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APPLICATIONS OF TIME-DELAYED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS TO PRICING, HEDGING AND PORTFOLIO MANAGEMENT IN INSURANCE AND FINANCE

Abstract. We investigate novel applications of a new class of equations which we call time-delayed backward stochastic differential equations. Time-delayed BSDEs may arise in insurance and finance in an attempt to find an investment strategy and an investment portfolio which should replicate a liability or meet a target depending on the strategy applied or the past values of the portfolio. In this setting, a managed investment portfolio serves simultaneously as the underlying security on which the liability/target is contingent and as a replicating portfolio for that liability/target. This is usually the case of capital-protected investments and performance-linked pay-offs. We give examples of pricing, hedging and portfolio management problems (asset-liability management problems) which could be investigated in the framework of time-delayed BSDEs. We focus on participating contracts and variable annuities. We believe that time-delayed BSDEs could offer a tool for studying investment life insurance contracts from a new and desirable perspective.

1. Introduction. Backward stochastic differential equations (BSDEs) were introduced by Pardoux and Peng (1990). Since then, theoretical properties of BSDEs have been thoroughly studied in the literature and BSDEs have found numerous applications in finance: see for example El Karoui et al. (1997) and Pham (2009).

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In this paper we study applications of a new class of backward stochastic differential equations driven by a Brownian motion. We consider the dynamics given by

$$Y(t) = \xi(\omega, Y_T, Z_T) + \int_t^T f(s, \omega, Y_s, Z_s)\, ds - \int_t^T Z(s)\, dW(s), \quad 0 \le t \le T. \tag{1.1}$$

Here, the generator f at time s and the terminal condition ξ depend on the past values of the solution $(Y_s, Z_s) = (Y(s+u), Z(s+u))_{-T \le u \le 0}$. The classical BSDE in the sense of Pardoux and Peng (1990) arises when ξ does not depend on (Y, Z), and f at time s depends only on the value of the solution at this time, $f(s, \omega, Y(s), Z(s))$. We introduce ω in (1.1) to point out that ξ and f can depend on an exogenously given source of uncertainty induced by the Brownian motion W.

The equation (1.1) can be called a time-delayed backward stochastic differential equation. This type of equation was introduced and investigated from the theoretical point of view in Delong and Imkeller (2010a), Delong and Imkeller (2010b), Dos Reis et al. (2011). However, no applications have been presented so far. The main contribution of this paper is to provide the first and novel applications of time-delayed BSDEs to problems related to pricing, hedging and investment portfolio management in insurance and finance. In particular, our goal is to show that time-delayed BSDEs may offer a tool for studying investment life insurance contracts from a new and desirable perspective. We point out that whereas forward SDEs with delays are well-studied (see for example David (2008) and the references therein), backward SDEs with delays are a new research field.

In financial applications of (1.1), Y stands for a replicating portfolio, Z denotes a replicating strategy and ξ is a terminal liability. The time-delayed BSDE (1.1) may arise when seeking an investment strategy Z and an investment portfolio Y which should replicate a liability or meet a target $\xi(Y_T, Z_T)$ depending on the investment strategy applied and the past values (past performance) of the investment portfolio. Time-delayed BSDEs may become useful when we face the problem of managing an investment portfolio which serves simultaneously as the underlying security on which the liability/target is contingent and as a replicating portfolio for that liability/target. Non-trivial dependencies $\xi(Y_T, Z_T)$ arise in the case of capital-protected investments (capital guarantees) and performance-linked pay-offs (profit-sharing rules). We point out that dependence of the value of the liability or target on the replicating portfolio does not occur in the classical financial mathematics where the claims are contingent only on exogenously given sources of uncertainty, and ξ is independent of (Y, Z). It should be noticed that time-delayed generator $f(t, Y_t, Z_t)$ may model a stream of liabilities depending on the past performance of the replicating portfolio.

We focus on participating contracts and variable annuities which are life insurance products with capital protection guarantees and benefits based on the performance of the underlying investment portfolio. Participating contracts and variable annuities are extensively studied in the actuarial literature but almost all papers treat the investment portfolio as an exogenously given stock, beyond the control of the insurer, and assume that bonuses and guarantees are contingent on the performance of that stock; see for example Bacinello (2001), Bauer et al. (2005), Dai et al. (2008), Huang et al. (2009), Milevsky and Posner (2001) and Milevsky and Salisbury (2006). These authors apply the methods of the classical mathematical finance and consider pricing and hedging of path-dependent European contingent claims. Under their approach dependencies and interactions between the investment portfolio and the liability are lost. The considerations do not focus on an important issue of risk management as the investment portfolio can be controlled internally by the insurer, who by choosing an appropriate investment strategy, can reduce or remove the financial risk of the issued guarantee. Our idea is to take into account interactions between assets and liabilities arising in participating contracts and variable annuities. We believe that this is the right direction in financial modelling of such insurance contracts.

The mutual dependence between the insurer's investment strategy and the pay-off arising under participating contracts was noticed by Kleinow and Wilder (2007), Kleinow (2009), Ballotta and Haberman (2009) and Sart (2010). Kleinow and Wilder (2007) and Kleinow (2009) considered perfect hedging of a participating contract with a guaranteed rate of return and a terminal bonus contingent on the return of a continuously rebalanced asset portfolio held by the insurer. Their strategy cannot be financed with the initial premium and the insurer has to provide additional capital to fulfill the obligation. In Ballotta and Haberman (2009) quadratic hedging is investigated for a participating policy with a terminal benefit equal to smoothed value, of an average type, of the insurer's asset portfolio. A static asset allocation strategy is found by a numerical experiment. Sart (2010) constructed, for a general participating contract, an investment portfolio which replicates the benefit contingent on the return earned by that portfolio. The investment strategy consists of bonds held to maturity and it is derived by solving a fixed point problem under which the value of the liability equals the asset value. Sart (2010) applies the amortized cost valuation of the bonds which makes the profit earned by the investment portfolio in subsequent periods deterministic. In this paper we combine the ideas from Kleinow and Wilder (2007), Kleinow (2009), Ballotta and Haberman (2009) and Sart (2010), and we aim at showing how to use time-delayed BSDEs to solve pricing, hedging, and portfolio management problems for participating contracts and variable annuities which take into account the feedback between assets and liabilities. We remark that in the context of variable annuities the optimal asset allocation has not been considered yet. Our investigation leads to new types of time-delayed BSDEs.

This paper is structured as follows. Section 2 presents a motivation for applying time-delayed BSDEs in insurance and finance. Sections 3–6 investigate investment strategies for portfolios subjected to ratchet options (in discrete and continuous time), bonuses based on the average portfolio value and withdrawal rates related to the maximum of the portfolio value.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ and a finite time horizon $T < \infty$. We assume that $\mathbb{F}$ is the natural filtration generated by a Brownian motion $W := (W(t), 0 \le t \le T)$ completed with sets of measure zero.

2. Pricing and hedging with time-delayed BSDEs. We consider a financial market which consists of two tradeable instruments: a risk-free asset and a risky bond. The price of the risk-free asset $B := (B(t),\, 0 \le t \le T)$ is given by the equation

$$\frac{dB(t)}{B(t)} = r(t)\,dt, \qquad B(0) = 1, \tag{2.1}$$

where $r := (r(t), 0 \le t \le T)$ denotes the risk-free interest rate. We assume that

(A1) the rate r is a non-negative, $\mathbb{F}$-progressively measurable, square integrable process.

The price of the risky bond $D := (D(t), 0 \le t \le T)$ with maturity T is given by the equation

$$\frac{dD(t)}{D(t)} = (r(t) + \sigma(t)\theta(t))\,dt + \sigma(t)\,dW(t), \qquad D(0) = d_0, \tag{2.2}$$

where $\theta := (\theta(t), 0 \le t \le T)$ and $\sigma := (\sigma(t), 0 \le t \le T)$ denote the risk premium and volatility. We assume that

(A2) θ and σ are non-negative, $\mathbb{F}$-progressively measurable, square integrable processes such that θ and σ^{-1} are a.s. uniformly bounded on $[0, T]$,

(A3) $0 < D(t) < 1$, $0 \le t < T$, $D(T) = 1$.

In particular, there exists a unique equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$.

Let us consider an investment portfolio $X := (X(t),\, 0 \le t \le T)$. Let $\pi := (\pi(t),\, 0 \le t \le T)$ denote the amount invested in the bond D. Any admissible strategy π should be an $\mathbb{F}$-predictable process satisfying $\int_0^T |\pi(s)\sigma(s)|^2\,ds$

$< \infty$, a.s. The dynamics or the value of the investment portfolio is given by the stochastic differential equation

$$dX(t) = \pi(t)((r(t) + \sigma(t)\theta(t))\,dt + \sigma(t)\,dW(t)) + (X(t) - \pi(t))r(t)\,dt, \qquad X(0) = x. \tag{2.3}$$

By the change of variables

$$Y(t) = X(t)e^{-\int_0^t r(s)\,ds}, \quad Z(t) = e^{-\int_0^t r(s)\,ds}\pi(t)\sigma(t), \quad 0 \le t \le T, \tag{2.4}$$

we arrive at the discounted portfolio process $Y := (Y(t), 0 \le t \le T)$ under the martingale measure $\mathbb{Q}$,

$$dY(t) = Z(t)\,dW^{\mathbb{Q}}(t), \qquad Y(0) = y, \tag{2.5}$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. We simultaneously work with the undiscounted portfolio X and the discounted portfolio Y and the reader should keep in mind (2.4).

We deal with the problem of finding an investment strategy π and an investment portfolio X which replicate a liability or meet a target $\xi(X_T, \pi_T)$ depending on the strategy applied or the past values of the portfolio. In the language of BSDEs our financial problem is equivalent to deriving a solution (Y, Z) to the time-delayed BSDE

$$Y(t) = \tilde{\xi}(Y_T, Z_T) - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \le t \le T, \tag{2.6}$$

which follows from (2.5). By tildes we denote discounted pay-offs: $\tilde{\xi} = e^{-\int_0^T r(s)\,ds}\xi$.

Let us start by giving a motivating example.

EXAMPLE 2.1 (Option Based Portfolio Insurance). Let us consider an investor who would like to invest x euros. They want to protect the initial capital and hope to gain an additional profit. Let $S := (S(t), 0 \le t \le T)$ represent a benchmark which the investor would like to follow. We set $S(0) = 1$. It is well-known that in order to meet the investor's target of protecting the initial capital, a financial institution should buy the bond which guarantees x at the terminal time which costs $xD(0)$ and the call option on a fraction λ of S which costs $C(x\lambda S(T) - x) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r(s)\,ds}(x\lambda S(T) - x)^+]$. From El Karoui et al. (2005) we know that there exists a unique λ (independent of x) such that

$$xD(0) + C(x\lambda S(T) - x) = x, \tag{2.7}$$

hence such a static investment strategy may be constructed. The strategy (2.7) is called Option Based Portfolio Insurance and has gained popularity in financial markets.

The key point is that the above well-known strategy can be obtained as a solution of a time-delayed BSDE. Let us now turn to dynamic investment strategies. The goal in our problem is to find (X,π) in (2.3) such that

$$X(T) = X(0) + (X(0)\lambda S(T) - X(0))^{+},$$

which leads to the time-delayed BSDE of the form

$$Y(t) = e^{-\int_0^T r(s)\,ds}\big(Y(0) + (Y(0)\lambda S(T) - Y(0))^{+}\big) - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \le t \le T, \tag{2.8}$$

where the terminal condition depends on the past values of Y via $Y(0)$. For given replicable benchmark S and some λ we expect to find multiple solutions (Y,Z) to (2.8) which differ in $Y(0)$. Taking a more general point of view, in order to meet our investment goal we have to find (X,π) in (2.3) such that

$$X(T) = X(0) + (X(T) - X(0))^{+},$$

which leads to the time-delayed BSDE

$$Y(t) = Y(0)e^{-\int_0^T r(s)\,ds} + \big(Y(T) - Y(0)e^{-\int_0^T r(s)\,ds}\big)^{+} - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \le t \le T, \tag{2.9}$$

where the terminal condition depends on the past values of Y via $Y(0)$ and the current value $Y(T)$. The BSDE (2.8) arises when the target or liability is contingent on the externally chosen benchmark S. However, in many financial problems the target or liability could be contingent on the performance of the internally managed assets and we end up with (2.9). In this case a financial institution can set its own (replicable) benchmark. We expect to obtain multiple solutions (Y,Z) to (2.9) which differ in $(Y(0),S)$.

We comment on three possible areas where the time-delayed BSDE (2.6), or more generally the time-delayed BSDE (1.1), may arise.

Application 1: Portfolio management. Example 2.1 shows that some portfolio management problems can be investigated in the framework of time-delayed BSDEs. The key example is a construction of a capital-protected investment as discussed in Example 2.1. A simple protection concerns only an initial investment, but a more sophisticated protection may be based on intermediate investment gain lock-ins and portfolio ratcheting or a draw-down constraint. Asian type guarantees paying back an average value of the investment portfolio are also common in portfolio management. All these capital protections depend on the past values of the investment portfolio, and portfolio management problems under such capital guarantees fit into the framework of time-delayed BSDEs.

Application 2: Participating contract. Under a participating contract the asset portfolio is set internally by the insurer who has full discretion over the choice of financial instruments. A policyholder earns a guaranteed return on the initial contribution and participates in the surplus return gained by the insurer's asset portfolio. The key feature of participating contracts is that the final pay-off from the policy is related (by the profit-sharing scheme) to the performance of the asset portfolio managed by the insurer which backs the liability (see TP.2.86-TP.2.93 in QIS5 (2010)). This implies that the insurer's investment strategy in the asset portfolio backing the liability and the past values of the asset portfolio affect the final value of the liability (via the profit-sharing scheme). The feedback between the asset allocation and the benefit makes participating contracts the key example of an insurance product which can be studied in the framework of time-delayed BSDEs.

Application 3: Variable annuity. Under a variable annuity (or a unit-linked contract) the policyholder's contributions are invested in different mutual funds. The key feature of variable annuities is that the final pay-off is related to the performance of the policyholder's investment account that is subject to a capital guarantee. This again implies that the investment strategy applied in the account and the past values of the investment account affect the final value of the liability. In general, the insurer does not have discretion over the allocation of the capital in the funds, which is to be decided by the policyholder. However, the insurer can propose an investment plan which is often accepted by the policyholder. In such a case, the insurer decides on the investment strategy and manages the policyholder's account to fulfill the capital guarantee which depends on the performance of the investment account.

In the cases of participating contracts and variable annuities the goal is to find a composition of the insurer's asset portfolio or the policyholder's investment account under which the embedded guarantee is fulfilled. Moreover, there should be a potential for additional profit to be next distributed to the policyholder. According to Solvency II Directive (see V.2.2 in QIS5 (2010)), the insurance reserve must include an estimate of the value of the liability arising under the contract including all possible guarantees, profits and bonuses. When valuating the liability under a participating contract, the future change of the allocations in the backing asset portfolio should be taken into account as the result of future management actions (see TP.2.92 in QIS5 (2010)). The value of the asset portfolio must match the reserve, and the assets held by the insurer must finance the liability which depends on the past and future performance of the asset portfolio and the allocation strategy. In participating contracts and variable annuities, there is a strong relation between the assets and liabilities as they affect each other. The problems investigated in this paper are examples of asset-liability management problems under which the assets and liabilities must be matched and the matching conditions involve fixed point equations. It should be noticed that by choosing an investment strategy, applying an appropriate asset-liability strategy, the insurer is able to fulfill the guarantee without deducting any fees needed to buy options and without setting separate hedge accounts as indicated in the papers mentioned in the Introduction.

We comment on the solvability of time-delayed BSDEs (1.1) and (2.6). Even for a Lipschitz generator f we may have three cases (see Delong and Imkeller (2010a) and Delong (2011)). There may be no solution to (1.1) or there may be a solution which is not interesting from the practical point of view, like a non-positive solution. These cases are interpreted as impossibility of hedging of a claim or unfairness of a contract. Recall from Example 2.1 that a positive solution to (2.8), the OBPI strategy, does not arise for all λ, S, which is clear in the context of the investment problem considered. There may be a unique practical solution to (1.1) which is interpreted as the existence of a unique hedging strategy under a uniquely determined premium (a unique asset-liability strategy). Finally, there may be multiple solutions to (1.1). In many portfolio management problems, including Example 2.1, participating contracts and variable annuities, we should not insist on obtaining a unique solution to a time-delayed BSDE, but we should try to find all multiple solutions instead. Apparently, in some financial applications multiple solutions are more meaningful than unique solutions. This is an important difference between time-delayed BSDEs and classical BSDEs which give unique solutions (in the Lipschitz case). It should be noticed that the existence of multiple solutions in Example 2.1 for (2.8) or (2.9) has a clear financial interpretation as it indicates the possibility of meeting the investment target for any initial premium and under different product designs (different asset-liability strategies) depending on the choice of value process S.

In the next sections we solve time-delayed BSDEs which may arise when dealing with participating policies and variable annuities. We denote participation factors by $\beta > 0, \gamma > 0$ and a guaranteed rate by $g \geq 0$. Integrability is understood as integrability under the martingale measure $\mathbb{Q}$ unless stated otherwise.

3. Hedging a ratchet contingent on the maximum value of the portfolio—the discrete time case. In this section we deal with the terminal liability of the form

$$\xi = \gamma \max\{X(0)e^{gT}, X(t_1)e^{g(T-t_1)}, \dots, X(t_{n-1})e^{g(T-t_{n-1})}, X(T)\}. \tag{3.1}$$

The pay-off (3.1) is called a *ratchet option* (or a *drawdown constraint*) and under this protection any intermediate investment gain earned by the investment portfolio X (a fraction of it) is locked in as the liability and guaranteed to be paid back at maturity (accumulated with a guaranteed rate). Ratchet options are very popular as death or survival benefits in variable annuities and are common forms of capital protections of investment funds. They can also be used as a profit sharing scheme in participating contracts.

We solve the corresponding time-delayed BSDE (2.6) and derive an investment strategy.

THEOREM 3.1. *Assume that* (A1)–(A3) *hold. Consider the time-delayed BSDE*

$$\begin{aligned} Y(t) = \gamma \max\big\{&Y(0)e^{-\int_0^T r(s)\,ds+gT}, Y(t_1)e^{-\int_{t_1}^T r(s)\,ds+g(T-t_1)}, \dots, \\ &Y(t_{n-1})e^{g(T-t_{n-1})}, Y(T)\big\} - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \quad 0 \le t \le T. \end{aligned} \tag{3.2}$$

Set

$$\mathcal{B} = \big\{\omega \in \Omega : \gamma \max\big\{e^{gT}D(0), e^{g(T-t_1)}D(t_1), \dots, e^{g(T-t_{n-1})}D(t_{n-1}), 1\big\} > 1\big\},$$
$$\mathcal{C} = \big\{\omega \in \Omega : \gamma \max\big\{e^{g(T-t_1)}D(t_1), \dots, e^{g(T-t_{n-1})}D(t_{n-1}), 1\big\} = 1\big\}.$$

The equation (3.2) *has the following square integrable solutions under the requirement that* $Y(0) \ge 0$*:*

1. *If* $\mathbb{P}(\mathcal{B}) > 0$, *then there exists a unique solution* $Y = Z = 0$.
2. *If* $\mathbb{P}(\mathcal{B}) = 0$, $\gamma e^{gT}D(0) = 1$, *then there exist multiple solutions* (Y, Z), *which differ in* $Y(0)$, *of the form*

$$Y(t) = \gamma Y(0)e^{gT-\int_0^t r(s)\,ds}D(t), \quad 0 \le t \le T,$$
$$Y(0)e^{gT-\int_0^T r(s)\,ds} = Y(0) + \int_0^T Z(s)\,dW(s).$$

3. *If* $\mathbb{P}(\mathcal{B}) = 0$, $\gamma e^{gT}D(0) < 1$, $\mathbb{P}(\mathcal{C}) = 1$, *then there exist multiple solutions* (Y, Z), *which differ in* $(Y(0), (\tilde{\eta}(t_{m+1}))_{m=0,1,\dots,n-1})$, *of the form*

$$Y(t_0) = Y(0),$$
$$\begin{aligned} Y(t_m) = \gamma &\max_{k=0,1,\dots,m} \big\{Y(t_k)e^{-\int_{t_k}^{t_m} r(u)\,du}e^{g(T-t_k)}\big\}D(t_m) \\ &+ \mathbb{E}^{\mathbb{Q}}[\tilde{\eta}(t_{m+1}) \mid \mathcal{F}_{t_m}], \end{aligned}$$

$$Y(t_{m+1}) = \gamma \max_{k=0,1,\ldots,m}\{Y(t_k)e^{-\int_{t_k}^{t_{m+1}} r(u)\,du}e^{g(T-t_k)}\}D(t_{m+1}) + \tilde{\eta}(t_{m+1})$$
$$= Y(t_m) + \int_{t_m}^{t_{m+1}} Z(s)\,dW^{\mathbb{Q}}(s),$$
$$Y(t) = \mathbb{E}^{\mathbb{Q}}[Y(t_{m+1})\,|\,\mathcal{F}_t], \quad t_m \le t \le t_{m+1},\ m = 0,1,\ldots,n-1,$$

with $t_0 = 0$*,* $t_n = T$*, and a sequence of square integrable non-negative random variables* $(\tilde{\eta}(t_{m+1}))_{m=0,1,\ldots,n-1}$ *such that* $\tilde{\eta}(t_{m+1}) \in \mathcal{F}_{t_{m+1}}$*.*

4. *If* $\mathbb{P}(\mathcal{B}) = 0$*,* $\gamma e^{gT}D(0) < 1$*,* $0 < \mathbb{P}(\mathcal{C}) < 1$*,* $\mathbb{P}(\tilde{\eta}(T) = 0\,|\,\mathcal{B}^c \setminus \mathcal{C}) = 1$*, then there exist multiple solutions* (Y,Z) *defined in item* 3*.*
5. *In the remaining cases, there exists a unique solution* $Y = Z = 0$*.*

The solution Y *is strictly positive provided that* $Y(0) > 0$*.*

Proof. 1. As $\mathcal{B} = \bigcup_{m=0,1,\ldots,n}\{\omega \in \Omega : \gamma e^{g(T-t_m)}D(t_m) > 1\}$ there exist t_k, $k = 0,1,\ldots,n$, such that $\mathbb{P}(\gamma e^{g(T-t_k)}D(t_k) > 1) > 0$. Taking the expected value of (3.2) we arrive at

$$Y(t_k) = \mathbb{E}^{\mathbb{Q}}[Y(T)\,|\,\mathcal{F}_{t_k}] \ge \gamma\mathbb{E}^{\mathbb{Q}}\big[Y(t_k)e^{-\int_{t_k}^{T} r(s)\,ds + g(T-t_k)}\,\big|\,\mathcal{F}_{t_k}\big]$$
$$= Y(t_k)\gamma e^{g(T-t_k)}D(t_k),$$

which results in a contradiction unless $Y(t_k) = 0$. As $\mathbb{E}^{\mathbb{Q}}[Y(T)\,|\,\mathcal{F}_{t_k}] = Y(t_k) = 0$, by non-negativity of $Y(T)$ we conclude first that $Y(T) = 0$ and next that $Y(t) = \mathbb{E}^{\mathbb{Q}}[Y(T)\,|\,\mathcal{F}_t] = 0$, $0 \le t \le T$. Finally, we get $Z(t) = 0$.

2. We first show that any solution to (3.2) must be a $(\mathbb{Q},\mathbb{F})$-square integrable martingale and have a representation

$$Y(t_{m+1}) = \gamma \max_{k=0,1,\ldots,m}\{Y(t_k)e^{-\int_{t_k}^{t_{m+1}} r(u)\,du}e^{g(T-t_k)}\}D(t_{m+1}) + \tilde{\eta}(t_{m+1}), \quad m = 0,1,\ldots,n-1, \tag{3.3}$$

with some sequence of non-negative square integrable random variables $(\tilde{\eta}(t_{m+1}))_{m=0,1,\ldots,n-1}$ such that $\tilde{\eta}(t_{m+1}) \in \mathcal{F}_{t_{m+1}}$. The martingale property of Y is obvious. By taking the expected value in (3.2) we arrive at

$$Y(t_{m+1}) = \gamma\mathbb{E}^{\mathbb{Q}}\Big[\max_{k=0,1,\ldots,n}\{Y(t_k)e^{-\int_{t_k}^{T} r(u)\,du}e^{g(T-t_k)}\}\,\Big|\,\mathcal{F}_{t_{m+1}}\Big]$$
$$\ge \gamma\mathbb{E}^{\mathbb{Q}}\Big[\max_{k=0,1,\ldots,m}\{Y(t_k)e^{-\int_{t_k}^{T} r(u)\,du}e^{g(T-t_k)}\}\,\Big|\,\mathcal{F}_{t_{m+1}}\Big]$$
$$= \gamma \max_{k=0,1,\ldots,m}\{Y(t_k)e^{-\int_{t_k}^{t_{m+1}} r(u)\,du}e^{g(T-t_k)}\}D(t_{m+1}),$$

and the statement (3.3) follows. Now we can prove item 2 of our theorem.

By taking the expected value of (3.3) we derive

$$
\begin{aligned}
(3.4)\qquad Y(0) &= \mathbb{E}^{\mathbb{Q}}[Y(t_{m+1})] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\gamma \max_{k=0,1,\ldots,m}\big\{Y(t_k)e^{-\int_{t_k}^{t_{m+1}} r(u)\,du}e^{g(T-t_k)}\big\}D(t_{m+1}) + \tilde{\eta}(t_{m+1})\Big], \\
&\geq \gamma Y(0)D(0)e^{gT} + \mathbb{E}^{\mathbb{Q}}[\tilde{\eta}(t_{m+1})] \\
&= Y(0) + \mathbb{E}^{\mathbb{Q}}[\tilde{\eta}(t_{m+1})], \qquad m = 0,1,\ldots,n-1,
\end{aligned}
$$

which immediately implies that $\tilde{\eta}(t_{m+1}) = 0$ for all $m = 0,1,\ldots,n-1$. Assume next that the maximum in (3.3) is not attained at $t_0 = 0$. If for some $m = 1,\ldots,n-1$,

$$
\mathbb{Q}\Big(\max_{k=0,1,\ldots,m}\big\{Y(t_k)e^{-\int_{t_k}^{t_{m+1}} r(u)\,du}e^{g(T-t_k)}\big\} > Y(0)e^{-\int_0^{t_{m+1}} r(u)\,du}e^{gT}\Big) > 0,
$$

then we would find as in (3.4) that $Y(0) = \mathbb{E}^{\mathbb{Q}}[Y(t_{m+1})] > \gamma Y(0)D(0)e^{gT}$, which is a contradiction. Hence, $Y(t_{m+1}) = \gamma Y(0)e^{gT}e^{-\int_0^{t_{m+1}} r(s)}D(t_{m+1})$ for all $m = 0,1,\ldots,n-1$ and a candidate solution (Y,Z) on $[0,T]$ could be defined as in item 2. It is easy to check that for our candidate solution the terminal condition is fulfilled: $\tilde{\xi} = \gamma \max_{m=0,1,\ldots,n}\{Y(t_m)e^{-\int_{t_m}^T r(s)\,ds + g(T-t_m)}\} = \gamma Y(0)e^{-\int_0^T r(s)\,ds + gT} = Y(T)$, hence a solution to (3.2) is derived.

3. The candidate solution follows from the representation (3.3). We check whether the solution constructed satisfies the terminal condition. We have

$$
Y(T) = \gamma \max_{k=0,1,\ldots,n-1}\big\{Y(t_k)e^{-\int_{t_k}^{t_n} r(u)\,du}e^{g(T-t_k)}\big\} + \tilde{\eta}(t_n),
$$

and

$$
\begin{aligned}
\tilde{\xi} &= \gamma \max_{k=0,1,\ldots,n}\big\{Y(t_k)e^{-\int_{t_k}^{t_n} r(u)\,du}e^{g(T-t_k)}\big\} \\
&= \gamma \max\Big\{\max_{k=0,1,\ldots,n-1}\big\{Y(t_k)e^{-\int_{t_k}^{t_n} r(u)\,du}e^{g(T-t_k)}\big\}, Y(T)\Big\} \\
&= \max\big\{Y(T) - \tilde{\eta}(T), \gamma Y(T)\big\}.
\end{aligned}
$$

Hence, $Y(T) = \tilde{\xi}$ if and only if $\gamma = 1$ or $\tilde{\eta}(T) = 0$. Since $\mathbb{P}(\mathcal{B}) = 0$ implies $\gamma \leq 1$, we investigate the case of $\tilde{\eta}(T) = 0$ and $\gamma < 1$. From (3.3) we see that $Y(t_1) = \gamma Y(0)e^{-\int_0^{t_1} r(u)\,du}e^{gT}D(t_1) + \tilde{\eta}(t_1)$ and $Y(0) = \gamma Y(0)e^{gT}D(0) + \mathbb{E}^{\mathbb{Q}}[\tilde{\eta}(t_1)]$. We choose $\tilde{\eta}(t_1)$ which yields a strictly positive pay-off with a positive probability (assuming $Y(0) > 0$). Consider $m = 1,\ldots,n-1$. Following (3.4) we derive

$$
\begin{aligned}
(3.5)\quad &\mathbb{E}^{\mathbb{Q}}[\tilde{\eta}(t_{m+1}) \,|\, \mathcal{F}_{t_m}]\\
&= Y(t_m) - \gamma \max_{k=0,1,\ldots,m}\big\{Y(t_k)e^{-\int_{t_k}^{t_m} r(u)\,du}e^{g(T-t_k)}\big\}D(t_m)\\
&= Y(t_m)\\
&\quad - \gamma \max\Big\{\max_{k=0,1,\ldots,m-1}\big\{Y(t_k)e^{-\int_{t_k}^{t_m} r(u)\,du}e^{g(T-t_k)}\big\}, Y(t_m)e^{g(T-t_m)}\Big\}D(t_m)\\
&= Y(t_m)\\
&\quad - \max\big\{Y(t_m) - \tilde{\eta}(t_m), \gamma Y(t_m)e^{g(T-t_m)}D(t_m)\big\}, \quad m = 1, \ldots, n-1,
\end{aligned}
$$

where we use the representation (3.3). We conclude that $\mathbb{P}(\mathcal{C}) = 1$ implies that $\mathbb{P}(\tilde{\eta}(T) = 0) = 1$. The candidate solution is a solution to (3.2). The case of $\gamma = 1$ is also included in assertion 3.

4. We have

$$\mathbb{P}(\tilde{\eta}(T) = 0) = \mathbb{P}(\tilde{\eta}(T) = 0 \,|\, \mathcal{C})\mathbb{P}(\mathcal{C}) + \mathbb{P}(\tilde{\eta}(T) = 0 \,|\, \mathcal{B}^c \setminus \mathcal{C})\mathbb{P}(\mathcal{B}^c \setminus \mathcal{C})$$

and $\mathbb{P}(\tilde{\eta}(T) = 0 \,|\, \mathcal{C}) = 1$. Hence, $\mathbb{P}(\tilde{\eta}(T) = 0) = 1$ if and only if $\mathbb{P}(\mathcal{C}) = 1$ or $\mathbb{P}(\tilde{\eta}(T) = 0 \,|\, \mathcal{B}^c \setminus \mathcal{C}) = 1$.

5. If $\mathbb{P}(\mathcal{C}) = 0$, then we deduce that $\mathbb{P}(\tilde{\eta}(T) > 0) > 0$ for any $\tilde{\eta}(T)$ satisfying (3.5). The assertion follows.

Strict positivity of the solution for $Y(0) > 0$ is obvious. ∎

The hedging of the ratchet (3.1) is only possible if at any time t we end up with a portfolio $X(t)$ which is sufficient to hedge at least its accumulated value $\gamma X(t)e^{g(T-t)}$. If we cannot guarantee that $\gamma e^{g(T-t)}D(t) \le 1$, then the terminal investment portfolio value may fall below the value of the ratchet. A financial institution would not issue the ratchet option under the assumptions of item 1 of Theorem 3.1. Hence, the time-delayed BSDE gives the zero solution.

Under the assumptions of item 5 of Theorem 3.1 the policyholder could receive only a part of the capital which he or she really owns as $\mathbb{P}(Y(T) > \tilde{\xi}) > 0$ (see the proof of item 3 of Theorem 3.1). The ratchet option is not fair from the point of view of the policyholder. Again, the zero solution arises.

If we set (g, γ) in such a way that the conditions of item 2 of Theorem 3.1 hold, then it is possible to hedge the ratchet (3.1) perfectly for any initial premium. It should be noticed that by choosing sufficiently large g and low γ the conditions stated in item 2 may be satisfied. In this case hedging the path-dependent ratchet option on the investment portfolio is equivalent to hedging the fraction γ of the guaranteed return g on the initial premium at the terminal time. This investment strategy yields a priori known return related to (γ, g) with no potential for additional profit. This is not a construction which would be implemented in real life as it is not very appealing to policyholders.

The most important solution to our time-delayed BSDE from the practical point of view is the solution constructed in item 3 of Theorem 3.1. If we set (g,γ) in such a way that the conditions of item 3 hold, then it is possible to hedge the ratchet (3.1) perfectly for any initial premium as in item 2. The key point is that in this case there is a potential for an unbounded growth in the portfolio value over the fixed guaranteed return g which is controlled by the sequence $\tilde{\eta}$. The most common guarantee of paying back the initial premium and the highest earned investment gain, $\gamma = 1, g = 0$, fits item 3.

To specify the solution completely we must decide on $\tilde{\eta}$. The choice of $\tilde{\eta}$ is crucial from the practical point of view. In order to choose $\tilde{\eta}$ an additional constraint for $\tilde{\eta}$ (or for Y, Z) is needed. We do not intend to discuss this step here. We remark that the discrete time process $\tilde{\eta}$ is an analogue of the value process of unconstrained allocation from El Karoui et al. (2005), where the authors suggest applying utility maximization to determine the target return $\tilde{\eta}$ (see El Karoui et al. (2005) for details).

The investment strategy from item 4 has a similar interpretation to the strategy from item 3. The difference is that under item 4 we have more freedom in choosing (g,γ) but an additional restriction on $\tilde{\eta}$ is imposed. For an example of a strategy under item 4 see Delong (2011). We conclude that multiple solutions to the time-delayed BSDE arise as the claim can be hedged for any initial premium (item 2) and under different product designs related to different choices of $\tilde{\eta}$ (items 3–4).

It should be noticed that item 3 of Theorem 3.1 gives a multi-period Option Based Portfolio Insurance strategy which has been derived from the time-delayed BSDE. For a semi-static version of the multi-period OBPI strategy we refer to Delong (2011).

4. Hedging a ratchet contingent on the maximum value of the portfolio—the continuous time case. We still consider the liability of a ratchet type contingent on the investment portfolio but now in continuous time. We investigate the following claim:

$$\xi = \gamma \sup_{s\in[0,T]} \{X(s)e^{g(T-s)}\}. \tag{4.1}$$

First, we present counterparts of items 1–2 of Theorem 3.1.

THEOREM 4.1. *Assume that* (A1)–(A3) *hold. Consider the time-delayed BSDE*

$$\begin{aligned} Y(t) = \gamma \sup_{0\le t\le T} \big\{Y(t)e^{-\int_t^T r(s)\,ds+g(T-t)}\big\} \\ - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \quad 0\le t\le T. \end{aligned} \tag{4.2}$$

Set

$$\mathcal{D} = \Big\{\omega \in \Omega : \gamma \sup_{0\le t\le T} \{e^{g(T-t)} D(t)\} > 1\Big\}.$$

The equation (4.2) *has the following square integrable solutions under the requirement that* $Y(0) \ge 0$*:*

1. *If* $\mathbb{P}(\mathcal{D}) > 0$*, then there exists a unique solution* $Y = Z = 0$.
2. *If* $\mathbb{P}(\mathcal{D}) = 0$*,* $\gamma e^{gT} D(0) = 1$*, then there exist multiple solutions* (Y, Z)*, which differ in* $Y(0)$*, of the form*

$$Y(t) = \gamma Y(0) e^{g - \int_0^t r(s)\, ds} D(t), \qquad 0 \le t \le T,$$
$$Y(0) e^{gT - \int_0^T r(s)\, ds} = Y(0) + \int_0^T Z(s)\, dW(s).$$

The solution Y *is strictly positive provided that* $Y(0) > 0$.

Proof. The proof is analogous to the proof of items 1–2 of Theorem 3.1. In particular, we can show that any solution Y to (4.2) must have a representation

$$Y(t) = \gamma \sup_{0\le s\le t} \big\{Y(s) e^{g(T-s)} e^{-\int_s^t r(u)\, du}\big\} D(t) + \tilde{\eta}(t), \qquad 0 \le t \le T, \tag{4.3}$$

with a square integrable, non-negative and $\mathbb{F}$-adapted process $(\tilde{\eta}(t))_{t\in[0,T]}$. ∎

In contrast to discrete time (see item 2 of Theorem 3.1), in continuous time we cannot always find (g, γ) satisfying the assumptions of item 2 of Theorem 4.1.

Proposition 4.1. *Let* $\mathcal{D}$ *be the set defined in Theorem* 4.1. *In the Cox–Ingersoll–Ross interest rate model we cannot find* $g \ge 0$ *and* $\gamma = 1/(e^{gT} D(0))$ *such that* $\mathbb{P}(\mathcal{D}) = 0$.

Proof. We have to find $g \ge 0$ such that $\sup_{0\le t\le T}\{e^{-gt} D(t)/D(0)\} \le 1$. If it were possible, then

$$e^{gt} \ge \frac{e^{n(t) - m(t) r(t)}}{e^{n(0) - m(0) r(0)}}, \qquad 0 \le t \le T,$$

would hold with some continuous functions n, m defining the bond price in the CIR model (see Cairns (2004)), and equivalently the following condition would hold:

$$r(t) \ge \frac{gt - n(t) + n(0) - m(0) r(0)}{-m(t)}, \qquad 0 \le t \le T_0 < T. \tag{4.4}$$

Consider the continuous function

$$h(t) = \frac{gt - n(t) + n(0) - m(0) r(0)}{-m(t)} \quad \text{on } [0, T_0] \quad \text{with} \quad h(0) = r(0) > 0.$$

For any finite g, due to continuity of $t \mapsto h(t)$, we have $h(t) \geq \epsilon > 0$ on some small time interval $[0, \lambda]$. However, $r(\lambda) < \epsilon$ with positive probability. Hence, the condition (4.4) is violated with positive probability. ■

What appears most interesting in continuous time is an extension of items 3 and 4 of Theorem 3.1. First, we show how to construct a process X which satisfies the condition $X(t) \geq \gamma \sup_{s\leq t}\{X(s)e^{g(T-s)}\}D(t)$, which has to be satisfied by (4.3).

Our next result concerns an extension of the dynamics under a drawdown constraint from Cvitanić and Karatzas (1995). Compared to Cvitanić and Karatzas (1995) we require that the controlled process X is above a fraction of its running maximum where the running maximum involves the process X accumulating with a growth rate and not the process X itself, and the fraction is a stochastic process and not a constant.

Proposition 4.2. *Assume that* (A1)–(A3) *hold, together with*

$$\gamma e^{gT} D(0) < 1, \qquad \gamma \sup_{0\leq t\leq T} \{e^{g(T-t)}D(t)\} \leq 1, \qquad \sup_{0\leq t\leq T} |\sigma(t)| \leq K.$$

Choose an $\mathbb{F}$-predictable process U such that

$$\mathbb{E}\left[\int_0^T \left|\frac{U(s)}{D(s)}\right|^2 ds\right] < \infty,$$

and consider a process S under the control U with the forward dynamics

$$dS(t) = U(t)\frac{dD(t)}{D(t)} + (S(t) - U(t))\frac{dB(t)}{B(t)}, \qquad S(0) = s_0 > 0.$$

There exists a unique, continuous, $\mathbb{P}$-square integrable solution to the forward SDE

$$\begin{aligned} dX(t) &= \Big(\gamma \sup_{0\leq s\leq t}\{X(s)e^{g(T-s)}\}D(t)\Big)\frac{dD(t)}{D(t)} \\ &\quad + \Big(X(t) - \gamma \sup_{0\leq s\leq t}\{X(s)e^{g(T-s)}\}D(t)\Big)\mathbf{1}\{S(t) > 0\}\frac{dS(t)}{S(t)}, \end{aligned}$$

$$X(0) = x > 0, \tag{4.5}$$

which satisfies $X(t) \geq \gamma \sup_{s\leq t}\{X(s)e^{g(T-s)}\}D(t)$ on $[0,T]$.

Proof. We follow the idea from Cvitanić and Karatzas (1995). We deal with the discounted processes $V(t) = X(t)/D(t)$ and $R(t) = S(t)/D(t)$. By Itô's formula we obtain the dynamics

$$dV(t) = \Big(V(t) - \gamma \sup_{0\leq s\leq t}\{V(s)D(s)e^{g(T-s)}\}\Big)\mathbf{1}\{R(t) > 0\}\frac{dR(t)}{R(t)}, \tag{4.6}$$

$$dR(t) = \Big(-R(t)\theta(t)\sigma(t) + R(t)\sigma^2(t) + \frac{U(t)}{D(t)}\theta(t)\sigma(t) - \frac{U(t)}{D(t)}\sigma^2(t)\Big)\,dt + \Big(\frac{U(t)}{D(t)}\sigma(t) - R(t)\sigma(t)\Big)\,dW(t). \tag{4.7}$$

Let

$$M(t) = \sup_{0\le s\le t}\Big\{\frac{V(s)D(s)e^{g(T-s)}}{D(0)e^{gT}}\Big\}.$$

Consider the sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times defined by $\tau_n = \tau_n^D \wedge \tau_n^R \wedge T$ where $\tau_n^D = \inf\{t : \gamma e^{g(T-t)}D(t) = 1 - 1/n\}$ and $\tau_n^R = \inf\{t : R(t) = 1/n\}$. We first solve the equation (4.5) on $[0, \tau_n]$. We rewrite the dynamics (4.6) as

$$dV(t) = (V(t) - \gamma M(t)D(0)e^{gT})\frac{dR(t)}{R(t)}. \tag{4.8}$$

By applying Itô's formula we derive

$$\begin{aligned} d\Big(\frac{V(t)}{M(t)}\Big) &= \Big(\frac{V(t)}{M(t)} - \gamma D(0)e^{gT}\Big)\frac{dR(t)}{R(t)} - V(t)\frac{dM(t)}{M^2(t)} \\ &= \Big(\frac{V(t)}{M(t)} - \gamma D(0)e^{gT}\Big)\frac{dR(t)}{R(t)} - \frac{D(0)e^{gt}}{D(t)}\,\frac{dM(t)}{M(t)}, \end{aligned}$$

and

$$\begin{aligned} d\Big(\log\Big(\frac{V(t)}{M(t)} - \gamma D(0)e^{gT}\Big)\Big) &= \frac{dR(t)}{R(t)} - \frac{1}{2}\frac{d[R](t)}{R^2(t)} - \frac{1}{\frac{V(t)}{M(t)} - \gamma D(0)e^{gT}}\,\frac{D(0)e^{gt}}{D(t)}\,\frac{dM(t)}{M(t)} \\ &= \frac{dR(t)}{R(t)} - \frac{1}{2}\frac{d[R](t)}{R^2(t)} - \frac{1}{1-\gamma D(t)e^{g(T-t)}}\,\frac{dM(t)}{M(t)}, \end{aligned}$$

where we use the localizing sequence

$$\tau_m = \inf\{t : V(t)/M(t) - \gamma D(0)e^{gT} = 1/m\}$$

and let $m \to \infty$. Notice that $V(t) > M(t)D(0)e^{gT}$ on $[0, \tau_n]$. We next obtain the key relation

$$\begin{aligned} &\log\Big(\frac{V(t)}{M(t)} - \gamma D(0)e^{gT}\Big) - \log\Big(\frac{D(0)}{D(t)}e^{gt} - \gamma D(0)e^{gT}\Big) \\ &= \log\big(1-\gamma D(0)e^{gT}\big) - \log\Big(\frac{D(0)}{D(t)}e^{gt} - \gamma D(0)e^{gT}\Big) + \log R(t) - \log R(0) \\ &\quad - \int_0^t \frac{1}{1-\gamma D(s)e^{g(T-s)}}\,\frac{dM(s)}{M(s)}, \qquad 0 \le t \le \tau_n. \end{aligned}$$

By applying the Skorokhod equation we find the unique processes (K, L) such that

$$L(t) = \log\big(1 - \gamma D(0)e^{gT}\big) - \log\Big(\frac{D(0)}{D(t)}e^{gt} - \gamma D(0)e^{gT}\Big) + \log R(t) - \log R(0), \tag{4.9}$$

$$K(t) = \int_0^t \frac{1}{1 - \gamma D(s)e^{g(T-s)}}\frac{dM(s)}{M(s)} = \sup_{0\le s\le t} L(t),$$

$$L(t) - K(t) = \log\Big(\frac{V(t)}{M(t)} - \gamma D(0)e^{gT}\Big) - \log\Big(\frac{D(0)}{D(t)}e^{gt} - \gamma D(0)e^{gT}\Big),$$

for $0 \le t \le \tau_n$. Notice that $L(0) = K(0) = 0$ and $K(t) \ge 0$. From the equations (4.9) we can derive a unique solution to (4.5) in the form

$$\begin{aligned} M(t) &= V(0)e^{\int_0^t (1-\gamma D(s)e^{g(T-s)})\,dK(s)}, \quad 0 \le t \le \tau_n, \\ V(t) &= M(t)\Big[\gamma D(0)e^{gT} \\ &\quad + (1 - \gamma D(0)e^{gT})\frac{R(t)}{R(0)}e^{-K(t)}\Big], \quad 0 \le t \le \tau_n. \end{aligned} \tag{4.10}$$

Let $\tau_\infty^{D,R} = \lim_{n\to\infty} \tau_n^D \wedge \tau_n^R$. We extend the solution to $[0, \tau_\infty^{D,R} \wedge T]$. By taking the limits we can consider measurable processes (L, K) and (M, V) on $[0, \tau_\infty^{D,R} \wedge T]$ defined by (4.9) and (4.10). It is straightforward to show that $M(t) \le V(0)e^{K(t)}$ on $[0, \tau_\infty^{D,R} \wedge T]$ under the condition that $\gamma \sup_{0\le t\le T}\{e^{g(T-t)}D(t)\} \le 1$. Hence

$$\begin{aligned} 0 \le \psi(t) &= V(t) - \gamma M(t)D(0)e^{gT} \\ &\le (1 - \gamma D(0)e^{gT})V(0)\frac{R(t)}{R(0)}, \quad 0 \le t \le \tau_\infty^{D,R} \wedge T. \end{aligned} \tag{4.11}$$

We now investigate the process $\tilde{V}$ defined by

$$\tilde{V}(t) = V(0) + \int_0^t \psi(s)\mathbf{1}\{R(s) > 0\}\frac{dR(s)}{R(s)}, \quad 0 \le t \le \tau_\infty^{D,R} \wedge T,$$

which coincides with the solution (4.10) to the equation (4.8) on $[0, \tau_n]$. One can show that $\tilde{V}$ is a continuous square integrable semimartingale under the assumptions of our proposition and the derived bound (4.11). Hence we obtain the convergence

$$\tilde{V}(t) = \lim_{n\to\infty} \tilde{V}(t \wedge \tau_n) = \lim_{n\to\infty} V(t \wedge \tau_n), \quad 0 \le t \le \tau_\infty^{D,R} \wedge T.$$

This also implies that M can be extended as an a.s. finite process to $[0,\tau_\infty^{D,R}\wedge T]$. We can now conclude that $V(t)\geq\gamma M(t)D(0)e^{gT}$ on $[0,\tau_\infty^{D,R}\wedge T]$, which follows from (4.10). If $\tau_\infty^{D,R}>T$, then $R(T)e^{-K(T)}>0$. If $\tau_\infty^{D,R}\leq T$, then $R(\tau_\infty^{D,R})e^{-K(\tau_\infty^{D,R})}=0$ and we end up with $V(\tau_\infty^{D,R})=\gamma M(\tau_\infty^{D,R})D(0)e^{gT}$. This implies that $dV(t)=0$ for $t>\tau_\infty^{D,R}$ and our solution V is defined to be constant after time $\tau_\infty^{D,R}$. One can easily check that $M(t)=M(\tau_\infty^{D,R})$ for $t\geq\tau_\infty^{D,R}$. Indeed, we have

$$\begin{aligned}M(t)&=\sup_{0\leq s\leq t}\Big\{\frac{V(s)D(s)e^{g(T-s)}}{D(0)e^{gT}}\Big\}\\&=\max\Big\{M(\tau_\infty^{D,R}),\sup_{\tau_\infty^{D,R}\leq s\leq t}\Big\{\frac{V(s)D(s)e^{g(T-s)}}{D(0)e^{gT}}\Big\}\Big\}\\&=\max\Big\{M(\tau_\infty^{D,R}),M(\tau_\infty^{D,R})\sup_{\tau_\infty^{D,R}\leq s\leq t}\{\gamma e^{g(T-s)}D(s)\}\Big\}\\&=M(\tau_\infty^{D,R}),\qquad t\geq\tau_\infty^{D,R}.\end{aligned}$$

Thus $V(t)\geq\gamma M(t)D(0)e^{gT}$ and $X(t)\geq\gamma\sup_{s\leq t}\{X(s)e^{g(T-s)}\}D(t)$ on $[0,T]$. Finally, square integrability of X is easily deduced from square integrability of V. ∎

We now give counterparts of items 3–5 of Theorem 3.1

THEOREM 4.2. *Let the assumptions of Proposition 4.2 hold. Consider the time-delayed BSDE*

$$\begin{aligned}dX(t)&=\pi(t)\frac{dD(t)}{D(t)}+(X(t)-\pi(t))\frac{dB(t)}{B(t)},\\X(T)&=\gamma\sup_{0\leq s\leq T}\{X(s)e^{g(T-s)}\}.\end{aligned}\tag{4.12}$$

Set

$$\begin{aligned}\mathcal{D}&=\Big\{\omega\in\Omega:\gamma\sup_{0\leq t\leq T}\{e^{g(T-t)}D(t)\}>1\Big\},\\\mathcal{E}&=\Big\{\omega\in\Omega:\gamma\sup_{0\leq t\leq T}\{e^{g(T-t)}D(t)\}=1\Big\}.\end{aligned}$$

The equation (4.12) *has the following $\mathbb{P}$-square integrable solutions under the requirement that $X(0)\geq 0$:*

1. *If $\mathbb{P}(\mathcal{D})=0$, $\gamma e^{gT}D(0)<1$, $\mathbb{P}(\mathcal{E})=1$, then there exist multiple solutions (X,π). The process X is defined in Proposition* 4.2 *and may have different $(X(0),S)$; the control process π is given by*

$$\pi(t) = \gamma \sup_{0\le s\le t}\{X(s)e^{g(T-s)}\}D(t) + \frac{U(t)}{S(t)}\Big(X(t) - \gamma \sup_{0\le s\le t}\{X(s)e^{g(T-s)}\}D(t)\Big)\mathbf{1}\{S(t)>0\}, \quad 0\le t\le T.$$

2. *If* $\mathbb{P}(\mathcal{D}) = 0$, $\gamma e^{gT}D(0) < 1$, $0 < \mathbb{P}(\mathcal{E}) < 1$, $\mathbb{P}(S(t) = 0$ *for some* $t\in[0,T] \,|\, \mathcal{D}^c\backslash\mathcal{E}) = 1$, *then there exist multiple solutions* (X,π) *defined in item* 1.
3. *In the remaining cases, there exists a unique solution* $Y = Z = 0$.

The solution X *is strictly positive provided that* $X(0) > 0$.

Proof. The result follows from Proposition 4.2. We have to investigate the terminal value

$$V(T) = M(T)\bigg[\gamma D(0)e^{gT} + (1-\gamma D(0)e^{gT})\frac{R(T)}{R(0)}e^{-K(T)}\bigg]\mathbf{1}\{\tau_\infty^{R,D} > T\} + M(T)\gamma D(0)e^{gT}\mathbf{1}\{\tau_\infty^{R,D} \le T\}$$

and the terminal condition $\tilde{\xi} = \gamma M(T)D(0)e^{gT}$. Hence, $V(T) = \tilde{\xi}$ if and only if $\mathbb{P}(\tau_\infty^{R,D} \le T) = 1$. We conclude as in the proof of Theorem 3.1. ∎

The conclusions and interpretations for hedging the ratchet in continuous time are analogous to those stated in Section 3. The most common guarantee of paying back the initial premium and the highest earned investment gain, $\gamma = 1, g = 0$, fits item 1 of Theorem 4.2.

5. Hedging a pay-off contingent on the average value of the portfolio. Apart from the ratchet studied in the previous sections, another common path-dependent pay-off in finance is the pay-off of Asian type. Many participating contracts have profit sharing schemes which are based on an average value of the asset portfolio, and such averaging of returns is called smoothing (see Ballotta and Haberman (2009)). There exist variable annuities/unit-linked products in the market under which a bonus as an average value of the policyholder's account is paid additionally at maturity.

Let us consider a participating contract or a unit-linked contract which provides a return linked to a benchmark process S and offers a terminal bonus under a profit-sharing scheme. The profit sharing scheme or bonus is based on the average value of the assets managed by the insurer within the duration of the policy. We deal with the claim

$$\xi = \beta X(0)S + \gamma\frac{1}{T}\int_0^T e^{\int_s^T r(u)\,du}X(s)\,ds. \tag{5.1}$$

We derive the corresponding investment strategy.

THEOREM 5.1. *Assume that* (A1)–(A3) *hold,* $S \geq 0$, $\mathbb{Q}(S > 0) > 0$ *and* $\mathbb{E}^{\mathbb{Q}}[|\tilde{S}|^2] < \infty$. *The time-delayed BSDE*

$$Y(t) = \beta Y(0)\tilde{S} + \gamma\frac{1}{T}\int_0^T Y(s)\,ds - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \leq t \leq T, \tag{5.2}$$

has the following square integrable solutions under the requirement that $Y(0) \geq 0$ *and* $\int_0^T Y(s)\,ds \geq 0$:

1. *If* $\beta\mathbb{E}^{\mathbb{Q}}[\tilde{S}] + \gamma = 1$, *then there exist multiple solutions* (Y, Z), *which differ in* $Y(0)$, *of the form*

$$Y(t) = Y(0) + \int_0^t Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \leq t \leq T,$$

with the $\mathbb{F}$*-predictable control*

$$Z(t) = \frac{1}{1-\gamma+\gamma\frac{t}{T}}M(t), \qquad 0 \leq t \leq T,$$

and the process M *derived from the martingale representation*

$$\beta Y(0)\tilde{S} = \beta Y(0)\mathbb{E}^{\mathbb{Q}}[\tilde{S}] + \int_0^T M(t)\,dW^{\mathbb{Q}}(t).$$

The solution Y *is strictly positive provided that* $Y(0) > 0$.

2. *If* $\beta\mathbb{E}^{\mathbb{Q}}[\tilde{S}] + \gamma \neq 1$, *then there exists a unique solution* $Y = Z = 0$.

REMARK. The requirement $\int_0^T Y(s)\,ds \geq 0$ is imposed as the bonus cannot be negative.

Proof. By evaluating (5.2) at $t = 0$ we conclude that (Y, Z) must satisfy

$$Y(0) + \int_0^T Z(s)\,dW^{\mathbb{Q}}(s) = \beta Y(0)\tilde{S} + \gamma\frac{1}{T}\int_0^T Y(s)\,ds. \tag{5.3}$$

By recalling the forward dynamics of the discounted portfolio value (2.6) we can calculate by Fubini's theorem for stochastic integrals that

$$\begin{aligned}\frac{1}{T}\int_0^T Y(s)\,ds &= \frac{1}{T}\int_0^T \Big(Y(0) + \int_0^s Z(u)\,dW^{\mathbb{Q}}(u)\Big)\,ds \\ &= Y(0) + \int_0^T \Big(1 - \frac{s}{T}\Big)Z(s)\,dW^{\mathbb{Q}}(s).\end{aligned} \tag{5.4}$$

By substituting the above relation into (5.3) we find that the pair (Y, Z) must satisfy

$$Y(0)(1-\gamma) + \int_0^T \Big(1-\gamma+\gamma\frac{s}{T}\Big) Z(s)\,dW^{\mathbb{Q}}(s) = \beta Y(0)\tilde{S}. \tag{5.5}$$

1. Choose the process M to satisfy the martingale representation of $\beta Y(0)\tilde{S}$ and the process Z to satisfy (5.5). We assume that $\beta > 0$, hence $1-\gamma > 0$. The denominator in the definition of Z is strictly positive and square integrability of M implies square integrability of Z. We now prove that Y satisfies the requirements of our proposition. This is trivial if $Y(0) = 0$. Assume that $Y(0) > 0$. We show that Y is strictly positive. By substituting the derived solution into (5.4) we can calculate

$$\begin{aligned}\frac{1}{T}\int_0^T Y(t)\,dt &= Y(0) + \int_0^T \Big(1-\frac{t}{T}\Big)\frac{1}{1-\gamma+\gamma\frac{t}{T}} M(t)\,dW^{\mathbb{Q}}(t) \\ &= Y(0) + \int_0^T h(t)M(t)\,dW^{\mathbb{Q}}(t),\end{aligned} \tag{5.6}$$

with

$$h(t) = \frac{T-t}{T-\gamma T+\gamma t}, \qquad 0 \le t \le T.$$

By integration by parts we obtain

$$\begin{aligned}0 &= h(T)\int_0^T M(t)\,dW^{\mathbb{Q}}(t) \\ &= \int_0^T h(t)M(t)\,dW^{\mathbb{Q}}(t) + \int_0^T\int_0^t M(s)\,dW^{\mathbb{Q}}(s)\,h'(t)\,dt,\end{aligned}$$

and using the martingale representation of $\beta Y(0)\tilde{S}$ we conclude that

$$\begin{aligned}\int_0^T h(t)M(t)\,dW^{\mathbb{Q}}(t) &= -\int_0^T\int_0^t M(s)\,dW^{\mathbb{Q}}(s)\,h'(t)\,dt \\ &= -\int_0^T (V(t)-V(0))h'(t)\,dt,\end{aligned} \tag{5.7}$$

where $V(t) = \beta Y(0)\mathbb{E}^{\mathbb{Q}}[\tilde{S}\,|\,\mathcal{F}_t]$. Rearranging (5.7) and substituting into (5.6) we arrive at

$$\begin{aligned}\frac{1}{T}\int_0^T Y(s)\,ds &= Y(0) + \beta Y(0)\mathbb{E}^{\mathbb{Q}}[\tilde{S}](h(T)-h(0)) - \int_0^T V(t)h'(t)\,dt \\ &= -\int_0^T V(t)h'(t)\,dt > 0.\end{aligned}$$

The inequality is deduced from non-negativity of V, continuity of $t \mapsto V(t)$, strict positivity of $V(0) = \beta Y(0)\mathbb{E}^{\mathbb{Q}}[\tilde{S}]$ and strict negativity of $h'(t) = -T/(T + \gamma t - \gamma T)^2$. Strict positivity of Y follows by taking the conditional expected value in (5.2), taking into account strict positivity of the bonus under the strategy and non-negativity of the benchmark return.

2. By taking the expected value in (5.5) we arrive at a contradiction unless $Y(0) = 0$. Taking the expected value in (5.3) we obtain $\mathbb{E}^{\mathbb{Q}}[\int_0^T Y(s)\,ds] = 0$ and by the non-negativity requirement we arrive at $Y(t) = 0$, $0 \le t \le T$.

We can allow $\beta = 0$ as well. In this case we conclude that if $\gamma = 1$ then $Z(t) = 0$, $Y(t) = Y(0)$, and if $\gamma \neq 1$ then $Z(t) = Y(t) = 0$. This case is included in our proposition. ■

We can also provide a similar result for the claim

$$\xi = S + \gamma \frac{1}{T} \int_0^T e^{\int_s^T r(u)\,du} X(s)\,ds. \tag{5.8}$$

THEOREM 5.2. *Assume that* (A1)–(A3) *hold,* $S \ge 0$, $\mathbb{Q}(S > 0) > 0$ *and* $\mathbb{E}^{\mathbb{Q}}[|\tilde{S}|^2] < \infty$. *The time-delayed BSDE*

$$Y(t) = \tilde{S} + \gamma \frac{1}{T} \int_0^T Y(s)\,ds - \int_t^T Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \le t \le T, \tag{5.9}$$

has the following square integrable solutions under the requirement that $Y(0) \ge 0$ *and* $\int_0^T Y(s)\,ds \ge 0$:

1. *If* $\gamma < 1$, *then there exists a unique solution* (Y, Z) *of the form*

$$Y(t) = \frac{\mathbb{E}^{\mathbb{Q}}[\tilde{S}]}{1-\gamma} + \int_0^t Z(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \le t \le T,$$

with the $\mathbb{F}$*-predictable control*

$$Z(t) = \frac{1}{1 - \gamma + \gamma\frac{t}{T}} M(t), \qquad 0 \le t \le T,$$

and the process M *derived from the martingale representation of*

$$\tilde{S} = \mathbb{E}^{\mathbb{Q}}[\tilde{S}] + \int_0^T M(t)\,dW^{\mathbb{Q}}(t).$$

The solution Y *is strictly positive provided that* $Y(0) > 0$.

2. *If* $\gamma \ge 1$, *then there exists no solution.*

Item 1 of Theorem 5.1 with $\beta \in (0,1)$ is practically relevant and provides a replicating strategy for the claim (5.1) for any initial premium. Our investment strategy is to split the contribution $X(0)$ into two parts: the first part $\beta X(0)$ is used to replicate the base return $\beta X(0) S$, the second part

$\gamma X(0)$ is used to hedge the bonus (notice that $\mathbb{E}^{\mathbb{Q}}[\tilde{S}] = 1$ as S models the benchmark return $S(T)/S(0)$ in the arbitrage-free market). The investment strategy H for hedging the participation bonus and the value of the corresponding replicating portfolio G can be derived by solving the time-delayed BSDE

$$G(t) = \gamma\beta X(0)\frac{1}{T}\int_0^T \tilde{S}(t)\,dt + \gamma\frac{1}{T}\int_0^T G(s)\,ds - \int_t^T H(s)\,dW^{\mathbb{Q}}(s), \qquad 0 \le t \le T,$$

which is of the form (5.9). The investment portfolio X backing the contract is of the form $X(t) = \beta X(0)S(t) + G(t)e^{\int_0^t r(s)\,ds}$ where $S(t)$ is the value of the benchmark investment providing the base return, and $G(t)e^{\int_0^t r(s)\,ds}$ is the value of the replicating portfolio hedging the participation bonus. Such a decomposition is important as Solvency II Directive requires a disclosure of the value of all guarantees and participation benefits (see TP.2.87 in QIS5 (2010)). By construction, the assets and the liabilities are matched.

6. Hedging a stream of payments based on the maximum value of the portfolio. In this final section we give an example of a claim which leads to an equation with a time delay entering the generator of a BSDE.

Under a variable annuity with a guaranteed minimum withdrawal benefit the policyholder is allowed to withdraw guaranteed amounts over the duration of the contract and receives the remaining value of the account at maturity. Huang et al. (2009) investigated a variable annuity with a withdrawal benefit set as a fraction of the running maximum of the account value. Inspired by Huang et al. (2009) we consider an insurance product under which the policyholder can withdraw a guaranteed amount set as a fraction γ of the running maximum of the investment account, and at maturity the remaining value is converted into a life-time annuity with a guaranteed consumption rate L. Such a product would allow for higher consumption in the times of booming financial markets before locking the accumulated money into a fixed life-time annuity. This is an example of an income drawdown option in retirement planning (see Emms and Haberman (2008)). We have to find an investment strategy π under which the dynamics of the investment account satisfies

$$\begin{aligned} dX(t) &= \pi(t)(\mu(t)\,dt + \sigma(t)\,dW(t)) + (X(t) - \pi(t))r(t)\,dt \\ &\quad - \gamma \sup_{s\in[0,t]}\{X(s)\}\,dt, \\ X(T) &= La(T), \end{aligned} \tag{6.1}$$

where a denotes the annuity factor

$$a(T) = \mathbb{E}^{\mathbb{Q}}\Big[\int_T^{T_0} e^{-\int_T^s r(u)\,du}\,ds \;\Big|\; \mathcal{F}_T\Big], \qquad T < T_0 < \infty.$$

In the traditional approach, the account value X follows an uncontrolled process and a hedging fee is deducted from X in order to cover the withdrawal rate and the annuity. Our approach is to find a strategy π under which X is controlled to cover the liabilities.

We can prove the following result.

THEOREM 6.1. *Assume that* (A1)–(A3) *hold. The time-delayed BSDE*

$$Y(t) = L\tilde{a}(T) + \int_t^T \gamma \sup_{u\in[0,s]} \left\{Y(u)e^{-\int_u^s r(v)dv}\right\} ds - \int_t^T Z(s)\, dW^{\mathbb{Q}}(s), \quad 0 \le t \le T, \tag{6.2}$$

has a unique square integrable solution (Y, Z) *for sufficiently small* γ *or* T. *The solution* Y *is strictly positive.*

Proof. The existence, uniqueness and integrability follow from Theorem 2.1 in Delong and Imkeller (2010a). The generator of the time-delayed BSDE (6.2) does not fit exactly the framework of that theorem but one can see that it is possible to replace the time-delayed generator of integral form with a time-delayed generator dependent on the supremum norm and prove the existence of a unique solution to (6.2) by the contraction method (see Theorem 2.1 in Delong (2011)). To prove strict positivity notice that the time-delayed BSDE

$$\hat{Y}(t) = L\tilde{a}(T) + \int_t^T \gamma \max\left\{0, \sup_{u\in[0,s]} \left\{\hat{Y}(u)e^{-\int_u^s r(v)dv}\right\}\right\} ds - \int_t^T \hat{Z}(s)\, dW^{\mathbb{Q}}(s), \quad 0 \le t \le T, \tag{6.3}$$

has a unique solution (by Theorem 2.1 in Delong and Imkeller (2010a) and Theorem 2.1 in Delong (2011)) which satisfies

$$\hat{Y}(t) = \mathbb{E}^{\mathbb{Q}}\left[L\tilde{a}(T) + \int_t^T \gamma \max\left\{0, \sup_{u\in[0,s]} \left\{\hat{Y}(u)e^{-\int_u^s r(v)\,dv}\right\}\right\} ds \,\Big|\, \mathcal{F}_t\right] > 0.$$

By the uniqueness of solution to (6.2), we conclude that $Y = \hat{Y}$. ■

We remark that for such a retirement product the withdrawal rate γ is usually low (see Milevsky and Posner (2001) and Milevsky and Salisbury (2006)).

Unfortunately, we cannot solve the equation (6.2) analytically. For an attempt at solving (6.2) numerically and related difficulties see Delong (2011).

Moving a step further, we could also try to find an investment strategy π under which the dynamics of the investment account satisfies

$$(6.4)\quad \begin{aligned} dX(t) &= \pi(t)(\mu(t)\,dt + \sigma(t)\,dW(t)) + (X(t) - \pi(t))r(t)\,dt \\ &\qquad - \gamma \sup_{s\in[0,t]}\{X(s)\}\,dt, \\ X(T) &= \gamma \sup_{0\le s\le T}\{X(s)\}a(T). \end{aligned}$$

Compared to (6.1) the last withdrawal rate is now locked in the life-time annuity (see Huang et al. (2009)). The unique solution derived under Theorem 2.1 of Delong and Imkeller (2010a) and Theorem 2.1 of Delong (2011) is $X = \pi = 0$ and it is not clear at the moment how to derive a non-zero solution (if any). If multiple solutions exist, then an additional criterion has to be introduced in order to make the pricing and hedging problem numerically well-posed.

7. Conclusion. In this paper we have investigated novel applications of time-delayed backward stochastic differential equations. Time-delayed BSDEs may arise when seeking an investment strategy and an investment portfolio which should replicate a liability or meet a target depending on the strategy applied or the past values of the portfolio. We have solved new pricing, hedging and portfolio management problems for participating contracts and variable annuities. Our results can be extended to cover to the cases where the time horizon and the time delay are different entities and such a differentiation may be useful in actuarial and financial applications.

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