multinomial models*, Decis. Econom. Finance 25 (2002), 145–155.

[10] W. Schachermayer, *A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time*, Insurance Math. Econom. 11 (1992), 249–257.

[11] M. Schulmerich and S. Trautmann, *Local expected shortfall-hedging in discrete time*, Eur. Finance Rev. 7 (2003), 75–102.

Marek Andrzej Kociński
Katedra Zastosowań Matematyki
Wydział Zastosowań Informatyki i Matematyki
Szkoła Główna Gospodarstwa Wiejskiego w Warszawie
Nowoursynowska 159
02-776 Warszawa, Polska
E-mail: marek_kocinski@sggw.pl

Received on 19.12.2007;
revised version on 29.8.2011 (1905)