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ON SOME NONLINEAR NONHOMOGENEOUS ELLIPTIC UNILATERAL PROBLEMS INVOLVING NONCONTROLLABLE LOWER ORDER TERMS WITH MEASURE RIGHT HAND SIDE

Abstract. We prove the existence of entropy solutions to unilateral problems associated to equations of the type $Au - \operatorname{div}(\phi(u)) = \mu \in L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$, where A is a Leray–Lions operator acting from $W_0^{1,p(\cdot)}(\Omega)$ into its dual $W^{-1,p(\cdot)}(\Omega)$ and $\phi \in C^0(\mathbb{R}, \mathbb{R}^N)$.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Let A be a nonlinear operator of the Leray–Lions type from $W_0^{1,p(\cdot)}(\Omega)$ into its dual $W^{-1,p(\cdot)}(\Omega)$ defined by $Au = -\operatorname{div}(a(x, u, \nabla u))$, where $a(x, u, \nabla u)$ is a Carathéodory vector valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ which satisfies suitable Leray–Lions conditions. Consider now the following nonlinear Dirichlet problem:

$$(1.1)\qquad \begin{cases} Au - \operatorname{div}(\phi(u)) = f - \operatorname{div}(F) & \text{in } \Omega, \\ u = 0 \quad \text{on } \Omega, \end{cases}$$

where $\phi = (\phi_1, \ldots, \phi_N) \in (C^0(\mathbb{R}))^N,\ f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$.

The study of problems with variable exponent is a new and interesting topic which raises many mathematical difficulties. One of our motivations for studying (1.1) comes from applications to electrorheological fluids (we refer to [12] for more details), an important class of non-Newtonian fluids (sometimes referred to as smart fluids). Other important applications are related to image processing (see [7]) and elasticity (see [15]). The function $\phi(u)$ does not belong in $(L^1_{\mathrm{loc}}(\Omega))^N$ because ϕ is just assumed to be contin-

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uous on $\mathbb{R}$, so that proving existence of a weak solution (i.e. in the sense of distributions) seems to be an arduous task. To overcome this difficulty we use the framework of entropy solutions.

The first objective of our paper is to study the problem (1.1) in the generalized Sobolev space with general right hand side μ which lies in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$.

The second objective is to treat unilateral problems; more precisely, the existence of an entropy solution for the following obstacle problem:

$$\begin{cases} u \in T_0^{1,p(\cdot)}(\Omega),\ u \geq \psi \quad \text{a.e. in } \Omega, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \\ \qquad \leq \int_\Omega f T_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx \\ \qquad\qquad \forall v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0, \end{cases} \tag{1.2}$$

is proved in Theorem 3.1 without assuming regularity of the obstacle ψ, in particular $\psi^+ \in K_\psi \cap L^\infty(\Omega)$ is not supposed.

The plan of the paper is as follows. In Section 2 we give some preliminaries and the definition of generalized Sobolev spaces. In Section 3 we make precise all the assumptions and give some technical results and we establish the existence of an entropy solution to problem (1.1). In Section 4 (Appendix) we give the proof of Lemma 3.5.

2. Preliminaries. For each open bounded subset Ω of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$C^+(\overline{\Omega}) = \{p : \overline{\Omega} \to \mathbb{R}^+ \text{ continuous} \mid 1 < p_- \leq p_+ < \infty\},$$

where $p_- = \inf_{x\in\overline{\Omega}} p(x)$ and $p_+ = \sup_{x\in\overline{\Omega}} p(x)$. We define the variable exponent Lebesgue space for $p \in C^+(\overline{\Omega})$ by

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable} \,\Big|\, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\}.$$

The space $L^{p(\cdot)}(\Omega)$ under the norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 \,\Big|\, \int_\Omega |u(x)/\lambda|^{p(x)}\,dx \leq 1\Big\}$$

is a uniformly convex, reflexive Banach space. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$.

Proposition 2.1 (cf. [8]).

(i) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

(ii) *For all $p_1, p_2 \in C^+(\overline{\Omega})$ such that $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, we have $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ and the embedding is continuous.*

PROPOSITION 2.2 (cf. [8]). *If we denote*

$$\rho(u) = \int_{\Omega} |u|^{p(x)}\, dx \qquad \forall u \in L^{p(\cdot)}(\Omega),$$

then:

(i) $\|u\|_{p(\cdot)} < 1$ *(resp.* $= 1, > 1$) $\Leftrightarrow$ $\rho(u) < 1$ *(resp.* $= 1, > 1$).
(ii) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p_-} \le \rho(u) \le \|u\|_{p(\cdot)}^{p_+}$ *and* $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p_+} \le \rho(u) \le \|u\|_{p(\cdot)}^{p_-}$.
(iii) $\|u\|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u) \to 0$ *and* $\|u\|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

We define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \qquad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and set $p^*(\cdot) = \frac{Np(\cdot)}{N-p(\cdot)}$ for $p(\cdot) < N$.

PROPOSITION 2.3 (cf. [8]).

(i) *Assuming* $1 < p_- \le p_+ < \infty$, *the spaces* $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are separable reflexive Banach spaces.*
(ii) *If* $q \in C^+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact and continuous.*
(iii) *There is a constant* $C > 0$ *such that* $\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)}$ *for all* $u \in W_0^{1,p(\cdot)}(\Omega)$.

REMARK 2.1. By Proposition 2.3(iii), $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.

LEMMA 2.1 (cf. [6]). *Let* $g \in L^{r(\cdot)}(\Omega)$ *and* $g_n \in L^{r(\cdot)}(\Omega)$ *with* $\|g_n\|_{r(\cdot)} \le C$ *for* $1 < r(\cdot) < \infty$. *If* $g_n(\cdot) \to g(\cdot)$ *a.e. on* Ω, *then* $g_n \rightharpoonup g$ *in* $L^{r(\cdot)}(\Omega)$.

3. Main general results

3.1. Basic assumptions and some lemmas. Let $a : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying the following conditions: for all $\xi, \eta \in \mathbb{R}^N$ and almost every $x \in \Omega$, we have

$$|a(x,s,\xi)| \le \beta(k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{3.1}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0 \quad \forall \xi \neq \eta, \tag{3.2}$$

$$a(x,s,\xi)\xi \ge \alpha|\xi|^{p(x)}, \tag{3.3}$$

where $k(\cdot)$ is a positive function in $L^{p'(\cdot)}(\Omega)$ and α and β are positive constants. Finally, consider the convex set

$$K_\psi = \{u \in W_0^{1,p(\cdot)}(\Omega) \mid u \ge \psi \text{ a.e. in } \Omega\}$$

where ψ is a measurable function such that

$$K_\psi \cap L^\infty(\Omega) \neq \emptyset. \tag{3.4}$$

We suppose that

$$\phi \in C^0(\mathbb{R}, \mathbb{R}^N), \tag{3.5}$$

$$f \in L^1(\Omega), \tag{3.6}$$

$$F \in (L^{p'(\cdot)}(\Omega))^N, \tag{3.7}$$

and $p \in C^+(\overline{\Omega})$ is such that there is a vector $l \in \mathbb{R}^N - \{0\}$ such that for any $x \in \Omega$,

$$g(t) = p(x + tl) \text{ is monotone for } t \in I_x = \{t \mid x + tl \in \Omega\}. \tag{3.8}$$

Lemma 3.1 (cf. [6]). *Assume that* (3.1)–(3.3) *hold, and let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0. \tag{3.9}$$

Then $u_n \to u$ *in* $W_0^{1,p(\cdot)}(\Omega)$.

Lemma 3.2. *Assume that* (3.8) *holds. Then there is a constant* $C > 0$ *such that*

$$\rho(u) \le C\rho(\nabla u) \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega) - \{0\}. \tag{3.10}$$

Proof. Let

$$\lambda_* = \inf_{u \in W_0^{1,p(\cdot)}(\Omega) - \{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx}.$$

By [9, Theorem 3.3], we have $\lambda_* > 0$, which implies that

$$0 < \lambda_* \le \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx} \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega) - \{0\};$$

consequently, there is a constant $C > 0$ such that $\rho(u) \le C\rho(\nabla u)$ for all $u \in W_0^{1,p(\cdot)}(\Omega) - \{0\}$. ■

REMARK 3.1. The inequality (3.10) holds true if we assume that there exists a function $\xi \geq 0$ such that $\nabla p \nabla \xi \geq 0$, with $|\nabla \xi| \neq 0$ in $\overline{\Omega}$ (cf. [3]).

LEMMA 3.3. *Let $F : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function with $F(0) = 0$, and $p \in C_+(\overline{\Omega})$. If $u \in W_0^{1,p(\cdot)}(\Omega)$, then $F(u) \in W_0^{1,p(\cdot)}(\Omega)$; moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega \mid u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega \mid u(x) \in D\}. \end{cases}$$

Proof. Consider first the case of $F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$. Let $u \in W_0^{1,p(\cdot)}(\Omega)$. Since $\overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)} = W_0^{1,p(\cdot)}(\Omega)$, there are $u_n \in C_0^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$, so $u_n \to u$ a.e. in Ω and $\nabla u_n \to \nabla u$ a.e. in Ω. Then $F(u_n) \to F(u)$ a.e. in Ω. On the other hand, $|F(u_n)| = |F(u_n) - F(0)| \leq \|F'\|_\infty |u_n|$, so

$$|F(u_n)|^{p(x)} \leq (\|F'\|_\infty + 1)^{p_+} |u_n|^{p(x)},$$
$$\left|\frac{\partial F(u_n)}{\partial x_i}\right|^{p(x)} = \left|F'(u_n)\frac{\partial u_n}{\partial x_i}\right|^{p(x)} \leq M\left|\frac{\partial u_n}{\partial x_i}\right|^{p(x)},$$

where $M = (\|F'\|_\infty + 1)^{p_+}$. We conclude that $F(u_n)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and so $F(u_n)$ converges to ν weakly in $W_0^{1,p(\cdot)}(\Omega)$. Then $F(u_n)$ converges to ν strongly in $L^{q(\cdot)}(\Omega)$ with $1 < q(x) < p^*(x)$ and $p^*(x) = Np(x)/(N - p(x))$, and since $F(u_n) \to \nu$ a.e. in Ω, we obtain

$$\nu = F(u) \in W_0^{1,p(\cdot)}(\Omega).$$

Let $F : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function. Then $F_n = F * \varphi_n \to F$ uniformly on each compact set, where φ_n is a regularizing sequence. We conclude that $F_n \in C^1(\mathbb{R})$ and $F_n' \in L^\infty(\mathbb{R})$. From the first part, we have $F_n(u) \in W_0^{1,p(\cdot)}(\Omega)$ and $F_n(u) \to F(u)$ a.e. in Ω. Since $(F_n(u))_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, it follows that $F_n(u) \rightharpoonup \overline{\nu}$ weakly in $W_0^{1,p(\cdot)}(\Omega)$ and a.e. in Ω, so $\overline{\nu} = F(u) \in W_0^{1,p(\cdot)}(\Omega)$. ∎

LEMMA 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$. If $u \in (W_0^{1,p(\cdot)}(\Omega))^N$ then*

$$\int_\Omega \operatorname{div}(u)\, dx = 0.$$

Proof. Fix $u = (u^1, \ldots, u^N) \in (W_0^{1,p(\cdot)}(\Omega))^N$. We have $\overline{D(\Omega)} = W_0^{1,p(\cdot)}(\Omega)$ and thus each u^i can be approximated by a suitable sequence $u_k^i \in D(\Omega)$ such that u_k^i converges to u^i strongly in $W_0^{1,p(\cdot)}(\Omega)$. Moreover, as $u_k^i \in$

$D(\Omega) \subset \overline{D(\Omega)}$, the Green formula gives

$$\int_{\Omega} \frac{\partial u_k^i}{\partial x_i}\, dx = \int_{\partial\Omega} u_k^i \vec{n}\, ds = 0. \tag{3.11}$$

On the other hand, $\partial u_k^i/\partial x_i \to \partial u^i/\partial x_i$ strongly in $L^{p(\cdot)}(\Omega)$. Thus $\partial u_k^i/\partial x_i \to \partial u^i/\partial x_i$ strongly in $L^1(\Omega)$, which gives $\int_\Omega \operatorname{div}(u)\, dx = 0$ by (3.11). ∎

3.2. General existence result. We now state our main result:

THEOREM 3.1. *Assume that* (3.1)–(3.8) *hold true. Then there exists a solution of the unilateral problem*

$$(P)\begin{cases} u \in T_0^{1,p(\cdot)}(\Omega), \quad u \ge \psi \quad \text{a.e. in } \Omega,\\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\, dx + \int_\Omega \phi(u)\nabla T_k(u-v)\, dx \\ \qquad \le \int_\Omega f T_k(u-v)\, dx + \int_\Omega F\nabla T_k(u-v)\, dx, \\ \qquad\qquad \forall v \in K_\psi \cap L^\infty(\Omega),\ \forall k>0, \end{cases}$$

where $T_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R}$ *measurable* $\mid T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ *for all* $k > 0\}$.

STEP 1: *The approximate problem*

THEOREM 3.2. *Let* $(f_n)_n$ *be a sequence in* $W^{-1,p'(\cdot)}(\Omega)\cap L^1(\Omega)$ *such that* $f_n \to f$ *in* $L^1(\Omega)$ *and* $\|f_n\|_1 \le \|f\|_1$, *and consider the approximate problem*

(3.12)

$$(P_n)\begin{cases} u_n \in K_\psi,\\ \langle Au_n, u_n - v\rangle + \int_\Omega \phi_n(u_n)\nabla(u_n-v)\, dx \\ \qquad \le \int_\Omega f_n(u_n-v)\, dx + \int_\Omega F\nabla(u_n-v)\, dx \quad \forall v \in K_\psi \cap L^\infty(\Omega), \end{cases}$$

where $\phi_n(s) = \phi(T_n(s))$. *Assume that* (3.1)–(3.8) *hold true. Then there exists a weak solution* u_n *of problem* (P_n).

Proof. We define the operator $G_n = -\operatorname{div}\phi_n : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ such that $\langle G_n(u), v\rangle = -\langle \operatorname{div}\phi_n(u), v\rangle = \int_\Omega \phi_n(u)\nabla v\, dx$ for all $u, v \in W_0^{1,p(\cdot)}(\Omega)$. From the Hölder inequality we have

$$\Big|\int_\Omega \phi_n(u)\nabla v\, dx\Big| \le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|\phi_n(u)\|_{p'(\cdot)}\|\nabla v\|_{p(\cdot)}$$

$$\leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\Big(\int_\Omega |\phi(T_n(u))|^{p'(x)}\,dx\Big)^{\gamma_0} \|v\|_{1,p(\cdot)}$$
$$\leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\Big(\operatorname{meas}(\Omega)\cdot\Big(\sup_{|s|\leq n}|\phi(s)|+1\Big)^{p_+}\Big)^{\gamma_0}\cdot\|v\|_{1,p(\cdot)}$$
$$\leq C_0\|v\|_{1,p(\cdot)}$$

where

$$\gamma_0 = \begin{cases} 1/p'_- & \text{if } \|\phi_n(u)\|_{p'(\cdot)} > 1,\\ 1/p'_+ & \text{if } \|\phi_n(u)\|_{p'(\cdot)} \leq 1,\end{cases}$$

and C_0 is a constant which depends only on ϕ, n and p. ∎

Lemma 3.5. *The operator $B_n = A + G_n$ is pseudo-monotone from the space $W_0^{1,p(\cdot)}(\Omega)$ into $W^{-1,p'(\cdot)}(\Omega)$. Moreover, B_n is coercive in the following sense: there exists $v_0 \in K_\psi$ such that*

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|_{1,p(\cdot)}} \to \infty \quad \text{if } \|v\|_{1,p(\cdot)} \to \infty,\ v \in K_\psi.$$

Proof. See the Appendix. ∎

In view of Lemma 3.5, there exists a solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$ of problem (P_n) (cf. [11]).

Step 2: *A priori estimate*

Proposition 3.1. *Assume that* (3.1)–(3.8) *hold true and let u_n be a solution of problem (P_n). Then for all $k \geq 0$, there exists a constant $c(k)$ (which does not depend on n) such that*

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \leq c(k). \tag{3.13}$$

Proof. Let $v_0 \in K_\psi \cap L^\infty(\Omega)$ and let $k \geq \|v_0\|_\infty$ be such that $v = T_h(u_n - T_k(u_n - v_0)) \in K_\psi \cap L^\infty(\Omega)$. Choosing v as a test function in (P_n) and letting $h \to \infty$, we obtain, for n large enough ($n \geq k + \|v_0\|_\infty$),

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n - v_0)\,dx + \int_\Omega \phi(u_n)\nabla T_k(u_n - v_0)\,dx \leq \int_\Omega f_n T_k(u_n - v_0)\,dx + \int_\Omega F\nabla T_k(u_n - v_0)\,dx.$$

This implies that

$$\begin{aligned}
&\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx \\
&\qquad \le \int_{\{|u_n-v_0|<k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla u_n|\,dx \\
&\qquad\quad + \int_{\{|u_n-v_0|<k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla v_0|\,dx \\
&\qquad\quad + k\|f\|_{L^1} + \int_{\{|u_n-v_0|<k\}} |F|\,|\nabla u_n|\,dx + \int_{\{|u_n-v_0|<k\}} |F|\,|\nabla v_0|\,dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\{|u_n-v_0|<k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \\
&\qquad \le \int_{\{|u_n-v_0|<k\}} |a(x,u_n,\nabla u_n)|\,|\nabla v_0|\,dx \\
&\qquad\quad + \int_{\{|u_n-v_0|<k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla u_n|\,dx \\
&\qquad\quad + \int_{\{|u_n-v_0|<k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla v_0|\,dx \\
&\qquad\quad + k\|f\|_{L^1} + \int_{\{|u_n-v_0|<k\}} |F|\,|\nabla u_n|\,dx + \int_{\{|u_n-v_0|<k\}} |F|\,|\nabla v_0|\,dx.
\end{aligned}$$

Since $\phi \in C^0(\mathbb{R},\mathbb{R}^N)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$, using Young's inequality we have

$$\begin{aligned}
\alpha \int_{\{|u_n-v_0|<k\}} |\nabla u_n|^{p(x)}\,dx &\le c_0 \int_{\{|u_n-v_0|<k\}} |a(x,u_n,\nabla u_n)|^{p'(x)}\,dx \\
&\quad + \frac{\alpha}{3} \int_{\{|u_n-v_0|<k\}} |\nabla u_n|^{p(x)}\,dx + c(k),
\end{aligned}$$

which implies, from (3.1) and (3.3),

$$\begin{aligned}
\alpha \int_{\{|u_n-v_0|<k\}} |\nabla u_n|^{p(x)}\,dx &\le \frac{\alpha}{6} \int_{\{|u_n-v_0|<k\}} (|u_n|^{p(x)} + |\nabla u_n|^{p(x)})\,dx \\
&\quad + \frac{\alpha}{3} \int_{\{|u_n-v_0|<k\}} |\nabla u_n|^{p(x)}\,dx + c(k),
\end{aligned}$$

hence

$$\frac{\alpha}{2} \int_{\{|u_n-v_0|<k\}} |\nabla u_n|^{p(x)}\,dx \le c(k),$$

where $c(k)$ is a constant which depends on k. Since $\{|u_n| \le k\} \subset \{|u_n - v_0| \le k + \|v_0\|_\infty\}$, we deduce that $\int_\Omega |\nabla T_k(u_n)|^{p(x)} dx \le c(k)$. ∎

STEP 3: *Strong convergence of truncations*

PROPOSITION 3.2. *Let u_n be a solution of problem (P_n). Then there exists a measurable function u such that*

$$T_k(u_n) \to T_k(u) \quad \textit{strongly in } W_0^{1,p(\cdot)}(\Omega).$$

We will use the following lemma:

LEMMA 3.6. *Assume that* (3.1)–(3.8) *hold true and let u_n be a solution of problem (P_n). Then*

$$\int_\Omega |\nabla T_k(u_n - T_h(u_n))|^{p(x)}\, dx \le kc \tag{3.14}$$

for all $k > h > \|v_0\|_\infty$, where c is a constant that does not depend on k, and $v_0 \in K_\psi \cap L^\infty(\Omega)$.

Proof. Let $l \ge \|v_0\|_\infty$. It is easy to see that $v = T_l(u_n - T_k(u_n - T_h(u_n)))$ $\in K_\psi \cap L^\infty(\Omega)$. By using v as a test function in (P_n) and letting $l \to \infty$, we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - T_h(u_n))\, dx + \int_\Omega \phi(T_h(u_n))\nabla T_k(u_n - T_h(u_n))\, dx$$
$$\le \int_\Omega f_n T_k(u_n - T_h(u_n))\, dx + \int_\Omega F\nabla T_k(u_n - T_h(u_n))\, dx.$$

Let us define

$$\chi_{hk}(t) = \begin{cases} 1 & \text{if } h < |t| < h + k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

We consider $\theta(t) = \phi(t)\chi_{hk}(t)$ and $\tilde\theta(t) = \int_0^t \theta(s)\, ds$. Then by Lemma 3.4,

$$\int_\Omega \phi(u_n)\nabla T_k(u_n - T_h(u_n))\, dx = \int_\Omega \phi(u_n)\chi_{hk}(u_n)\nabla u_n\, dx$$
$$= \int_\Omega \theta(u_n)\nabla u_n\, dx = \int_\Omega \operatorname{div}(\tilde\theta(u_n))\, dx = 0.$$

Thus, the second term on the left side of (3.2) vanishes for n large enough, which implies that

$$\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - T_h(u_n))\, dx \le k\|f\|_{L^1(\Omega)} + \int_\Omega F\nabla T_k(u_n - T_h(u_n))\, dx.$$

By Young's inequality,

$$\int_{\Omega} a(x, u_n, \nabla u_n)\nabla T_k(u_n - T_h(u_n))\, dx \\ \le k\|f\|_{L^1(\Omega)} + c_1 + \frac{\alpha}{2}\int_{\Omega} |\nabla T_k(u_n - T_h(u_n))|^{p(x)}\, dx.$$

Since $\nabla T_k(u_n - T_h(u_n)) = \nabla u_n \chi(hk)$ we have

$$\int_{\Omega} a(x, u_n, \nabla T_k(u_n - T_h(u_n)))\nabla T_k(u_n - T_h(u_n))\, dx \\ \le kc_2 + \frac{\alpha}{2}\int_{\Omega} |\nabla T_k(u_n - T_h(u_n))|^{p(x)}\, dx.$$

Finally, from (3.3), we deduce that $\int_{\Omega} |\nabla T_k(u_n - T_h(u_n))|^{p(x)} dx \le kc$, which concludes the proof of Lemma 3.6. ■

Proof of Proposition 3.2. We will show first that $(u_n)_n$ is a Cauchy sequence in measure. Let $k > 2h > 2\|v_0\|_\infty$. Then

$$k\, \mathrm{meas}\{|u_n - T_h(u_n)| > k\} \le \int_{\{|u_n - T_h(u_n)| > k\}} |T_k(u_n - T_h(u_n))|\, dx.$$

By Hölder's inequality, Poincaré's inequality and (3.14) one has

$$k\, \mathrm{meas}\{|u_n - T_h(u_n)| > k\} \\ \le \int_{\Omega} |T_k(u_n - T_h(u_n))|\, dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|1\|_{p'(\cdot)}\|T_k(u_n - T_h(u_n))\|_{p(\cdot)} \\ \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)(\mathrm{meas}(\Omega) + 1)^{1/p'_-}\|T_k(u_n - T_h(u_n))\|_{p(\cdot)} \le C_4 k^{1/\gamma},$$

where

$$\gamma = \begin{cases} 1/p_- & \text{if } \|\nabla T_k(u_n - T_h(u_n))\|_{p(\cdot)} > 1, \\ 1/p_+ & \text{if } \|\nabla T_k(u_n - T_h(u_n))\|_{p(\cdot)} \le 1. \end{cases} \tag{3.16}$$

Finally, for $k > 2h > 2\|v_0\|_\infty$, we have

$$\mathrm{meas}\{|u_n| > k\} \le \mathrm{meas}\{|u_n - T_h(u_n)| > k - h\} \le \frac{c}{(k-h)^{1-1/\gamma}}, \tag{3.17}$$

so

$$\mathrm{meas}(\{|u_n| > k\} \to 0 \quad \text{as } k \to \infty, \tag{3.18}$$

and, for all $\delta > 0$,

$$\mathrm{meas}\{|u_n - u_m| > \delta\} \le \mathrm{meas}\{|u_n| > k\} \\ + \mathrm{meas}\{|u_m| > k\} + \mathrm{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.$$

By (3.18), for each $\varepsilon > 0$, there exists k_0 such that

$$\text{meas}\{|u_n| > k\} \le \varepsilon/3 \quad \text{and} \quad \text{meas}\{|u_m| > k\} \le \varepsilon/3 \quad \forall k \ge k_0. \tag{3.19}$$

By (3.13), the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, so a subsequence (not relabeled) converges to η_k weakly in $W_0^{1,p(\cdot)}(\Omega)$ as $n \to \infty$, and by the compact embedding, $T_k(u_n)$ converges to η_k strongly in $L^{p(\cdot)}(\Omega)$ a.e. in Ω. Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω. Then there exists n_0 which depends on δ and ε such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \le \varepsilon/3 \quad \forall m, n \ge n_0 \text{ and } k \ge k_0. \tag{3.20}$$

In view of (3.19) and (3.20), we obtain

$$\forall \delta > 0,\ \exists \varepsilon > 0: \quad \text{meas}\{|u_n - u_m| > \delta\} \le \varepsilon \quad \forall n, m \ge n_0(k_0, \delta).$$

Thus $(u_n)_n$ is a Cauchy sequence in measure in Ω, so there exists a subsequence still denoted u_n which converges almost everywhere to some measurable function u. Then u_n converges to u a.e. in Ω, and by Lemma 2.1, we obtain:

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^{1,p(\cdot)}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{3.21}$$

Now, we choose $v \equiv T_l(u_n - h_m(u_n - v_0)(T_k(u_n) - T_k(u)))$ as a test function in (P_n), where

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m, \\ 0 & \text{if } |s| \ge m+1, \\ m+1-|s| & \text{if } m \le |s| \le m+1. \end{cases} \tag{3.22}$$

For $n > m+1$, by letting $l \to \infty$ we get

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n)\nabla\big(h_m(u_n - v_0)(T_k(u_n) - T_k(u))\big)\,dx \\ + \int_\Omega \phi(u_n)\nabla\big(h_m(u_n - v_0)(T_k(u_n) - T_k(u))\big)\,dx \\ \le \int_\Omega f_n h_m(u_n - v_0)(T_k(u_n) - T_k(u))\,dx \\ + \int_\Omega F\nabla\big(h_m(u_n - v_0)(T_k(u_n) - T_k(u))\big)\,dx, \end{aligned}$$

which implies that

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\,dx \\ + \int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v_0)h_m'(u_n - v_0)(T_k(u_n) - T_k(u))\,dx \end{aligned}$$

$$+\int_{\Omega} \phi(u_n)\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx$$
$$+\int_{\Omega} \phi(u_n)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx$$
$$\le \int_{\Omega} f_n h_m(u_n-v_0)(T_k(u_n)-T_k(u))\,dx$$
$$+\int_{\Omega} F\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx$$
$$+\int_{\Omega} F\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx.$$

By almost everywhere convergence of u_n, $h_m(u_n-v_0)(T_k(u_n)-T_k(u))$ converges to 0 weakly* in $L^\infty(\Omega)$ as $n\to\infty$, so

$$\int_{\Omega} f_n h_m(u_n-v_0)(T_k(u_n)-T_k(u))\,dx=\epsilon(n). \tag{3.23}$$

Moreover, by Lebesgue's theorem, $\phi(u_n)h_m(u_n-v_0)$ tends to $\phi(u)h_m(u-v_0)$ strongly in $L^{p'(\cdot)}(\Omega)$, and since $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $L^{p(\cdot)}(\Omega)$ we can deduce that

$$\int_{\Omega} \phi(u_n)\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx=\epsilon(n). \tag{3.24}$$

Similarly,

$$\int_{\Omega} F\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx=\epsilon(n). \tag{3.25}$$

On the other hand,

$$\Big|\int_{\Omega} F\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx\Big|$$
$$=\Big|\int_{\Omega} F\nabla(u_n-v_0)(T_k(u_n)-T_k(u))\chi_{\{m<|u_n-v_0|<m+1\}}\,dx\Big|$$
$$\le\int_{\Omega} |F\nabla(T_M(u_n)-v_0)(T_k(u_n)-T_k(u))|\,dx$$

with $M=m+1+\|v_0\|_\infty$. Then by Lebesgue's theorem, $F(T_k(u_n)-T_k(u))$ converges to 0 strongly in $L^{p'(\cdot)}(\Omega)$, and since $\nabla(T_M(u_n)-v_0)$ converges to $\nabla(T_M(u)-v_0)$ weakly in $(L^{p(\cdot)}(\Omega))^N$, we have

$$\int_{\Omega} F\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx=\epsilon(n). \tag{3.26}$$

Similarly,

$$\int_{\Omega} \phi(u_n)\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx=\epsilon(n). \tag{3.27}$$

We claim that

$$(3.28)\quad \int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx=\epsilon(n).$$

Indeed,

$$\begin{aligned}\Big|\int_\Omega & a(x,u_n,\nabla u_n)\nabla(u_n-v_0)h_m'(u_n-v_0)(T_k(u_n)-T_k(u))\,dx\Big|\\ &=\Big|\int_{\{m\le|u_n-v_0|\le m+1\}} a(x,u_n,\nabla u_n)\nabla(u_n-v_0)(T_k(u_n)-T_k(u))\,dx\Big|\\ &\le 2k\int_{\{m\le|u_n-v_0|\le m+1\}}|a(x,u_n,\nabla u_n)\nabla(u_n-v_0)|\,dx\\ &\le 2k\Big(\int_{\{l\le|u_n|\le l+s\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx+\int_{\{l\le|u_n|\le l+s\}}|a(x,\nabla u_n)|\,|\nabla v_0|\,dx\Big)\end{aligned}$$

where $l=m-\|v_0\|_\infty$ and $s=2\|v_0\|_\infty+1$. We take $v\equiv\ u_n-T_s(u_n-T_l(u_n))$ as a test function in (P_n) to get

$$\begin{aligned}\int_{\{l\le|u_n|\le l+s\}} & a(x,u_n,\nabla u_n)\nabla u_n\,dx+\int_\Omega \operatorname{div}(\tilde\theta_s(u_n))\,dx\\ &\le\int_\Omega f_nT_s(u_n-T_l(u_n))\,dx+\int_\Omega F\nabla T_s(u_n-T_l(u_n))\,dx\end{aligned}$$

where $\tilde\theta_s(t)=\int_0^t\theta_s(z)\,dz$ and $\theta_s(z)=\phi(z)\chi_{sl}(z)$ with

$$\chi_{sl}=\begin{cases}1, & l\le t\le l+s,\\ 0, & \text{otherwise.}\end{cases}$$

Using the fact that $\tilde\theta(u_n)\in(W_0^{1,p(\cdot)}(\Omega))^N$ and Lemma 3.4, we get

$$(3.29)\quad \begin{aligned}\int_{\{l\le|u_n|\le l+s\}} & a(x,u_n,\nabla u_n)\nabla u_n\,dx\\ &\le s\int_{\{|u_n|>l\}}|f_n|\,dx+\int_{\{l\le|u_n|\le l+s\}}F\nabla u_n\,dx.\end{aligned}$$

Firstly, we will show that

$$\int_{\{l\le|u_n|\le l+s\}}F\nabla u_n\,dx=\epsilon(n,m).$$

Indeed, by (3.29) and Young's inequalities, we get

$$\int_{\{l\le|u_n|\le l+s\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \le s\int_{\{|u_n|>l\}}|f_n|\,dx + c\int_{\{|u_n|>l\}}|F|^{p'(x)}\,dx + \frac{\alpha}{2}\int_{\{l\le|u_n|\le l+s\}}|\nabla u_n|^{p(x)}\,dx,$$

which yields, thanks to (3.3),

$$\frac{\alpha}{2}\int_{\{l\le|u_n|\le l+s\}}|\nabla u_n|^{p(x)}\,dx \le s\int_{\{|u_n|>l\}}|f_n|\,dx + c\int_{\{|u_n|>l\}}|F|^{p'(x)}\,dx,$$

which implies that

$$\int_\Omega|\nabla T_s(u_n-T_l(u_n))|^{p(x)}dx \le \frac{2s}{\alpha}\int_{\{|u_n|>l\}}|f_n|\,dx + \frac{2c}{\alpha}\int_{\{|u_n|>l\}}|F|^{p'(x)}\,dx.$$

Consequently, by the strong convergence in $L^1(\Omega)$ of f_n and since $F\in L^{p'(\cdot)}(\Omega)$, by Lebesgue's theorem we have

$$\lim_{l\to\infty}\lim_{n\to\infty}\int_\Omega|\nabla T_s(u_n-T_l(u_n))|^{p(x)}\,dx = 0,$$

which implies by Hölder's inequality, that

$$\lim_{l\to\infty}\lim_{n\to\infty}\int_\Omega F\nabla T_s(u_n-T_l(u_n))\,dx = 0.$$

Hence

$$\int_{\{l\le|u_n|\le l+s\}} F\nabla u_n dx = \epsilon(n,l). \tag{3.30}$$

Finally by (3.29) and (3.30) we deduce

$$\int_{\{l\le|u_n|\le l+s\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = \epsilon(n,l). \tag{3.31}$$

On the other hand,

$$\begin{aligned}\int_{\{l\le|u_n|\le l+s\}} &|a(x,u_n,\nabla u_n)|\,|\nabla v_0|\,dx\\ &\le c\Big(\int_\Omega|a(x,\nabla T_s(u_n-T_l(u_n)))|^{p'(x)}\,dx\Big)^\gamma\|\nabla v_0\chi_{\{|u_n|>l\}}\|_{p(\cdot)}\\ &\le c\Big(\int_\Omega\big(|k(x)+|\nabla T_s(u_n-T_l(u_n))|^{p(x)}+|T_s(u_n-T_l(u_n))|^{p(x)}\big)\,dx\Big)^\gamma\\ &\quad\times\|\nabla v_0\chi_{\{|u_n|>l\}}\|_{p(\cdot)}\end{aligned}$$

where

$$\gamma = \begin{cases} 1/p'_- & \text{if } \|a(x, \nabla T_s(u_n - T_l(u_n)))\|_{p'(\cdot)} \geq 1, \\ 1/p'_+ & \text{if } \|a(x, \nabla T_s(u_n - T_l(u_n)))\|_{p'(\cdot)} < 1. \end{cases}$$

Furthermore by Lemma 3.6, we have

$$\int_\Omega |\nabla T_s(u_n - T_l(u_n))|^{p(x)}\, dx \leq c(s), \tag{3.32}$$

$$\int_\Omega |T_s(u_n - T_l(u_n))|^{p(x)}\, dx \leq c'(s), \tag{3.33}$$

where $c(s)$ and $c'(s)$ are constants independent of l. By (3.2), (3.32) and (3.33), we obtain

$$\int_{\{l\leq|u_n|\leq l+s\}} |a(x, u_n, \nabla u_n)|\, |\nabla v_0|\, dx = \epsilon(n, l). \tag{3.34}$$

Finally, (3.31) and (3.34) yield the estimate (3.28). Combining (3.23)–(3.28) and $l = m - \|v_0\|_\infty$, we get

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\, dx \leq \epsilon(n, m). \tag{3.35}$$

Splitting the first integral on the left hand side of (3.35) where $|u_n| \leq k$ and $|u_n| > k$, we can write

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n)\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\, dx \\ &\quad = \int_{\{|u_n|\leq k\}} a(x, T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\, dx \\ &\qquad - \int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n)\nabla T_k(u)h_m(u_n - v_0)\, dx \\ &\quad \geq \int_{\{|u_n|\leq k\}} a(x, T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u))h_m(u_n - v_0)\, dx \\ &\qquad - \int_\Omega |a(x, T_M(u_n), \nabla T_M(u_n))|\, |\nabla T_k(u)|\chi_{\{|u_n|>k\}}\, dx \end{aligned}$$

where $M = m + \|v_0\|_\infty + 1$. Since $a(x, T_M(u_n), \nabla T_M(u_n))$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, for a subsequence we have $a(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup l_m$ weakly in $(L^\infty(\Omega))^N$ as $n \to \infty$. Since $|\frac{\partial T_k(u_n)}{\partial x_i}|\chi_{\{|u_n|>k\}}$ converges to $|\frac{\partial T_k(u)}{\partial x_i}|\chi_{\{|u|>k\}} = 0$ strongly in $L^{p(\cdot)}(\Omega)$, we get

$$\int_\Omega |a(x, T_M(u_n), \nabla T_M(u_n))|\, |\nabla T_k(u)|\chi_{\{|u_n|>k\}}\, dx = \epsilon(n). \tag{3.36}$$

From (3.35) and (3.36) we have

$$(3.37)\quad \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx \le \epsilon(n,m).$$

It is easy to see that

$$\begin{aligned}(3.38)\quad &\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx\\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\qquad \times \nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx\\ &\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx\\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\qquad \times \nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx\\ &\quad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u_n)h_m(u_n-v_0)\,dx\\ &\quad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)h_m(u_n-v_0)\,dx.\end{aligned}$$

By the continuity of the Nemytskiĭ operator, $a(x,T_k(u_n),\nabla T_k(u))h_m(u_n-v_0)$ converges to $a(x,T_k(u),\nabla T_k(u))h_m(u-v_0)$ strongly in $(L^{p'(\cdot)}(\Omega)^N$ while $\partial T_k(u_n)/\partial x_i$ converges to $\partial T_k(u)/\partial x_i$ weakly in $L^{p(\cdot)}(\Omega)$. The second and third terms of the right hand side of (3.38) tend respectively to $\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)h_m(u-v_0)\,dx$ and $-\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)h_m(u-v_0)\,dx$. So (3.37) and (3.38) yield

$$\begin{aligned}(3.39)\quad &\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\qquad \times \nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx \le \epsilon(n,m),\end{aligned}$$

which implies that

$$\begin{aligned}(3.40)\quad &\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u))\,dx\\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\qquad \times \nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx\\ &\quad + \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\qquad \times \nabla(T_k(u_n)-T_k(u))(1-h_m(u_n-v_0)).\end{aligned}$$

Since $1-h_m(u_n-v_0)=0$ in $\{x\in\Omega \mid |u_n-v_0|<m\}$ and $\{x\in\Omega \mid |u_n|<k\} \subset \{x\in\Omega \mid |u_n-v_0|<m\}$ for m large enough, we deduce from (3.40) that

$$\begin{aligned}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u))\,dx \\ = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,\nabla T_k(u))] \\ \times \nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx \\ - \int_{\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u))\nabla T_k(u)\,dx.\end{aligned}$$

It is easy to see that the last term tends to zero as $n\to\infty$, which implies that

$$\begin{aligned}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u))\,dx \\ = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\ \times \nabla(T_k(u_n)-T_k(u))h_m(u_n-v_0)\,dx \\ + \epsilon(n).\end{aligned}$$

Combining (3.39) and (3.41), we have

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u))\,dx \le \epsilon(n,m).$$

By passing to the lim sup over n and letting m tend to infinity, we obtain

$$\limsup_{n\to\infty} \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \times \nabla(T_k(u_n)-T_k(u))\,dx = 0.$$

Thus by Lemma 3.1, $T_k(u_n)$ converges to $T_k(u)$ strongly in $W_0^{1,p(\cdot)}(\Omega)$. ∎

Proof of Theorem 3.1. Let $v\in K_\psi\cap L^\infty(\Omega)$ and take $T_l(u_n-T_k(u_n-v))$ as a test function in (P_n). Letting $l\to\infty$, we can write, for n large enough $(n>k+\|v\|_\infty)$,

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v)\,dx + \int_\Omega \phi(u_n)\nabla T_k(u_n-v)\,dx \\ \le \int_\Omega f_n T_k(u_n-v)\,dx + \int_\Omega F\nabla T_k(u_n-v)\,dx.\end{aligned}$$

We get

$$\int_\Omega a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx + \int_\Omega \phi(T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx \le \int_\Omega f_n T_k(u_n - v)\,dx + \int_\Omega F\nabla T_k(u_n - v)\,dx.$$

By Fatou's lemma and the fact that $a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))$ converges to $a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))$ weakly in $(L^{p'(\cdot)}(\Omega))^N$, it is easy to see that

$$\int_\Omega a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))\nabla T_k(u_-v)\,dx \le \liminf_{n\to\infty} \int_\Omega a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx.$$

On the other hand, by using $F \in (L^{p'(\cdot)}(\Omega))^N$, we deduce that

$$\int_\Omega F\nabla T_k(u_n - v)\,dx \to \int_\Omega F\nabla T_k(u - v)\,dx \quad \text{as } n \to \infty. \tag{3.41}$$

Moreover, by Lebesgue's theorem, $\phi(T_{k+\|v\|_\infty}(u_n))$ tends to $\phi(T_{k+\|v\|_\infty}(u))$ strongly in $(L^{p'(\cdot)}(\Omega))^N$ as $n \to \infty$, and $\nabla T_k(u_n - v)$ converges to $\nabla T_k(u - v)$ weakly in $(L^{p(\cdot)}(\Omega))^N$, so that

$$\int_\Omega \phi(T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx \to \int_\Omega \phi(T_{k+\|v\|_\infty}(u))\nabla T_k(u - v)\,dx \quad \text{as } n \to \infty. \tag{3.42}$$

Similarly,

$$\int_\Omega f_n T_k(u_n - v)\,dx \to \int_\Omega f T_k(u - v)\,dx. \tag{3.43}$$

By using (3.43), (3.42), we can pass to the limit in (3.41) to obtain

$$\int_\Omega a(x, \nabla u_n)\nabla T_k(u_n - v)\,dx + \int_\Omega \phi(u_n)\nabla T_k(u_n - v)\,dx \le \int_\Omega f_n T_k(u_n - v)\,dx + \int_\Omega F\nabla T_k(u_n - v)\,dx,$$

which completes the proof of Theorem 3.1. ∎

Remark 3.2. Note that the condition (3.8) is used essentially to prove the coercivity of the operator B_n.

We can prove the coercivity of B_n if we replace the condition (3.8) by

$$p_+ - p_- < 1. \tag{3.44}$$

This is the objective of the following theorem:

THEOREM 3.3. *Assume that* (3.1)–(3.7) *and* (3.44) *hold true. Then there exists a solution of problem* (P).

Proof. Following the same steps of argument of the proof of Theorem 3.1, it suffices to show the coercivity of the operator B_n.

Indeed, let $v_0 \in K_\psi$. From the Hölder inequality and the growth condition, we have

$$\begin{aligned}
\langle Av, v_0\rangle &= \int_\Omega a(x,v,\nabla v)\nabla v_0\,dx \\
&\le C\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\Big(\int_\Omega |a(x,v,\nabla v)|^{p'(x)}\,dx\Big)^{\gamma'}\|v_0\|_{W_0^{1,p(\cdot)}(\Omega)} \\
&\le C\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|v_0\|_{W_0^{1,p(\cdot)}(\Omega)}\Big(\int_\Omega \beta(k(x)^{p'(x)}+|v|^{p(x)}+|\nabla v|^{p(x)})\,dx\Big)^{\gamma'} \\
&\le C_0(C_1+\rho(v)+\rho(\nabla v))^{\gamma'} \le C_0(C_1+C(\rho(\nabla v))^{p_+/p_-}+\rho(\nabla v))^{\gamma'}
\end{aligned}$$

where

$$\gamma' = \begin{cases} 1/p'_- & \text{if } \|a(x,v,\nabla v)\|_{L^{p'(\cdot)}(\Omega)} > 1, \\ 1/p'_+ & \text{if } \|a(x,v,\nabla v)\|_{L^{p'(\cdot)}(\Omega)} \le 1. \end{cases} \tag{3.45}$$

From (3.3) we have

$$\begin{aligned}
&\frac{\langle Av,v\rangle}{\|v\|_{1,p(\cdot)}} - \frac{\langle Av,v_0\rangle}{\|v\|_{1,p(\cdot)}} \\
&\qquad \ge \frac{1}{\|v\|_{1,p(\cdot)}}\big(\alpha\rho(\nabla v) - C_0(C_1+C(\rho(\nabla v))^{p_+/p_-}+\rho(\nabla v))^{\gamma'}\big).
\end{aligned} \tag{3.46}$$

Since $\|v\|_{1,p(\cdot)} \to \infty$ we have $\|a(x,v,\nabla v)\|_{L^{p'(\cdot)}(\Omega)} > 1$; then $\gamma' = 1/p'_-$, and as $p_+ - p_- < 1$, we have $\frac{p_+}{p'_- p_-} < 1$, so

$$\frac{\langle Av,v\rangle}{\|v\|_{1,p(\cdot)}} - \frac{\langle Av,v_0\rangle}{\|v\|_{1,p(\cdot)}} \to \infty \quad \text{as } \|v\|_{1,p(\cdot)} \to \infty.$$

Since $\langle G_n v, v\rangle/\|v\|_{1,p(\cdot)}$ and $\langle G_n v, v_0\rangle/\|v\|_{1,p(\cdot)}$ are bounded, we have

$$\frac{\langle B_n v, v-v_0\rangle}{\|v\|_{1,p(\cdot)}} = \frac{\langle Av, v-v_0\rangle}{\|v\|_{1,p(\cdot)}} + \frac{\langle G_n v, v\rangle}{\|v\|_{1,p(\cdot)}} - \frac{\langle G_n v, v_0\rangle}{\|v\|_{1,p(\cdot)}} \to \infty$$

as $\|v\|_{1,p(\cdot)} \to \infty$. ∎

4. Appendix

Proof of Lemma 3.5. Let $v_0 \in K_\psi$. From the Hölder inequality and the growth condition, we have

$$\begin{aligned}\langle Av, v_0\rangle &= \int_\Omega a(x,v,\nabla v)\nabla v_0\,dx\\ &\le C\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\Big(\int_\Omega |a(x,v,\nabla v)|^{p'(x)}\,dx\Big)^{\gamma'}\,dx\,\|v_0\|_{W_0^{1,p(\cdot)}(\Omega)}\\ &\le C\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|v_0\|_{W_0^{1,p(\cdot)}(\Omega)}\Big(\int_\Omega \beta(k(x)^{p'(x)}+|v|^{p(x)}+|\nabla v|^{p(x)})\,dx\Big)^{\gamma'}\\ &\le C_0(C_1+\rho(v)+\rho(\nabla v))^{\gamma'} \le C_0(C_1+C\rho(\nabla v)+\rho(\nabla v))^{\gamma'}\end{aligned}$$

where

$$\gamma' = \begin{cases} 1/p'_- & \text{if } \|a(x,v,\nabla v)\|_{L^{p'(\cdot)}(\Omega)} \ge 1,\\ 1/p'_+ & \text{if } \|a(x,v,\nabla v)\|_{L^{p'(\cdot)}(\Omega)} \le 1.\end{cases} \tag{4.1}$$

From (3.3) we have

$$\frac{\langle Av, v\rangle}{\|v\|_{1,p(\cdot)}} - \frac{\langle Av, v_0\rangle}{\|v\|_{1,p(\cdot)}} \ge \frac{1}{\|v\|_{1,p(\cdot)}}\big(\alpha\rho(\nabla v) - C_0(C_1+C\rho(\nabla v)+\rho(\nabla v))^{\gamma'}\big). \tag{4.2}$$

Hence $\rho(\nabla v)/\|v\|_{1,p(\cdot)} \to \infty$ as $\|v\|_{1,p(\cdot)} \to \infty$, and we have

$$\frac{\langle Av, v\rangle}{\|v\|_{1,p(\cdot)}} - \frac{\langle Av, v_0\rangle}{\|v\|_{1,p(\cdot)}} \to \infty \quad \text{as } \|v\|_{1,p(\cdot)} \to \infty.$$

Since $\langle G_n v, v\rangle/\|v\|_{1,p(\cdot)}$ and $\langle G_n v, v_0\rangle/\|v\|_{1,p(\cdot)}$ are bounded, we have

$$\frac{\langle B_n v, v-v_0\rangle}{\|v\|_{1,p(\cdot)}} = \frac{\langle Av, v-v_0\rangle}{\|v\|_{1,p(\cdot)}} + \frac{\langle G_n v, v\rangle}{\|v\|_{1,p(\cdot)}} - \frac{\langle G_n v, v_0\rangle}{\|v\|_{1,p(\cdot)}} \to \infty$$

as $\|v\|_{1,p(\cdot)} \to \infty$. It remains to show that B_n is pseudo-monotone.

Let $(u_k)_k$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,p(\cdot)}(\Omega),\\ B_n u_k \rightharpoonup \chi & \text{in } W^{-1,p'(\cdot)}(\Omega),\\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{4.3}$$

We will prove that $\chi = B_n u$ and $\langle B_n u_k, u_k\rangle$ converges to $\langle \chi, u\rangle$ as $k \to \infty$.

Firstly, since $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we have $u_k \to u$ in $L^{p(\cdot)}(\Omega)$ for a subsequence still denoted by $(u_k)_k$. Since $(u_k)_k$ is a bounded sequence in $W_0^{1,p(\cdot)}(\Omega)$, by the growth condition, $(a(x,u_k,\nabla u_k))_k$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, therefore there exists $\varphi \in (L^{p'(\cdot)}(\Omega))^N$ such that

$$a(x, u_k, \nabla u_k) \rightharpoonup \varphi \quad \text{in } (L^{p'(\cdot)}(\Omega))^N \ \text{ as } \ k \to \infty. \tag{4.4}$$

Then $\phi_n = \phi \circ T_n$ is a continuous function, and since $u_k \to u$ in $L^{p(\cdot)}(\Omega)$ we have

$$\phi_n(u_k) \to \phi_n(u) \quad \text{in } (L^{p'(\cdot)}(\Omega))^N \text{ as } k \to \infty. \tag{4.5}$$

It is clear that, for all $v \in W_0^{1,p(\cdot)}(\Omega)$,

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla v\, dx - \lim_{k\to\infty} \int_\Omega \phi_n(u_k)\nabla v\, dx \\ &= \int_\Omega \varphi\nabla v\, dx - \int_\Omega \phi_n(u)\nabla v\, dx. \end{aligned} \tag{4.6}$$

On the one hand, by (4.5) we have

$$\int_\Omega \phi_n(u_k)\nabla u_k\, dx \to \int_\Omega \phi_n(u)\nabla u\, dx \quad \text{as } k \to \infty. \tag{4.7}$$

Combining (4.3) and (4.6), we have

(4.8)

$$\begin{aligned} \limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big\{ \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx - \int_\Omega \phi_n(u_k)\nabla u_k\, dx \Big\} \\ &\le \int_\Omega \varphi\nabla u\, dx - \int_\Omega \phi_n(u)\nabla u\, dx. \end{aligned}$$

Therefore

$$\limsup_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx \le \int_\Omega \varphi\nabla u\, dx. \tag{4.9}$$

On the other hand, thanks to (3.3), we have

$$\int_\Omega (a(x, u_k, \nabla u_k) - a(x, u_k, \nabla u))(\nabla u_k - \nabla u)\, dx > 0, \tag{4.10}$$

so

$$\begin{aligned} \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx \ge & -\int_\Omega a(x, u_k, \nabla u)\nabla u\, dx \\ & + \int_\Omega a(x, u_k, \nabla u_k)\nabla u\, dx + \int_\Omega a(x, u_k, \nabla u)\nabla u_k\, dx, \end{aligned}$$

and by (4.4), we get

$$\liminf_{k\to\infty} \int_\Omega a(x, u_k, \nabla u_k)\nabla u_k\, dx \ge \int_\Omega \varphi\nabla u\, dx.$$

This implies, by using (4.9), that

$$\lim_{k\to\infty}\int_{\Omega} a(x,u_k,\nabla u_k)\nabla u_k\,dx = \int_{\Omega}\varphi\nabla u\,dx. \tag{4.11}$$

By combining (4.6), (4.7) and (4.11), we find that $\langle B_n u_k, u_k\rangle$ converges to $\langle \chi, u\rangle$ as $k\to\infty$.

On the other hand, by (4.11) and the fact that $a(x,u_k,\nabla u)$ converges to $a(x,u,\nabla u)$ in $(L^{p'(\cdot)}(\Omega))^N$ we deduce that

$$\lim_{k\to\infty}\int_{\Omega}(a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))(\nabla u_k-\nabla u)\,dx = 0,$$

and by Lemma 3.1, u_k converges to u in $W_0^{1,p(\cdot)}(\Omega)$ and a.e. in Ω. We deduce that $a(x,u_k,\nabla u_k)$ converges to $a(x,u_k,\nabla u)$ in $(L^{p'(\cdot)}(\Omega))^N$, and $\phi_n(u_k)$ converges to $\phi_n(u)$ in $(L^{p'(\cdot)}(\Omega))^N$. Hence $\chi = B_n u$, which completes the proof of Lemma 3.5. ∎

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