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Dariusz Socha (Warszawa)

DISCRETE TIME OPTIMAL DIVIDEND PROBLEM WITH CONSTANT PREMIUM AND EXPONENTIALLY DISTRIBUTED CLAIMS

Abstract. An optimal dividend problem is studied consisting in maximisation of expected discounted dividend payments until ruin time. A solution of this problem for constant premium d and exponentially distributed claims is presented. It is shown that an optimal policy is a barrier policy. Moreover, an analytic way to solve this problem is sketched.

1. Introduction. We consider the surplus of an insurance company in the form

$$X_{n+1} = X_n - U_n + Y_{n+1}, \tag{1.1}$$

with initial capital $X_0 = x$. In the formula above, X_n denotes the assets of the insurance company at the beginning of the nth period. The company collects premiums from customers and has to pay claims. The balance between premiums and claims may be negative and is modelled by an i.i.d. sequence Y_{n+1}. Denote the probability distribution of Y_{n+1} by ν and by Y a generic random variable with this distribution. We assume that $\mathbb{E}|Y| < \infty$. The insurance company decides about the amount of dividend payout U_n which is viewed as a control variable. This decision is based on available information. Denote by $\mathcal{F}_n$ the filtration generated by X_n, i.e. $\mathcal{F}_n = \sigma(X_n)$, and let $\bigcup_{k=0}^{n} \mathcal{F}_k = \mathcal{G}_n$. Then we assume that the stochastic process $\{U_n : n = 1, 2, \dots\}$ is adapted to $\mathcal{G}_n$, and moreover $0 \le U_n \le X_n$ for all $n \in \mathbb{N}$. The procedure is stopped at the time of ruin $\tau = \inf\{n : X_n < 0\}$. Denote by $D_n(X_n)$ the interval $[0, X_n]$ if $X_n \ge 0$, and let $D_n(X_n) = \{0\}$ if $X_n < 0$.

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Definition 1.1.

(i) $\mathcal{G}_n$-measurable random variables $U_n \in D_n(X_n)$ are called *decisions* at time n. We denote by Π_n the set of all decisions at time n.
(ii) A sequence $u = (f_0, f_1, \dots)$ of decisions $f_i \in \Pi_i$ is called an *admissible strategy.*
(iii) We denote by Π the set $\Pi_1 \times \Pi_2 \times \cdots$.

For a fixed strategy $\{U_n\}$ define the value function to be the expected value of discounted dividend payments until ruin. For a discount factor $\gamma \in (0,1)$ we have

$$V(\{U_n\}, x) = \mathbb{E}\Big[\sum_{n=0}^{\tau-1} \gamma^n U_n \;\Big|\; X_0 = x\Big].$$

The problem is to find an admissible strategy U_n^* that maximises the above functional and to characterise the value function defined as

$$V(x) = \sup_{U_n \in \Pi} V(\{U_n\}, x) = \sup_{U_n \in \Pi} \mathbb{E}\Big[\sum_{n=0}^{\tau-1} \gamma^n U_n \;\Big|\; X_0 = x\Big]. \tag{1.2}$$

The above model, first proposed by de Finetti [5], was intensively studied in the 1960s. The most important results were obtained by Miyasawa [10], Gerber [6], Shubik and Thompson [14], and Morill [11]. These results essentially use the assumption that the random variables Y_{n+1} in (1.1) are integer-valued. It seems that this discrete model is more complicated than the similar one in continuous time. The main reason is that in the latter case we can use Lévy processes to consider an equivalent problem. The problem of finding an optimal dividend strategy has been studied extensively in the Brownian motion setting [2, 15] and in the Cramér–Lundberg setting [3, 7]. The main problem consists in showing that optimal strategies are of barrier form, i.e. there is a unique point a such that below a we do not pay a dividend while above a we pay a dividend equal to the difference between the surplus and a. The problem was solved in the continuous time setting by Loeffen [9]. A similar problem in which dividend pay-outs are restricted to random discrete times was solved by Albrecher, Bauerle and Thonhauser [1].

In this paper optimality of barrier strategies in the case of exponential claims is shown. Also the value function $V(\cdot)$ is constructed using elementary methods.

2. Bellman equation. One can show heuristically that the value function $V(\cdot)$ is a solution to the Bellman equation

$$V(x) = \sup_{0 \le u \le x} \Big\{ u + \gamma \int_{-(x-u)}^{\infty} V(x-u+y)\, \nu(dy) \Big\}. \tag{2.1}$$

We show that this equation has a unique solution and an optimal strategy exists. For this purpose we observe that

$$V(x) = x + \sup_{0\le u\le x}\Big\{u - x + \gamma \int_{-(x-u)}^{\infty} V(x-u+y)\,\nu(dy)\Big\}.$$

Replacing u by $x-u$ we get

$$V(x) = x + \sup_{0\le u\le x}\Big\{-u + \gamma \int_{-u}^{\infty} V(u+y)\,\nu(dy)\Big\}. \tag{2.2}$$

Define $W(x) = V(x) - x$. Then

$$W(x) = \sup_{0\le u\le x}\Big\{-u + \gamma \int_{-u}^{\infty} W(u+y)\,\nu(dy) + \gamma \int_{-u}^{\infty} (u+y)\,\nu(dy)\Big\}. \tag{2.3}$$

We introduce an operator T as

$$(TW)(x) = \sup_{0\le u\le x}\Big\{-u + \gamma \int_{-u}^{\infty} W(u+y)\,\nu(dy) + \gamma \int_{-u}^{\infty} (u+y)\,\nu(dy)\Big\}. \tag{2.4}$$

We will show that T is a contraction:

THEOREM 2.1.

(i) *The operator T given by* (2.4) *is a contraction on* $C([0,\infty),\mathbb{R})$.
(ii) *There is a unique continuous bounded solution* $W(\cdot)$ *to* (2.3).
(iii) $W(x) \le \frac{\gamma \mathbb{E} Y^+}{1-\gamma}$.

Proof. To see that T acts on the space of bounded continuous functions, note that the continuity of $v(u)$ implies the continuity of $\sup_{0\le u\le x} v(u)$. Since

$$\begin{aligned} Tv(x) &\le \sup_{0\le u\le x}\{-u + \gamma\|v\| + \gamma\mathbb{E}(Y+u)^+\} \\ &\le \sup_{0\le u\le x}\{-u(1-\gamma) + \gamma\|v\| + \gamma\mathbb{E}|Y|\} \\ &\le \gamma\|v\| + \gamma\mathbb{E}|Y|, \end{aligned}$$

we see that $T: C([0,\infty)) \to C([0,\infty))$. Let $f,g \in C([0,\infty))$. By (2.4) we have

$$Tf(x) - Tg(x) \le \sup_{0\le u\le x}\Big\{\gamma \int_{-u}^{\infty} (f(u+y) - g(u+y))\,\nu(dy)\Big\} \le \gamma\|f-g\|.$$

Applying the Banach fixed-point theorem we complete the proof of (ii).

For (iii) first observe that by (2.4),

$$T0(x) = \sup_{0\le u\le x}\Big\{-u + \gamma \int_{-u}^{\infty} (u+y)\,\nu(dy)\Big\} = \gamma\mathbb{E}Y^+.$$

This implies that

$$T^2 0(x) \le \gamma^2 \mathbb{E} Y^+ + \gamma \mathbb{E} Y^+.$$

Proceeding by induction, we see that

$$T^n 0(x) \sum_{n=1}^{\infty} \le \gamma^n \mathbb{E} Y^+.$$

Passing to the limit and using the fact that T is a contraction we obtain

$$W(x) \le \frac{\gamma \mathbb{E} Y^+}{1-\gamma}. \ \blacksquare$$

Now we prove the existence of a maximiser:

THEOREM 2.2. *Let $W(\cdot)$ be the solution to* (2.3) *and let*

$$f(u) = -u + \gamma \int_{-u}^{\infty} W(u+y)\,\nu(dy) + \gamma \int_{-u}^{\infty} (u+y)\,\nu(dy).$$

Define

$$\underline{u}(x) = \inf\{s \ge 0 : W(x) = f(s)\},$$
$$\overline{u}(x) = \sup\{0 \le s \le x : W(x) = f(s)\}.$$

Then $\underline{u}(x)$ is a l.s.c. maximiser of the RHS in (2.3) *while $\overline{u}(x)$ is an u.s.c. maximiser.*

Proof. Fix $x_0 \ge 0$ and let x_n be any sequence that converges to x_0. Assume that $s_n = \underline{u}(x_n)$ and a subsequence of s_n converges to s. We have

$$W(x_{n_k}) = f(s_{n_k}).$$

Taking into account that $W(\cdot)$ is continuous we can pass to the limit to get

$$W(x) = \liminf_{k\to\infty} W(x_{n_k}) = \liminf_{k\to\infty} f(s_{n_k}) = f(s).$$

Moreover $\overline{u}(x) \le s$. Since this holds for any limit of a convergent subsequence of s_n we see that $\underline{u}(x)$ is l.s.c. The proof of the upper semicontinuity of $\overline{u}(x)$ is similar. ■

REMARK 2.3. Assume that $\underline{u}(x)$ (resp. $\overline{u}(x)$) is a l.s.c. (resp. u.s.c.) maximiser of (2.3). Then the function $\overline{u}^\star(x) := x - \underline{u}(x)$ (resp. $\underline{u}^\star(x) := x - \overline{u}(x)$) is an u.s.c. (resp. l.s.c.) maximiser in (2.1). Consequently, $\overline{u}^\star$ is the largest and $\underline{u}^\star$ is the smallest maximiser in (2.1).

We show that the solution of the Bellman equation is the value function of the optimal dividend problem. We start with the following lemma.

LEMMA 2.4. *If $V(\cdot)$ is a solution to the Bellman equation* (2.1) *and $u^\star$ is a measurable maximiser in* (2.1) *then*

(2.5)
$$V(x) = \mathbb{E}\Big[\sum_{k=0}^{\min(n,\tau)-1} \gamma^k u^\star(X_k) \,\Big|\, X_0 = x\Big] + \gamma^{\min(n,\tau)}\mathbb{E}[V(X_{\min(n,\tau)}) \mid X_0 = x].$$

Proof. The statement holds for $n = 1$. Indeed, letting $V(x) = 0$ for $x < 0$ we see that

$$\begin{aligned} V(x) &= \mathbb{E}[u^\star(X_0) \mid X_0 = x] + \gamma\mathbb{E}[V(X_{\min(1,\tau)}) \mid X_0 = x] \\ &= u^\star(x) + \gamma\mathbb{E}[V(X_{\min(1,\tau)}) \mid X_0 = x]. \end{aligned}$$

Furthermore,

$$X_{\min(1,\tau)} = (X_0 + Y_1 - u^\star(X_0))\chi_{\{\tau>1\}}.$$

Hence

$$\begin{aligned} V(x) &= u^\star(x) + \gamma\mathbb{E}[V((X_0 + Y_1 - u^\star(X_0))\chi_{\{\tau>1\}}) \mid X_0 = x] \\ &= u^\star(x) + \gamma \int_{-(x-u^\star(x))}^{\infty} V(x + y - u^\star(x))\,\nu(dy) \\ &= \sup_{0\le u\le x}\Big\{u + \gamma \int_{-(x-u)}^{\infty} V(x + y - u)\,\nu(dy)\Big\}. \end{aligned}$$

Assume that the statement holds for some $l = n$. We shall now prove it for $l = n + 1$. We have

$$\begin{aligned} &\mathbb{E}\Big[\sum_{k=0}^{\min(n,\tau-1)} \gamma^k u^\star(X_k) \,\Big|\, X_0 = x\Big] + \gamma^{\min(n+1,\tau)}\mathbb{E}[V(X_{\min(n+1,\tau)}) \mid X_0 = x] \\ &= \mathbb{E}\Big[\sum_{k=0}^{\min(n-1,\tau-1)} \gamma^k u^\star(X_k) \,\Big|\, X_0{=}x\Big] + \mathbb{E}[\gamma^{\min(n,\tau-1)}u^\star(X_{\min(n,\tau-1)}) \mid X_0{=}x] \\ &\quad + \gamma^{\min(n+1,\tau)}\mathbb{E}[V(X_{\min(n+1,\tau)}) \mid X_0 = x]. \end{aligned}$$

Therefore

$$\begin{aligned} V(x) &- \gamma^{\min(n,\tau-1)}\mathbb{E}[V(X_{\min(n,\tau-1)}) \mid X_0 = x] \\ &\qquad + \mathbb{E}[\gamma^{\min(n,\tau-1)}u^\star(X_{\min(n,\tau-1)}) \mid X_0 = x] \\ &\qquad + \gamma^{\min(n+1,\tau)}\mathbb{E}[V(X_{\min(n+1,\tau)}) \mid X_0 = x] \\ &= V(x) - \gamma^{\min(n,\tau-1)}\mathbb{E}[V(X_{\min(n,\tau-1)}) \mid X_0 = x] \\ &\qquad + \gamma^{\min(n,\tau-1)}\mathbb{E}[u^\star(X_{\min(n,\tau-1)}) + \gamma V(X_{\min(n+1,\tau)}) \mid X_0 = x] \end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[u^\star(X_{\min(n,\tau-1)}) + \gamma V(X_{\min(n+1,\tau)}) \mid X_0 = x] \\
&\quad = \mathbb{E}\big[\mathbb{E}[u^\star(X_{\min(n,\tau-1)}) + \gamma V(X_{\min(n+1,\tau)}) \mid X_{\min(n,\tau-1)}] \bigm| X_0 = x\big] \\
&\quad = \mathbb{E}\big[\mathbb{E}[u^\star(X_{\min(n,\tau-1)}) \\
&\qquad + \gamma V(X_{\min(n,\tau-1)} - u^\star(X_{\min(n,\tau-1)}) + Y_{n+1}) \mid X_{\min(n,\tau-1)}] \bigm| X_0 = x\big] \\
&\quad = \mathbb{E}[V(X_{\min(n,\tau-1)}) \mid X_0 = x].
\end{aligned}$$

Hence

$$\begin{aligned}
&V(x) - \gamma^{\min(n,\tau-1)}\mathbb{E}[V(X_{\min(n,\tau-1)}) \mid X_0 = x] \\
&\qquad + \gamma^{\min(n,\tau-1)}\mathbb{E}[u^\star(X_{\min(n,\tau-1)}) + \gamma V(X_{\min(n+1,\tau)}) \bigm| X_0 = x] = V(x),
\end{aligned}$$

which completes the proof. ■

THEOREM 2.5. *Let $V(x)$ be a solution to* (2.1). *Then $V(\cdot)$ coincides with the value function defined in* (1.2). *Furthermore the optimal strategy $u \in \Pi$ is a stationary strategy, i.e. there exists a measurable $u^\star$ such that*

$$u = (u^\star(X_0), u^\star(X_1), u^\star(X_2), \ldots),$$

where $u^\star$ is a measurable selector for which we have equality in (2.1).

Proof. From Theorem 2.2 there exists a measurable $u^\star$ that

$$V(x) = u^\star + \gamma \int_{-(x-u^\star)}^{\infty} V(x - u^\star + y)\,\nu(dy).$$

By Lemma 2.4 we have

$$V(x) = \mathbb{E}\Big[\sum_{k=0}^{\min(n-1,\tau-1)} \gamma^k u^\star(X_k) \Bigm| X_0 = x\Big] + \gamma^n \mathbb{E}[V(X_{\min(n,\tau-1)}) \mid X_0 = x].$$

Since $U_{\min(n,\tau-1)} \le X_{\min(n,\tau-1)}$ and the conditional distribution of $X_{\min(n,\tau-1)}$ is the same for all $n \ge 1$ under the strategy $U_{\min(n,\tau-1)} = X_{\min(n,\tau-1)}$, we see by Theorem 2.1(iii) that

$$V(X_{\min(n,\tau-1)}) \le X_{\min(n,\tau-1)} + \frac{\gamma \mathbb{E}Y^+}{1-\gamma} \quad \text{a.s.}$$

It is easily seen that when the dividend is not paid then

$$X_{\min(n,\tau-1)} = X_0 + Y_1 + Y_2 + \cdots + Y_{\min(n,\tau-1)} \quad \text{a.s.}$$

This implies that

$$\gamma^n \mathbb{E}[V(X_{\min(n,\tau-1)}) \mid X_0 = x] \le \gamma^n\Big(x + \min(n,\tau-1)\mathbb{E}Y^+ + \frac{\gamma \mathbb{E}Y^+}{1-\gamma}\Big).$$

Letting n tend to ∞ we see that

$$\gamma^n\left(x+\min(n,\tau-1)\mathbb{E}Y+\frac{\gamma\mathbb{E}Y^+}{1-\gamma}\right)\to 0.$$

Therefore

$$V(x)=\mathbb{E}\Big[\sum_{k=0}^{\tau-1}\gamma^k u^\star(X_k)\,\Big|\,X_0=x\Big]. \blacksquare$$

2.1. Properties of solution. We will prove several basic properties of the value function $V(\cdot)$ of problem (1.2).

Proposition 2.6.

(i) *The value function* $V(x)$ *is bounded:*

$$x+\frac{\mathbb{E}Y^+\gamma}{1-p_+\gamma}\le V(x)\le x+\frac{\mathbb{E}Y^+\gamma}{1-\gamma}. \tag{2.6}$$

(ii) *If* $x\ge y\ge 0$ *then*

$$V(x)-V(y)\ge x-y. \tag{2.7}$$

(iii) *For all* $x\ge 0$,

$$V(x)-u^\star(x)=V(x-u^\star(x))$$

and furthermore $\underline{u}^\star(x-u^\star(x))=0$.

(iv) *For all* $x\ge 0$,

$$\underline{u}^\star(x-\underline{u}^\star(x))=0,$$
$$\overline{u}^\star(x-\overline{u}^\star(x))=0,$$

where $\underline{u}^\star$ *and* $\overline{u}^\star$ *are respectively the l.s.c. and u.s.c. maximisers of* (2.1) (*see Remark* 2.3).

Proof. We follow the considerations from [12].

(i) The upper bound in (i) follows directly from Theorem 2.1(i). Using the strategy $U_0=x$ and $U_i=Y_i^+$ until ruin we have $P(\tau=n+1)=(1-p_+)p_+^n$. Then

$$\begin{aligned}V(x)&\ge x+\mathbb{E}\Big[\sum_{n=1}^{\tau-1}\gamma^n Y_n^+\Big]=(1-p_+)\sum_{n=1}^{\infty}p_+^n\sum_{k=1}^{n}\gamma^k\mathbb{E}[Y\,|\,Y>0]\\&=(1-p_+)\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}p_+^n\gamma^k\mathbb{E}[Y|Y>0]=\sum_{k=1}^{\infty}p_+^k\gamma^k\mathbb{E}[Y\,|\,Y>0]\\&=\frac{\mathbb{E}[Y\,|\,Y>0]p_+\gamma}{1-p_+\gamma}=\frac{\mathbb{E}[Y^+]\gamma}{1-p_+\gamma}.\end{aligned}$$

(ii) The desired inequality is equivalent to $W(x)\ge W(y)$, which is obvious by the definition (2.3).

(iii) Taking into account that $0 \le u^\star(x) \le x$ and setting $y = x - u^\star(x)$ in (2.7) we obtain

$$V(x) \ge u^\star(x) + V(x - u^\star(x)).$$

To prove the opposite inequality we put $x - u^\star(x)$ in (2.1) to find that

$$\begin{aligned} (2.8)\qquad & V(x - u^\star(x)) \\ &= \sup_{0 \le u \le x - u^\star(x)} \Big\{ u + \gamma \int_{-(x - u^\star(x) - u)}^{\infty} V(x - u^\star(x) - u + y)\, \nu(dy) \Big\}. \end{aligned}$$

Hence

$$V(x - u^\star(x)) \ge \gamma \int_{-(x - u^\star(x))}^{\infty} V(x - u^\star(x) + y)\, \nu(dy) = V(x) - u^\star(x).$$

Thus we obtain the desired equality, and the smallest u for which the supremum in (2.8) is attained is 0.

(iv) The fact that $\underline{u}^\star(x - \underline{u}^\star(x)) = 0$ follows directly from (iii). Assume now that $\overline{u}^\star(x - \overline{u}^\star(x)) = k > 0$. Then $\overline{u}^\star(x) = \overline{u}^\star(x) + k$, which contradicts the maximality of $\overline{u}^\star$. ■

Corollary 2.7. *If $P(Y > 0) = 0$ then $V(x) = x$ and the optimal strategy has the form $u_0^\star(x) = x$, $u_n^\star(x) = 0$ for $n \ge 1$ and $x \ge 0$. If $P(Y \le 0) = 1$ then*

$$V(x) = x + \frac{\gamma \mathbb{E} Y^+}{1-\gamma} \quad \textit{and} \quad u_n^\star(x) = x \quad \textit{for } n \ge 0 \textit{ and } x \ge 0.$$

2.2. Properties of the optimal strategy

Lemma 2.8. *Consider the l.s.c. maximiser $\underline{u}^\star(\cdot)$ of (2.1) and let $a = \sup\{x \ge 0 : \underline{u}^\star(x) = 0\}$. Then $a < \infty$ and*

$$\underline{u}^\star(x) = x - a \quad \textit{for } x \ge a.$$

Proof. Assume that $\underline{u}^\star(x) = 0$. Then using (2.1) and (2.6) we deduce that

$$\begin{aligned} x + \frac{\mathbb{E} Y^+ \gamma}{1 - p_+ \gamma} &\le V(x) = \gamma \int_{-x}^{\infty} V(x+y)\, \nu(dy) \\ &\le \gamma \int_{-x}^{\infty} \Big(x + y + \frac{\mathbb{E} Y^+ \gamma}{1-\gamma} \Big)\, \nu(dy) \le \gamma x + \gamma \mathbb{E} Y^+ + \frac{\mathbb{E} Y^+ \gamma^2}{1-\gamma}. \end{aligned}$$

Therefore

$$x(1-\gamma) \le \gamma \mathbb{E}Y^+ + \frac{\mathbb{E}Y^+\gamma^2}{1-\gamma} - \frac{\mathbb{E}Y^+\gamma}{1-p_+\gamma} = \frac{\mathbb{E}Y^+\gamma}{1-\gamma} - \frac{\mathbb{E}Y^+\gamma}{1-p_+\gamma}$$
$$= \frac{-\mathbb{E}Y^+\gamma^2 p_+ + \mathbb{E}Y^+\gamma^2}{(1-p_+\gamma)(1-\gamma)},$$

and consequently

$$x \le \frac{(1-p_+)\mathbb{E}[Y^+]\gamma^2}{(1-p_+\gamma)(1-\gamma)^2}.$$

Since $\underline{u}^\star$ is l.s.c. by the definition of a, we must have $\underline{u}^\star(a) = 0$. ∎

The derived inequalities mean that if the capital is large enough the dividend should be paid.

Remark 2.9. Let Y_n be i.i.d. integer-valued random variables. Suppose additionally that $n, c_0, \dots, c_n, d_1, \dots, d_n \in \mathbb{N}$ are such that $d_k - c_{k-1} \ge 2$ for all $k = 1, \dots, n$ and $0 \le c_0 < d_1 \le c_1 < d_2 \le \dots < d_n \le c_n$. The strategy

$$u^\star(x) = \begin{cases} 0 & \text{if } 0 \le x \le c_0, \\ x - c_k & \text{if } c_k < x < d_{k+1}, \\ 0 & \text{if } d_k \le x \le c_k, \\ x - c_n & \text{if } x \ge c_n, \end{cases}$$

which we will call a *barrier strategy*, is an optimal strategy. The proof can be found in [4].

The case when the random variable is not integer-valued is much more subtle. Now we restrict our attention to random variables with an analytic probability distribution function. This is not very restrictive since a lot of distributions considered in applications have this regularity (see e.g. [13]). The most important examples are the normal distribution or Erlang distributions. We have

Theorem 2.10. *Suppose that the Y_n are i.i.d. random variables with probability density function $g(x)$ which is analytic in some neighbourhood $\mathcal{U}$ of the real line. Then every optimal strategy is a barrier strategy.*

Proof. We use tools from complex analysis. For more details see e.g. [13]. From Lemma 2.8 we will deduce that for any probability distribution with finite first moment there exists the last barrier of the strategy $u^\star$. It is sufficient to show that for $x < a$ we have

$$u^\star(x) = \begin{cases} 0 & \text{if } x \le c_0, \\ x - c_k & \text{if } c_k < x < d_{k+1}, \\ 0 & \text{if } d_k \le x \le c_k. \end{cases} \qquad k = 0, 1, \dots, n.$$

If Y has a probability distribution function $g(y)$ then (2.2) takes the form

$$V(x) = x + \sup_{0\le u\le x}\Big\{-u + \gamma \int_{-u}^{\infty} V(u+y)g(y)\,dy\Big\}.$$

Define

$$f(z) = -z + \gamma \int_{0}^{\infty} V(y)g(y-z)\,dy. \tag{2.9}$$

Then $V(x) = x + \sup_{0\le z\le x} f(z)$. If f is decreasing on $\mathbb{R}$ then $u(x) = 0$ and the conclusion holds. If f is increasing, then $u(x) = x$. It is easy to observe that the conclusion holds when f has finitely many extremes. Now suppose there is no partition of $[0, a]$ into finitely many intervals in each of which f is monotone. From the continuity of the derivative we infer the existence of a sequence c_n with limit point c such that

$$f'(c_n) = 0. \tag{2.10}$$

Suppose that $z \in \mathcal{U}$. Then (2.9) is an analytic function, since the integral is uniformly convergent. Then its derivative

$$f'(z) = -1 - \gamma \int_{0}^{\infty} V(y)g'(y-z)\,dy$$

is also holomorphic. From the uniqueness theorem for analytic functions and from (2.10) we deduce that for $z \in \mathcal{U}$,

$$\int_{0}^{\infty} V(y)g'(y-z)\,dy = -\frac{1}{\gamma}.$$

Differentiating the above with respect to z we obtain

$$\int_{0}^{\infty} V(y)g''(y-z)\,dy = 0.$$

We can rewrite this as

$$V(y) \star g''(y) = 0.$$

From the Titchmarch Theorem (see for example [17] for more details) we conclude that $g''(z) = 0$ implies $g(z)$ is not a probability distribution. This contradiction guarantees the existence of finitely many roots of the derivative and the barrier form of $u^\star(x)$ as stated in the theorem. ∎

3. Exponential distribution. Now we consider the case when the claims are exponentially distributed and fix the premium d paid at the end of each period. Then Y (1.1) has probability density function

$$g(x) = \lambda e^{\lambda(x-d)}\chi_{\{x<d\}}.$$

Obviously we have

$$\mathbb{E}Y = d - \frac{1}{\lambda}.$$

Of course we should have $\lambda d - 1 \geq 0$. Then equation (2.1) can be written as

$$V(x) = x + \sup_{0\leq u\leq x}\Big\{-u + \gamma \int_0^{d+u} V(y)\lambda e^{\lambda(y-d-u)}\, dy\Big\}. \tag{3.1}$$

Define

$$f(u) = -u + \gamma \int_0^{d+u} V(y)\lambda e^{\lambda(y-d-u)}\, dy. \tag{3.2}$$

We have

Lemma 3.1. $\lim_{u\to\infty} f(u) = -\infty$.

Proof. Under the bounds (2.6) we obtain

$$\begin{aligned} f(u) &= -u + \gamma \int_0^{d+u} V(y)\lambda e^{\lambda(y-d-u)}\, dy \\ &\leq -u + \gamma \int_0^{d+u} \Big(y + \frac{\gamma \mathbb{E}Y^+}{1-\gamma}\Big)\lambda e^{\lambda(y-d-u)}\, dy. \end{aligned}$$

Integrating by parts we have

$$\begin{aligned} f(u) &\leq -u + \gamma\Big(\frac{\mathbb{E}Y^+\gamma(1-e^{\lambda(-(d+u))})}{1-\gamma} + \frac{\lambda(d+u) + e^{\lambda(-(d+u))} - 1}{\lambda}\Big) \\ &= -u(1-\gamma) + C_1 e^{-u} + C_2 \to -\infty. \end{aligned}$$

Passing to the limit is possible since the constants C_1 and C_2 do not depend of u. ∎

Lemma 3.2. *If f is non-increasing in some neighbourhood of 0, then f has a global maximum at 0.*

Proof. From (3.2), $f(u)$ is differentiable on $\mathbb{R}_+$ and

$$f'(u) = -1 + \gamma\lambda V(d+u) - \gamma\lambda^2 \int_0^{d+u} V(y)e^{\lambda(y-d-u)}\, dy.$$

From our assumption we know that $f'(0) \leq 0$. Therefore

$$\begin{aligned} f'(0) &= -1 + \gamma\lambda V(d) - \gamma\lambda^2 \int_0^{d} V(y)e^{\lambda(y-d)}\, dy \\ &= -1 + \gamma\lambda V(d) - \lambda V(0) \leq 0. \end{aligned} \tag{3.3}$$

From the property (2.7) we also know that

$$d \le V(d) - V(0).$$

Consequently,

$$0 \ge -1 + \lambda(\gamma V(d) - V(0)) \ge -1 + \lambda(\gamma(d + V(0)) - V(0)).$$

Hence

$$V(0) \ge \frac{\gamma d - \frac{1}{\lambda}}{1-\gamma}. \tag{3.4}$$

Assume now that there exists $p > 0$ where f has a global maximum. Then for x in some left neighbourhood of p we get

$$V(x) = \gamma \int_0^{d+x} V(y)\lambda e^{\lambda(y-d-x)}\, dy.$$

Differentiating the above with respect to x we obtain

$$V'(x) + \lambda V(x) = \gamma\lambda V(x+d).$$

Making the neighbourhood of p small enough, we have $x + d > p$ and

$$V'(x) + \lambda V(x) = \lambda\gamma(x + d + V(p) - p).$$

On the other hand, for $x \ge p$,

$$V(x) = x + V(p) - p.$$

Taking into account that the function $V(x)$ is differentiable at p we have $V'(p) = 1$ and

$$V(p) = \frac{\gamma d - \frac{1}{\lambda}}{1-\gamma}.$$

By (3.4) we obtain

$$V(0) \ge \frac{\gamma d - \frac{1}{\lambda}}{1-\gamma} = V(p),$$

which is a contradiction, since $V(x)$ is strictly increasing. ∎

LEMMA 3.3. *If $f'(0) \le 0$ then the optimal strategy $u^\star(x) = x - u(x)$ is trivial, i.e. $u^\star(x) = x$.*

Proof. From (3.1) we conclude that

$$V(x) = x + \sup_{0\le u\le x} f(u) = x + f(0).$$

This implies that $u(x) = 0$ for all $x \ge 0$, hence $u^\star(x) = x$.

In this way we have proven that $f'(0) \le 0$ implies that $f(0) > f(u)$ for all $u \ge 0$. ∎

LEMMA 3.4. *If $f'(0) > 0$ then there exists a point p such that f increases in $[0, p]$ and has a global maximum at p.*

Proof. From Lemma 3.2 we conclude that f is increasing on $(0,p)$ and has a local maximum at p. Therefore $f'(p)=0$. Taking into account (3.1) we have

$$V(x)=\gamma\int_0^{d+x} V(y)\lambda e^{\lambda(y-d-x)}\,dy \qquad \text{for } x\le p.$$

In particular

$$V(p)=\gamma\int_0^{d+p} V(y)\lambda e^{\lambda(y-d-p)}\,dy.$$

Differentiating (3.2) we get

$$f'(u)=-1+\gamma\lambda V(d+u)-\gamma\lambda^2\int_0^{d+u} V(y)e^{\lambda(y-d-u)}\,dy.$$

From the fact that $f'(p)=0$ we see that

$$\begin{aligned} f'(p)=0&=-1+\gamma\lambda V(d+p)-\gamma\lambda^2\int_0^{d+p} V(y)e^{\lambda(y-d-p)}\,dy\\ &=-1+\gamma\lambda V(d+p)-\lambda V(p). \end{aligned}$$

Since $V(d+p)\ge d+V(p)$ we obtain

$$V(p)\ge\frac{\gamma d-\frac{1}{\lambda}}{1-\gamma}. \tag{3.5}$$

Assume that there exists $p_1>p$ at which f has a global maximum. For any x in some left neighbourhood of p_1 we have, by (3.1),

$$V(x)=\gamma\int_0^{d+x} V(y)\lambda e^{\lambda(y-d-x)}\,dy.$$

Differentiating the above, we obtain

$$V'(x)+\lambda V(x)=\lambda\gamma V(x+d).$$

Since $V'(p_1)=1$ and $V(x+d)=V(p_1)+x+d-p_1$ for $x+d>p_1$ we obtain a contradiction, which completes the proof of the theorem. ∎

COROLLARY 3.5. *If $f'(0)>0$ then the optimal strategy is a barrier strategy.*

Proof. In fact, Theorem 3.4 implies that f is increasing on $[0,p]$ and has a global maximum at p. Hence

$$u(x)=\begin{cases} x & \text{for } x\in(0,p],\\ p & \text{for } x>p.\end{cases}$$

Recalling that $u^\star(x) = x - u(x)$ we get

$$u^\star(x) = \begin{cases} 0 & \text{for } x \in (0,p], \\ x-p & \text{for } x > p. \end{cases} \blacksquare$$

COROLLARY 3.6. *If the expected value of Y is non-positive, then the trivial strategy is an optimal strategy.*

Proof. Suppose that this is not true. Then there exists a point $p > 0$ at which

$$V(p) = \frac{\gamma d - \frac{1}{\lambda}}{1-\gamma}.$$

But then

$$V(p) < \frac{d - \frac{1}{\lambda}}{1-\gamma} = \frac{\mathbb{E}Y}{1-\gamma} \le 0.$$

This contradicts the fact that $V(x)$ is positive. ∎

The results we have derived so far involve conditions on $f'(0)$. It is not clear, however, whether or not the condition $f'(0) > 0$ is ever fulfilled. We have

LEMMA 3.7. *If*

$$\gamma > \frac{1}{(\lambda d - 1)e^{-\lambda d} + 1}$$

then the optimal strategy is a barrier strategy.

Proof. It suffices to show that $f'(0) > 0$ for sufficiently large γ. As in (3.3), we have

$$f'(0) = -1 + \gamma\lambda V(d) - \lambda V(0) = -1 + \lambda\gamma(V(d) - V(0)) - \lambda(1-\gamma)V(0). \tag{3.6}$$

To estimate the above expression from below, we first estimate $V(0)$. From (3.1) we have

$$V(0) = \gamma\lambda \int_0^d V(y)e^{\lambda(y-d)}\,dy \le \gamma V(d)(1 - e^{-\lambda d}). \tag{3.7}$$

Applying this inequality to (3.6) gives

$$\begin{aligned} f'(0) &\ge -1 + \lambda\gamma(V(d) - V(0)) - \lambda(1-\gamma)\gamma V(d)(1 - e^{-\lambda d}) \\ &= -1 + \lambda\gamma(V(d) - V(0)) - (1-\gamma)\lambda\gamma(V(d) - V(0))(1 - e^{-\lambda d}) \\ &\quad - (1-\gamma)\lambda\gamma V(0)(1 - e^{-\lambda d}). \end{aligned}$$

Iterating (3.7), we obtain

$$\begin{aligned} f'(0) \ge -1 + \gamma\lambda(V(d) - V(0)) - (1-\gamma)\lambda(V(d) - V(0)) \sum_{k=1}^{n} \gamma^k (1 - e^{-\lambda d})^k \\ - (1-\gamma)\gamma^n V(0)(1 - e^{-\lambda d})^n. \end{aligned}$$

Letting $n \to \infty$ yields

$$\begin{aligned} f'(0) &\ge -1 + \gamma\lambda(V(d) - V(0)) - (1-\gamma)\lambda(V(d) - V(0)) \sum_{k=1}^{\infty} \gamma^k (1 - e^{-\lambda d})^k \\ &= -1 + \gamma\lambda(V(d) - V(0)) - (1-\gamma)\lambda(V(d) - V(0)) \frac{\gamma(1-e^{-\lambda d})}{1-\gamma(1-e^{-\lambda d})} \\ &= -1 + (V(d) - V(0))\gamma\lambda\left(1 - \frac{(1-\gamma)(1-e^{-\lambda d})}{1-\gamma(1-e^{-\lambda d})}\right) \\ &= -1 + (V(d) - V(0))\gamma\lambda \frac{e^{-\lambda d}}{1-\gamma(1-e^{-\lambda d})} \\ &\ge -1 + d\gamma\lambda \frac{e^{-\lambda d}}{1-\gamma(1-e^{-\lambda d})}. \end{aligned}$$

After simple algebraic transformations, we obtain

$$\gamma > \frac{1}{(\lambda d - 1)e^{-\lambda d} + 1} \;\Rightarrow\; f'(0) > 0.$$

Recalling that $\mathbb{E}Y > 0 \Leftrightarrow \lambda d - 1 > 0$ we obtain

$$\frac{1}{(\lambda d - 1)e^{-\lambda d} + 1} = a < 1.$$

From this we finally obtain $f'(0) > 0$. ∎

Taking into account these results we can state our main theorem.

THEOREM 3.8. *The optimal strategy for the optimal dividend problem with constant premium and exponentially distributed claims is a barrier strategy, i.e. there exists $p > 0$ such that for all $x \ge 0$ we have*

$$u^\star(x) = \max(x - p, 0),$$

and $u^\star(x)$ is a maximiser of (2.1).

An interesting issue appears when we are dealing with the estimation of the optimal barrier p. The following result holds:

PROPOSITION 3.9. *Let $V(x)$ be a solution to* (3.1) *and p an optimal barrier. Then*

$$\frac{\gamma - 1 + \gamma e^{-\lambda d}}{\lambda(1-\gamma)} \le p \le \frac{\gamma d - \frac{1}{\lambda}}{1-\gamma} - \frac{\gamma(\lambda d - 1 + e^{-\lambda d})}{\lambda(1-\gamma(1-e^{-\lambda d}))}.$$

Proof. Using the inequality (3.5) we obtain

$$\frac{\gamma d - \frac{1}{\lambda}}{1-\gamma} = V(p) \le p + \frac{\gamma(\lambda d - 1 + e^{-\lambda d})}{\lambda(1-\gamma)}.$$

This implies that

$$\frac{\gamma - 1 + \gamma e^{-\lambda d}}{\lambda(1-\gamma)} \le p.$$

To prove the opposite inequality we use the lower bound in (2.6) to obtain

$$\frac{\gamma d - \frac{1}{\lambda}}{1-\gamma} = V(p) \ge p + \frac{\gamma(\lambda d - 1 + e^{-\lambda d})}{\lambda(1 - \gamma(1 - e^{-\lambda d}))}.$$

We finally conclude that

$$p \le \frac{\gamma d - \frac{1}{\lambda}}{1-\gamma} - \frac{\gamma(\lambda d - 1 + e^{-\lambda d})}{\lambda(1 - \gamma(1 - e^{-\lambda d}))}. \blacksquare$$

Now we briefly sketch the procedure of finding a solution of the optimal dividend problem with exponential claims. Without loss of generality, we can assume that $f'(0) > 0$ where $f(u)$ is given by (3.2). Recalling the proof of Lemma 3.4 we obtain

$$V'(x) + \lambda V(x) = \lambda\gamma V(x+d) \quad \text{for } x < p.$$

We can use Proposition 3.9 to determine the value of p.

First assume that $p < d$. Then

$$V'(x) + \lambda V(x) = \lambda\gamma(V(p) + x + d - p), \quad x < p.$$

This is a non-homogeneous linear equation which can be easily solved by standard methods. As a result, we have

$$V(x) = \frac{\gamma\left(-p\lambda + \lambda V(p) + d\lambda + \lambda x - 1\right)}{\lambda} + Ce^{-\lambda x}.$$

For $x > p$ we obviously have $V(x) = V(p) + x - p$. Taking into account the continuity and differentiability of $V(x)$ at p we obtain

$$C = \frac{(\gamma - 1)e^{p\lambda}}{\lambda}, \quad V(p) = \frac{\gamma d\lambda - 1}{(1-\gamma)\lambda}.$$

Hence

(3.8)

$$V(x) = \begin{cases} \dfrac{\gamma(\gamma + \lambda(d + (\gamma-1)p) - 2)}{(1-\gamma)\lambda} + x\gamma - \left(\dfrac{1-\gamma}{\lambda}\right)e^{\lambda p - \lambda x} & \text{for } x \in (0,p], \\ x - p + \dfrac{\gamma d\lambda - 1}{(1-\gamma)\lambda} & \text{for } x > p. \end{cases}$$

In order to determine the threshold p, we rewrite (3.1) for $x < p$:

$$V(x) = \gamma \int_0^{d+x} V(y)\lambda e^{\lambda(y-d-x)}\,dy$$
$$= \gamma \int_0^{p} V(y)\lambda e^{\lambda(y-d-x)}\,dy + \gamma \int_p^{d+x} V(y)\lambda e^{\lambda(y-d-x)}\,dy.$$

Calculating these integrals and recalling (3.8) we find that the right hand side above is equal to

$$\gamma x + \frac{\gamma(\gamma + \lambda(d + (\gamma-1)p) - 2)}{(1-\gamma)\lambda}$$
$$+ e^{\lambda(p-x)}\left(\frac{(\gamma-1)\gamma(\lambda p - 2)}{\lambda}e^{-\lambda d} + e^{-\lambda(p+d)}\frac{\gamma^2(d\lambda + \gamma(\lambda p + 2) - \lambda p - 3)}{(\gamma-1)\lambda}\right).$$

The coefficient of $e^{\lambda(p-x)}$ in the above formula should be equal to the coefficient in the same expression in (3.8). Continuing these calculations we finally obtain

$$\frac{(\gamma-1)^2 + \gamma(\gamma-1)^2\left(-e^{-d\lambda}\right)(\lambda p - 2)}{\gamma^2(2\gamma + \lambda(d + (\gamma-1)p) - 3)} = e^{\lambda(-d-p)}.$$

Let us now consider the case when $p > d$. Then

$$V'(x) + \lambda V(x) = \lambda\gamma V(x+d). \tag{3.9}$$

For $x \in (p-d, p)$ we have

$$V'(x) + \lambda V(x) = \lambda\gamma V(x+d) = \lambda\gamma(V(p) + x + d - p). \tag{3.10}$$

Finally, for $x > p - d$ we see that

$$V(x) = V(p) + x - p.$$

Solving the first equation (3.10) we conclude that for $x \in (p-d, p)$,

$$V(x) = \frac{\gamma(-p\lambda + \lambda V(p) + d\lambda + \lambda x - 1)}{\lambda} + Ce^{-\lambda x}.$$

Knowing the solution in this interval, we will focus on (3.9). Note that solving it requires knowledge of the solution $V(x)$ for $x \in (d, p)$. Assume for simplicity that $p - d < d$. Then (3.9) reads

$$V'(x) + \lambda V(x) = \lambda\gamma\left[\frac{\gamma\left(-p\lambda + \lambda V(p) + d\lambda + \lambda(x+d) - 1\right)}{\lambda} + Ce^{-\lambda(x+d)}\right].$$

For $x \in [0, p-d)$ we have

$$V(x) = -\frac{2\gamma^2}{\lambda} + C\gamma\lambda x e^{\lambda(-(d+x))} + De^{\lambda(-x)} + 2\gamma^2 d - \gamma^2 p + \gamma^2 V(p) + \gamma^2 x.$$

In order to find the constants C, D and $V(p)$ we use the differentiability of $V(x)$ at p and $d - p$. Then, in order to find p we proceed as in the previous case. The procedure of finding $p > 2d$ is as follows. We have

$$V'(x) + \lambda V(x) = \begin{cases} \lambda\gamma V(x+d) & \text{for } x \in (0, p-2d], \\ \lambda\gamma V(x+d) & \text{for } x \in (p-2d, p-d], \\ \lambda\gamma(x+d+V(p)-p) & \text{for } x \in (p-d, p]. \end{cases}$$

We solve the equation in $(p-d,p]$, and then in $(p-2d,pd]$. Thus the equation is solved in $(0, p-2d]$. In the general case, the procedure is as follows:

$$V'(x)+\lambda V(x)= \begin{cases} \lambda\gamma V(x+d) & \text{for } x \in (0, p-nd], \\ \lambda\gamma V(x+d) & \text{for } x \in (p-id, p-(i-1)d],\ i = n, \dots, 1, \\ \lambda\gamma(x+d+V(p)-p) & \text{for } x \in (p-d, p], \end{cases}$$

Obviously the procedure terminates since $p < \infty$ (Lemma 2.8).

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Dariusz Socha
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-662 Warszawa, Poland
E-mail: d.socha@mini.pw.edu.pl

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