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BLOW-UP FOR THE ENERGY-CRITICAL NONLINEAR WAVE EQUATION AND SCHRÖDINGER EQUATION WITH INVERSE-SQUARE POTENTIAL

Abstract. We give a sufficient condition under which the solutions of the energy-critical nonlinear wave equation and Schrödinger equation with inverse-square potential blow up. The method is a modified variational approach, in the spirit of the work by Ibrahim et al. [Anal. PDE 4 (2011), 405–460].

1. Introduction. Consider the Cauchy problem

$$\begin{cases} \partial_t^2 u + P_a u = |u|^{1+4/(d-2)}, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x) \in \dot{H}^1(\mathbb{R}^d), \\ \partial_t u(0,x) = u_1(x) \in L^2(\mathbb{R}^d), \end{cases} \tag{1.1}$$

$$\begin{cases} \mathrm{i}\partial_t v - P_a v = -|v|^{4/(d-2)} v, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ v(0,x) = v_0(x) \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{1.2}$$

where $P_a = -\Delta + a/|x|^2$, $d \geq 3$, u is a real-valued function and v is a complex valued function defined on some space-time slab.

The wave equation (1.1) arises from the study of wave propagation on conic manifolds [4], while the Schrödinger equation (1.2) is a model used in quantum mechanics (see for example [2, 3, 6]). The operator $P_a = -\Delta + a/|x|^2$ has been studied in combustion theory (see [17]).

Planchon et al. [13] proved that Strichartz-type L^p estimates hold for the linear wave equation with inverse-square potential, under the assumption that the Cauchy data are spherically symmetric. Then these authors used these estimates to obtain the global well-posedness for the energy-critical nonlinear wave equation. In [14], they established a dispersive estimate for

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the wave equation with inverse-square potential. Furthermore, Burq et al. [1] proved spacetime weighted L^2 estimates for the Schrödinger and wave equation with inverse-square potential, and then deduced Strichartz estimate for these equations.

Miao et al. [11] studied maximal estimates for solutions to an initial value problem for the Schrödinger equation with inverse-square potential, and obtained the corresponding pointwise convergence result. In [12] they further studied Strichartz-type estimates of the solution for the linear wave equation with inverse-square potential and improved the range of admissible pairs under some condition. As an application, they showed the global well-posedness of the semilinear inverse-square potential wave equation with some small power in the radial case.

By a limiting argument, energy solutions of (1.1) and (1.2), respectively, obey the following energy conservation laws:

$$\begin{aligned} E(u,\partial_t u)(t) &\\ &:= \int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{2}|\partial_t u(x,t)|^2 + \frac{a}{2}\,\frac{|u(x,t)|^2}{|x|^2} - \frac{1}{2^*}|u(x,t)|^{2^*}\right] dx \\ &\equiv E(u_0,u_1), \\ E(v(t)) &:= \int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla v(x,t)|^2 + \frac{a}{2}\,\frac{|v(x,t)|^2}{|x|^2} - \frac{1}{2^*}|v(x,t)|^{2^*}\right] dx \equiv E(v_0). \end{aligned}$$

For the energy-critical nonlinear Schrödinger equation without potential,

$$iu_t + \Delta u = -|u|^{4/(d-2)}u, \tag{1.3}$$

Kenig and Merle [7] first obtained the blow-up and scattering theory of the radial solutions with energy below that of the ground state of

$$-\Delta W = |W|^{4/(d-2)}W. \tag{1.4}$$

Subsequently, Killip and Visan [10] made use of a double Duhamel argument from [9, 15] to remove the radial assumption. In these works, a finite time blow-up result was proved by a virial argument with the assumption that the corresponding energy is below the ground state and $\|\nabla u_0\|_2 \geq \|\nabla W\|_2$. For the energy-critical nonlinear wave equation without potential, a similar scattering and blow-up result was established by Kenig and Merle [8].

For the nonlinear wave and Schrödinger equation with inverse-square potential, there is a stationary solution W satisfying

$$P_a W = |W|^{1+4/(d-2)}. \tag{1.5}$$

The existence and uniqueness of solutions to (1.5) has been proved in [16]. In particular, the uniqueness in $L^{2^*}(B_1) \cup L^{2^*}(\mathbb{R}^d \cap B_1)$ was shown for $a \in (-(d-2)^2/4, 0]$, where B_1 is a unit ball in $\mathbb{R}^d$, and it was pointed out that under the same condition, the infimum

$$S_a = \inf_{u \in \mathcal{D}^{1,2} \setminus \{0\}} \frac{Q(u)}{\|u\|_{2^*}^2} \tag{1.6}$$

is attained at W.

Basing on the result for stationary solutions in [16], we shall give a sufficient condition for blow-up in our context.

1.1. Variational setting. Define the static energy

$$\mathcal{H}(\varphi) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla\varphi|^2 + \frac{a}{2} \frac{|\varphi|^2}{|x|^2} - \frac{1}{2^*} |\varphi|^{2^*} \right) dx,$$

the functional

$$\mathcal{K}(\varphi) = \int_{\mathbb{R}^d} \left(|\nabla\varphi|^2 + a \frac{|\varphi|^2}{|x|^2} - |\varphi|^{2^*} \right) dx,$$

and the quadratic form

$$Q(\varphi) = \int_{\mathbb{R}^d} \left(|\nabla\varphi|^2 + a \frac{|\varphi|^2}{|x|^2} \right) dx.$$

Note that $\mathcal{K} : \mathcal{H} \to \mathbb{R}$ is Fréchet differentiable.

Define

$$m_c = \inf\{\mathcal{H}(\varphi);\ 0 \neq \varphi \in \dot{H}^1,\ \mathcal{K}(\varphi) = 0\}. \tag{1.7}$$

Our main results are the following:

THEOREM 1.1. *Assume $a \in (-(d-2)^2/4, 0]$. Let $(u_0, u_1) \in \dot{H}^1 \times L^2$. Define*

$$\mathbb{W}^- = \{(u_0, u_1) \in \dot{H}^1 \times L^2;\ E(u_0, u_1) < m_c,\ \mathcal{K}(u_0) < 0\}.$$

If $(u_0, u_1) \in \mathbb{W}^-$, then the corresponding solution u of (1.1) *blows up in finite time.*

THEOREM 1.2. *Assume $a \in (-(d-2)^2/4, 0]$. Let $v_0 \in H^1$. Define*

$$\mathbb{S}^- = \{v_0 \in \dot{H}^1;\ E(v_0) < m_c,\ \mathcal{K}(v_0) < 0\}.$$

Let $v_0 \in \mathbb{S}^-$ and assume that either $xv_0 \in L^2$ or $v_0 \in H^1$ is radial. Then the corresponding solution v of (1.2) *blows up in finite time.*

This paper is organized as follows. In Section 3, we shall show some variational estimates. The main theorems are proved in Section 4.

2. Preliminaries. In this section, we introduce notation and recall some known results.

2.1. Notation. We write $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$.

2.2. Local theory. The small data global existence for the wave equation with inverse-square potential,

$$\begin{cases} \partial_t^2 u + P_a u = \pm|u|^k, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x) \in \dot{H}^{s_c}(\mathbb{R}^d), \\ \partial_t u(0,x) = u_1(x) \in \dot{H}^{s_c-1}(\mathbb{R}^d), \end{cases} \tag{2.8}$$

has been established in [13, Theorem 4.1]. We shall consider the energy-critical case, that is, $s_c = 1$ and $k = 1 + 4/(d-2)$. First we summarize the result of [13]:

Theorem 2.1 (Small data global existence for NLW). *Let $d \geq 3$, and suppose $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ with small norms. Then there exists a unique solution to* (1.1) *such that $u(t,x) \in \mathcal{C}_t(\mathbb{R}; \dot{H}^1(\mathbb{R}^d)), \partial_t u(t,x) \in \mathcal{C}_t(\mathbb{R}; L^2(\mathbb{R}^d))$.*

A similar result holds true for the Schrödinger equation (1.2) (see [11]):

Theorem 2.2 (Small data global existence for NLS). *Let $d \geq 3$, and suppose $v_0 \in \dot{H}^1(\mathbb{R}^d)$ with small norm. Then there exists a unique solution to* (1.2) *such that $v(t,x) \in \mathcal{C}_t(\mathbb{R}; \dot{H}^1(\mathbb{R}^d))$.*

Now we give some estimates which will be used in our proof.

Lemma 2.1. *Let $(u_0, u_1) \in \dot{H}^1 \times L^2$ with $\|(u_0,u_1)\|_{\dot{H}^1 \times L^2} \leq A$. Assume that I is the maximal lifespan of the corresponding solution to* (1.2)*. There exists $\varepsilon_0 > 0$ such that if for some $M > 0$ and $0 < \varepsilon < \varepsilon_0$, we have $\int_{|x| \geq M} (|\nabla_x u_0|^2 + |u_1|^2)\,dx \leq \varepsilon$, then for all $t \in I_+ = [0,\infty) \cap I$,*

$$\int_{|x| \geq 3M/2+t} \left(\frac{|u|^2}{|x|^2} + |\nabla_x u(x,t)|^2 + |\partial_t u(x,t)|^2 \right) dx \leq C\varepsilon.$$

The proof of this lemma can be found in [8, Lemma 2.17]. We will also need the following weighted radial Sobolev embedding inequality:

Lemma 2.2 (Weighted radial Sobolev embedding, [10]). *Let ω and f be radial functions and $0 \leq \omega \lesssim 1$. Then*

$$\big\| |x|^{(d-1)/2} \omega^{1/4} f \big\|^2_{L_x^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)} \|\omega^{1/2} \nabla f\|_{L_x^2(\mathbb{R}^d)}.$$

3. Variational estimates. In this section, we define the energy threshold m_c by the variational method, and we prove various estimates for solutions with energy below m_c. First, to study the behavior of $\mathcal{K}$ near the origin we set some notation. By the definitions of $\mathcal{K}$ and Q, we have

$$\mathcal{K}(\varphi) = Q(\varphi) + N(\varphi),$$

where

$$Q(\varphi) = \int_{\mathbb{R}^d} \left(|\nabla\varphi|^2 + a\frac{|\varphi|^2}{|x|^2} \right) dx \quad \text{and} \quad N(\varphi) = -\int_{\mathbb{R}^d} |\varphi|^{2^*}\, dx.$$

Lemma 3.1. *For any $\varphi \in \dot{H}^1(\mathbb{R}^d)$, set $\varphi_\lambda = e^{d\lambda}\varphi(e^{2\lambda})$. Then*

$$\lim_{\lambda\to-\infty} Q(\varphi_\lambda) = 0.$$

Proof. This is obvious by the definition of Q. ∎

Lemma 3.2. *For any bounded sequence $\varphi_n \in \dot{H}^1(\mathbb{R}^d) \setminus \{0\}$ with*

$$\lim_{n\to\infty} Q(\varphi_n) = 0,$$

for large n we have

$$\mathcal{K}(\varphi_n) > 0.$$

Proof. From $Q(\varphi_n) \to 0$ and the Hardy inequality

$$\int_{\mathbb{R}^d} a\frac{|\varphi_n|^2}{|x|^2}\, dx \lesssim \|\nabla\varphi_n\|_{L^2}^2,$$

we know that

$$\lim_{n\to\infty} \|\nabla\varphi_n\|_{L^2}^2 = 0.$$

Then by the Sobolev inequality, we have

$$\|\varphi_n\|_{L_x^{2^*}}^{2^*} \lesssim \|\nabla\varphi_n\|_{L_x^2}^{2^*} = o(\|\nabla\varphi_n\|_{L^2}^2).$$

Hence, for large n,

$$\mathcal{K}(\varphi_n) = \int_{\mathbb{R}^d} \left(|\nabla\varphi_n|^2 + a\frac{|\varphi_n|^2}{|x|^2} - |\varphi_n|^{2^*} \right) dx \approx \int_{\mathbb{R}^d} |\nabla\varphi_n|^2\, dx > 0. \; ∎$$

According to the above analysis, we will replace the functional $\mathcal{H}$ in (1.7) with a positive functional $\mathcal{J}$, and the constraint $\mathcal{K}(\varphi) = 0$ with $\mathcal{K}(\varphi) \le 0$. Let

$$\mathcal{J}(\varphi) = \mathcal{H}(\varphi) - \frac{1}{2}\mathcal{K}(\varphi) = \frac{1}{d}\int_{\mathbb{R}^d} |\varphi|^{2^*}\, dx.$$

Then $\mathcal{J}(\varphi) \ge 0$.

Now we can characterize the minimization problem (1.7) by making use of $\mathcal{J}$.

Lemma 3.3. *For the infimum m_c in* (1.7)*, we have*

$$m_c = \inf\{\mathcal{J}(\varphi);\, 0 \ne \varphi \in \dot{H}^1,\, \mathcal{K}(\varphi) \le 0\}.$$

Moreover, m_c is attained at Φ, the solution to (1.5).

Proof. Denote the right hand side above by m_1. Obviously, $m_1 \le m_c$, since $\mathcal{H}(\varphi) = \mathcal{J}(\varphi)$ whenever $\mathcal{K}(\varphi) = 0$. It now suffices to show $m_1 \ge m_c$.

For all φ with $\mathcal{K}(\varphi)<0$, set $\varphi_\lambda=e^{d\lambda}\varphi(e^{2\lambda})$; then there exists a $\lambda_0<0$ such that

$$\mathcal{K}(\varphi_{\lambda_0})=0.$$

This implies that $\mathcal{H}(\varphi_{\lambda_0})\geq m_c$. So, $\mathcal{J}(\varphi_{\lambda_0})\geq m_c$. Note that $\mathcal{J}(\varphi_\lambda)$ is nondecreasing in λ. Thus,

$$\mathcal{J}(\varphi)\geq\mathcal{J}(\varphi_{\lambda_0})\geq m_c,$$

which gives $m_1\geq m_c$. ■

Now we give another characterization of m_c in (1.7).

LEMMA 3.4. *The energy threshold satisfies*

$$m_c=\inf_{0\neq\varphi}\frac{1}{d}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx=\inf_{0\neq\varphi}\frac{1}{d}\bigg(\frac{Q(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx=\frac{1}{d}S_a^{d/2}.$$

Proof. Let

$$\widetilde{m}\triangleq\inf_{0\neq\varphi}\frac{1}{d}\bigg(\frac{Q(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx.$$

First, we prove $m_c\geq\widetilde{m}$. By the definition of $\mathcal{K}(\varphi)$, we have $Q(\varphi)/\|\varphi\|_{2^*}^{2^*}\leq 1$. Thus

$$\frac{1}{d}\bigg(\frac{Q(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx\leq\frac{1}{d}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx.$$

Taking the infimum on both sides we obtain $m_c\geq\widetilde{m}$.

Next, we prove $m_c\leq\widetilde{m}$. By the definition of $\widetilde{m}$, for every $\varepsilon\in(0,1)$, there exists φ such that

$$\widetilde{m}+\varepsilon>\frac{1}{d}\bigg(\frac{Q(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx.$$

By homogeneity and scaling $\varphi\mapsto\mu\varphi$, we know

$$\frac{1}{d}\bigg(\frac{Q(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}\int_{\mathbb{R}^d}|\varphi|^{2^*}\,dx=\frac{1}{d}\bigg(\frac{Q(\mu\varphi)}{\|\mu\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}\int_{\mathbb{R}^d}|\mu\varphi|^{2^*}\,dx.\tag{3.9}$$

If we let

$$\mu=\Big(1-\frac{\varepsilon}{m_c}\Big)^{-\frac{1}{2^*}}\bigg(\frac{Q(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{1}{2^*-2}},$$

then

$$\bigg(\frac{Q(\mu\varphi)}{\|\mu\varphi\|_{2^*}^{2^*}}\bigg)^{\frac{2^*}{2^*-2}}=1-\frac{\varepsilon}{m_c}.$$

Thus,

$$\begin{aligned}\widetilde{m}+\varepsilon &> \frac{1}{d}\Big(1-\frac{\varepsilon}{m_c}\Big)\int_{\mathbb{R}^d}|\mu\varphi|^{2^*}\,dx\\ &\ge \Big(1-\frac{\varepsilon}{m_c}\Big)\inf_{0\neq\varphi}\frac{1}{d}\int_{\mathbb{R}^d}|\mu\varphi|^{2^*}\,dx = m_c-\varepsilon.\end{aligned}$$

Then we obtain $\widetilde{m}\ge m_c-2\varepsilon$, and letting $\varepsilon\to 0$ yields the desired result.

The last equality of the statement is obtained by a direct computation and the definition of S_a in (1.6). ∎

LEMMA 3.5. *Suppose $(u_0,u_1)\in\mathbb{W}^-$. Then the corresponding solution u with maximal lifespan I satisfies $u(t)\in\mathbb{W}^-$ for all $t\in I$. An analogous result holds true for a solution v to the Schrödinger equation* (1.2).

Proof. This is a consequence of the energy conservation law and the continuity of the nonlinear flow of (1.1). ∎

To end this section, we give uniform bounds on the Fréchet derivative $\mathcal{K}$ with energy below m_c, which plays an important role for the blow-up analysis in Section 4.

LEMMA 3.6. *Let $(u_0,u_1)\in W^-$, and let u be the corresponding solution of* (1.1) *with maximal lifespan I. Then $\mathcal{K}(u)\le -4(m_c-E(u,\partial_t u))$ for all $t\in I$. An analogous result holds true for a solution v to the Schrödinger equation* (1.2).

Proof. Denote $h(\lambda)=\mathcal{H}(u_\lambda)$, where $u_\lambda=e^{d\lambda}u(e^{2\lambda}x)$. Then

$$\begin{aligned}h'(\lambda)&=2e^{4\lambda}\|\nabla u\|_2^2+2e^{4\lambda}\|x^{-1}u\|_2^2-2e^{2\cdot 2^*\lambda}\|u\|_{2^*}^{2^*},\\ h''(\lambda)&=8e^{4\lambda}\|\nabla u\|_2^2+8e^{4\lambda}\|x^{-1}u\|_2^2-42^*e^{2\cdot 2^*\lambda}\|u\|_{2^*}^{2^*}\\ &=4h'(\lambda)-\frac{16}{d-2}e^{2\cdot 2^*\lambda}\|u\|_{2^*}^{2^*},\end{aligned}$$

which implies

$$h''(\lambda)\le 4h'(\lambda). \tag{3.10}$$

By Lemmas 3.1 and 3.2, and the continuity of $\mathcal{K}$ in λ, there exists $\lambda_0<0$ such that $\mathcal{K}(u_{\lambda_0})=0$ and $\mathcal{K}(u_\lambda)<0$ for $\lambda_0<\lambda\le 0$. Therefore, $\mathcal{H}(u_{\lambda_0})\ge\mathcal{H}(Q)$. Thus, integrating (3.10) over $(\lambda_0,0)$, we obtain

$$\mathcal{K}(u)\le 4(\mathcal{H}(u)-\mathcal{H}(u_{\lambda_0})).$$

Since $\mathcal{H}(u_{\lambda_0})=J(u_{\lambda_0})+\frac{1}{2}\mathcal{K}(u_{\lambda_0})=J(u_{\lambda_0})\ge m_c$, and $\mathcal{H}(u)\le E(u)$, we obtain the desired result. ∎

4. Finite time blow-up. In this section, we prove Theorems 1.1 and 1.2.

4.1. The proof of Theorem 1.1. We first give the proof of Theorem 1.1 under the assumption that $u_0 \in L^2$.

THEOREM 4.1. *Let* $(u_0, u_1) \in \dot{H}^1 \times L^2$, $u_0 \in L^2$, *and let* u *be the solution of* (1.1) *with maximal lifespan* I. *Assume that* $E(u_0, u_1) < m_c$ *and* $\mathcal{K}(u_0) < 0$. *Then* I *is a finite interval.*

Proof. Define

$$y(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 \, dx.$$

Then $y'(t) = 2\int_{\mathbb{R}^d} uu_t \, dx$ and

$$y''(t) = 2 \int_{\mathbb{R}^d} \left((u_t)^2 + |u|^{2^*} - a\frac{|u|^2}{|x|^2} - |\nabla u|^2 \right) dx = 2 \int_{\mathbb{R}^d} (u_t)^2 \, dx - 2\mathcal{K}(u).$$

Since $\mathcal{K}(u) < 0$, we have

$$y''(t) \geq 2 \int_{\mathbb{R}^d} (u_t)^2 \, dx.$$

Assume now that $[0, \infty) \subseteq I$. Then, as $y''(t) > 0$, there exists t_0 such that $y'(t_0) > 0$, and hence $y'(t) > 0$ for $t > t_0$.

Hence, for $t > t_0$,

$$y''(t)y(t) \geq 2 \int_{\mathbb{R}^d} (u_t)^2 \, dx \int_{\mathbb{R}^d} u^2 \, dx \geq 2\Big(\int_{\mathbb{R}^d} uu_t \, dx \Big)^2 = 2y'(t)^2,$$

so that, for $t \geq t_0$,

$$\frac{y''(t)}{y'(t)} \geq 2\frac{y'(t)}{y(t)}, \quad \text{i.e.} \quad (\log y'(t))' \geq 2(\log y(t))'.$$

Consequently, for $t \geq t_0$, we have

$$\text{either} \quad \log y' \geq 2\log y - C_0, \quad \text{or} \quad y' \geq Cy^2,$$

which leads to finite-time blow-up of y, contradicting the hypothesis $[0, \infty) \subseteq I$. ∎

REMARK 4.1. Notice that Theorem 1.1 is an extension of Theorem 4.1.

Now, we are in a position to complete the proof of Theorem 1.1:

Proof of Theorem 1.1. Take $\phi \in C_0^\infty(B_2)$ such that $\phi \equiv 1$ for $|x| < 1$, and $0 \leq \phi \leq 1$. Define

$$y_R(t) = \int \phi\Big(\frac{x}{R}\Big) u^2(t,x) \, dx.$$

Then

$$y_R'(t) = 2 \int_{\mathbb{R}^d} \phi\Big(\frac{x}{R}\Big) u\partial_t u \, dx$$

and

$$\begin{aligned}
y_R''(t) &= 2\int_{\mathbb{R}^d}(\partial_t u)^2\phi\Big(\frac{x}{R}\Big)\,dx + 2\int_{\mathbb{R}^d}\phi\Big(\frac{x}{R}\Big)u\Big(\Delta u - a\frac{u}{|x|^2} + |u|^{1+\frac{4}{d-2}}\Big)\,dx\\
&= 2\int_{\mathbb{R}^d}(\partial_t u)^2\phi\Big(\frac{x}{R}\Big)\,dx + 2\int_{\mathbb{R}^d}\phi\Big(\frac{x}{R}\Big)|u|^{2^*} - 2\int_{\mathbb{R}^d}\phi\Big(\frac{x}{R}\Big)|\nabla u|^2\,dx\\
&\quad + \frac{1}{R^2}\int_{\mathbb{R}^d}|u|^2(\Delta\phi)\Big(\frac{x}{R}\Big)\,dx - 2a\int_{\mathbb{R}^d}\phi\Big(\frac{x}{R}\Big)\frac{|u|^2}{|x|^2}\,dx\\
&= 2\int_{\mathbb{R}^d}(\partial_t u)^2\phi\Big(\frac{x}{R}\Big)\,dx - 2\int_{\mathbb{R}^d}\Big(|\nabla u|^2 + a\frac{|u|^2}{|x|^2} - |u|^{2^*}\Big)\,dx\\
&\quad + 2\int_{\mathbb{R}^d}\Big(1-\phi\Big(\frac{x}{R}\Big)\Big)\Big(|\nabla u|^2 + a\frac{|u|^2}{|x|^2} - |u|^{2^*}\Big)\,dx\\
&\quad + \frac{1}{R^2}\int_{\mathbb{R}^d}|u|^2(\Delta\phi)\Big(\frac{x}{R}\Big)\,dx.
\end{aligned}$$

By our choice of $\phi(x)$, we have $\operatorname{supp}\Delta\phi(x/R)\subset\{R<|x|\le 2R\}$, and $\Delta\phi$ is bounded. Then

$$\frac{1}{R^2}\int_{\mathbb{R}^d}|u|^2(\Delta\phi)\Big(\frac{x}{R}\Big)\,dx \lesssim \frac{1}{R^2}\int_{R<|x|\le 2R}|u|^2\,dx \lesssim \int_{|x|>R}\frac{|u|^2}{|x|^2}\,dx.$$

Thus,

$$\begin{aligned}
y_R''(t) &= 2\int_{\mathbb{R}^d}(\partial_t u)^2\phi\Big(\frac{x}{R}\Big)\,dx\\
&\quad - 2\int_{\mathbb{R}^d}\Big(|\nabla u|^2 + a\frac{|u|^2}{|x|^2} - |u|^{2^*}\Big)\,dx + O_r(R),
\end{aligned}\tag{4.11}$$

where

$$O_r(R) = O\Big(\int_{|x|>R}\Big(|\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^{2^*}\Big)\,dx\Big).$$

By Lemma 3.6, we know that $\mathcal{K}(u)\le -4(m_c - E(u))$. Then $-2\mathcal{K}(u)\ge 8(m_c - E(u)) \triangleq \delta$. Recall (Lemma 2.1) that there exists $\varepsilon_0>0$ such that, for $0<\varepsilon<\varepsilon_0$, there exists $M_0 = M_0(\varepsilon)$ so that

$$\int_{|x|\ge M_0+t}\Big(|\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^{2^*}\Big)\,dx \le \varepsilon$$

for $t\in I_+ = [0,\infty)\cap I$, where I is the maximal lifespan of the solution. Now we choose ε_1 small and R so large that $R>2M_0$ and $O_r(R)\le\varepsilon_1\le\frac{1}{2}\delta$.

Then, for $0 < t < \frac{1}{2}R$,

$$(4.12) \qquad y_R''(t) \geq \frac{1}{2}\delta \quad \text{and} \quad y_R''(t) \geq 2 \int_{\mathbb{R}^d} (\partial_t u)^2 \phi\left(\frac{x}{R}\right) dx.$$

Note also that

$$(4.13) \qquad y_R(0) \leq CM_0^2 A^2 + \varepsilon_1 R^2 \quad \text{and} \quad |y_R'(0)| \leq CM_0 A^2 + \varepsilon_1 R.$$

Now we define

$$T = \frac{2CM_0^2 A^2 + 2\varepsilon_1 R^2 + 2CM_0 A^2 + 2\varepsilon_1 R}{\delta}.$$

Then, if $T < \frac{1}{2}R$,

$$y_R'(t) \geq y_R'(0) + \frac{1}{2}T\delta \geq CM_0^2 A^2 + \varepsilon_1 R^2.$$

Thus, there exists $0 < t_1 < T$ such that $y_R'(t_1) = CM_0^2 A^2 + \varepsilon_1 R^2$. And for $0 < t < t_1$, we have $y_R'(t) \leq CM_0^2 A^2 + \varepsilon_1 R^2$. Note that, if $t_1 \leq t \leq \frac{1}{2}R$, then $y_R'(t) > y_R'(t_1)$, and also

$$y_R(t_1) = y_R(0) + \int_0^{t_1} y_R'\, dt \leq y_R(0) + t_1 y_R'(t_1).$$

By the second inequality of (4.12),

$$(4.14) \qquad \begin{aligned} y_R''(t) y_R(t) &\geq 2 \int_{\mathbb{R}^d} (\partial_t u)^2 \phi\left(\frac{x}{R}\right) dx \int_{\mathbb{R}^d} |u|^2 \phi\left(\frac{x}{R}\right) dx \\ &\geq 2\left(\int_{\mathbb{R}^d} (\partial_t u) u \phi\left(\frac{x}{R}\right) dx \right)^2 = 2(y_R'(t))^2. \end{aligned}$$

When $t_1 < t < \frac{1}{2}R$,

$$\frac{y_R''(t)}{y_R'(t)} \geq \frac{y_R'(t)}{y_R(t)}.$$

Integration from t_1 to t gives

$$\frac{y_R'(t)}{y_R^2(t)} \geq \frac{y_R'(t_1)}{y_R^2(t_1)}.$$

A direct computation yields

$$y_R(t) \geq \frac{1}{\frac{1}{y_R(t_1)} + (t_1 - t)\frac{y_R'(t)}{y_R^2(t)}},$$

which leads to blow-up in finite time.

4.2. The proof of Theorem 1.2. To prove that the solution v blows up in finite time (in either of the two cases described in Theorem 1.2), we will use the convexity method [5, 18].

Proof of Theorem 1.2. Let us first treat the case when $xv_0 \in L^2_x(\mathbb{R}^d)$. Define the virial quantity

$$V(t) = \int_{\mathbb{R}^d} |x|^2 |v(t,x)|^2 \, dx.$$

A direct calculation gives

$$\partial_t^2 V(t) = 8 \int_{\mathbb{R}^d} (|\nabla v(t,x)|^2 + |x|^{-2}|v(t,x)|^2 - |v(t,x)|^{2^*}) \, dx.$$

By Lemma 3.6, we have

$$\partial_t^2 V(t) = 8\mathcal{K}(v) \le -32(m_c - E(v)) < 0.$$

Thus, v has to blow up in finite time.

We next consider the case when $v_0 \in H^1_x(\mathbb{R}^d)$ is radial. Blow-up for the energy-critical nonlinear Schrödinger equation was addressed by Killip and Visan [10]. We define the truncated virial quantity

$$V_R(t) = \int_{\mathbb{R}^d} \psi_R(x) |v(t,x)|^2 \, dx,$$

where $\psi_R(x) = R^2 \psi(|x|/R)$, $\psi''(x/R) < 0$, and ψ is a radially smooth cut-off function with

$$\psi(x) = \begin{cases} |x|^2/2, & |x| \le 2, \\ 0, & |x| \ge 3. \end{cases}$$

Then by direct computations and the choice of a,

$$\begin{aligned} (4.15) \quad & \partial_t^2 V_R(t) \\ &= \int_{\mathbb{R}^d} (-\Delta^2)\psi_R(x)|v(t,x)|^2 \, dx + 4\Re \int_{\mathbb{R}^d} \partial_{jk}\psi_R(x) \bar{v}_j v_k \, dx \\ &\quad + 4a \int_{\mathbb{R}^d} \partial_j \psi_R(x) \frac{x_j}{|x|^4} |v|^2 \, dx - \frac{4}{d} \int_{\mathbb{R}^d} \Delta\psi_R(x) |v|^{2^*} \, dx \\ &= \frac{1}{R^2} O\Big(\int_{|x|\sim R} |v(t,x)|^2 \, dx \Big) + 4 \int_{|x|<2R} \Big(|\nabla v|^2 + a\frac{|v|^2}{|x|^2} - |v|^{2^*} \Big) \, dx \\ &\quad + 4 \int_{|x|\ge 2R} \Big(\psi''\Big(\frac{x}{R}\Big) |\nabla v|^2 + a\frac{R}{r}\psi'\Big(\frac{x}{R}\Big)\frac{|v|^2}{|x|^2} - \frac{1}{d}\psi''\Big(\frac{x}{R}\Big)|v|^{2^*} \Big) \, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{R^2} O\Big(\int\limits_{|x|\sim R} |v(t,x)|^2 \, dx \Big) + 4 \int\limits_{\mathbb{R}^d} \Big(|\nabla v|^2 + a \frac{|v|^2}{|x|^2} - |v|^{2^*} \Big) \, dx \\
&\quad - 4 \int\limits_{|x|>2R} \Big(|\nabla v|^2 + a \frac{|v|^2}{|x|^2} - |v|^{2^*} \Big) \, dx \\
&\quad + 4 \int\limits_{|x|\geq 2R} \Big(\psi''\Big(\frac{x}{R}\Big) |\nabla v|^2 + a \frac{R}{r} \psi'\Big(\frac{x}{R}\Big) \frac{|v|^2}{|x|^2} - \frac{1}{d} \psi''\Big(\frac{x}{R}\Big) |v|^{2^*} \Big) \, dx \\
&= \frac{1}{R^2} O\Big(\int\limits_{|x|\sim R} |v(t,x)|^2 \, dx \Big) + 4\mathcal{K}(v) \\
&\quad + 4 \int\limits_{|x|>2R} (|\nabla v|^2 - |v|^{2^*}) \Big(\psi''\Big(\frac{x}{R}\Big) - 1 \Big) \, dx \\
&\quad + 4 \int\limits_{|x|>2R} a \Big(\frac{|v|^2}{|x|^2} \Big(\frac{R}{r} \psi'\Big(\frac{x}{R}\Big) - 1 \Big) \Big) \, dx \\
&\quad + 4 \int\limits_{|x|>2R} \Big(|v|^{2^*} \Big(1 - \frac{1}{d} \Big) \psi''\Big(\frac{x}{R}\Big) \Big) \, dx \\
&=: I_1 + 4\mathcal{K}(v) + I_2 + I_3 + I_4.
\end{aligned}
$$

As $\psi''(x/R) \leq 0$, we have $I_4 \leq 0$. Since $v \in L^2_x(\mathbb{R}^d)$, we can choose R sufficiently large (depending on the mass of v) so that I_1 is less than ζ, where $\mathcal{K}(v) \leq -4(m_c - E(u)) \triangleq -\zeta$. For the estimate of I_3,

$$
\begin{aligned}
I_3 &= 4 \int\limits_{|x|>2R} a \frac{|v|^2}{|x|^2} \Big(\frac{R}{r} \psi'\Big(\frac{x}{R}\Big) - 1 \Big) \, dx \\
&\leq \int\limits_{\mathbb{R}^d} |a| \frac{|v|^2}{4R^2} \Big(\Big| \frac{1}{2} \psi'\Big(\frac{x}{R}\Big) \Big| + 1 \Big) \, dx \lesssim \frac{1}{R^2} \int\limits_{\mathbb{R}^d} |v|^2 \, dx.
\end{aligned}
$$

Thus I_3 can be made arbitrarily small by choosing R sufficiently large.

Now, we estimate I_2. Let $\omega(x) = 1 - \psi''(x/R)$. Note that $0 \leq \omega \lesssim 1$ is radial with $\operatorname{supp}(\omega) \subseteq \{|x| \geq 2R\}$. Then by the weighted radial Sobolev embedding inequality (2.2), we have

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d} |v(x,t)|^{2^*} \omega(x) \, dx &\lesssim \|\omega^{1/4} u(t)\|_{L^\infty_x}^{\frac{4}{d-2}} \int\limits_{\mathbb{R}^d} |u(t,x)|^2 \, dx \\
&\lesssim R^{-\frac{2(d-1)}{d-2}} \big\| |x|^{\frac{d-1}{2}} \omega^{1/4} u(t) \big\|_{L^\infty_x}^{\frac{4}{d-2}} \|u_0\|_{L^2_x}^2 \\
&\lesssim R^{-\frac{2(d-1)}{d-2}} \|\omega^{1/2} \nabla u(t)\|_{L^2_x}^{\frac{2}{d-2}} \|u_0\|_{L^2_x}^{\frac{2(d-1)}{d-2}} \\
&\lesssim (R^{-1} \|u_0\|_{L^2_x})^{\frac{2(d-1)}{d-2}} (\|\omega^{1/2} \nabla u(t)\|_{L^2_x}^2 + 1).
\end{aligned}
$$

Noting that $u_0 \in L^2_x(\mathbb{R}^d)$, I_2 can also be made small enough by taking R sufficiently large depending on the mass of u. Thus (4.15) yields $\partial_{tt} V_R < 0$. This finishes the proof of the theorem.

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