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ORTHOGONAL SERIES ESTIMATION OF BAND-LIMITED REGRESSION FUNCTIONS

Abstract. The problem of nonparametric function fitting using the complete orthogonal system of Whittaker cardinal functions s_k, $k = 0, \pm 1, \ldots$, for the observation model $y_j = f(u_j) + \eta_j$, $j = 1, \ldots, n$, is considered, where $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ for $\Omega > 0$ is a band-limited function, u_j are independent random variables uniformly distributed in the observation interval $[-T, T]$, η_j are uncorrelated or correlated random variables with zero mean value and finite variance, independent of the observation points. Conditions for convergence and convergence rates of the integrated mean-square error $E\|f - \hat{f}_n\|^2$ and the pointwise mean-square error $E(f(x) - \hat{f}_n(x))^2$ of the estimator $\hat{f}_n(x) = \sum_{k=-N(n)}^{N(n)} \hat{c}_k s_k(x)$ with coefficients $\hat{c}_k$, $k = -N(n), \ldots, N(n)$, obtained by the Monte Carlo method are studied.

1. Introduction. *Band-limited functions*, i.e. functions whose Fourier transform is zero outside a bounded interval $[-\Omega, \Omega]$, where $\Omega > 0$, occur frequently in signal processing, communication and information theory [3], [17]. The class of such functions will be denoted by $BL(\Omega)$. The problem of recovering band-limited functions from their observations in the presence of random errors was the subject of several recent works [4], [6], [8]–[13], [18] in which a fixed design observation model was considered. In [7] quasi-random sampling was applied for recovery of band-limited functions under noise.

Two of the above mentioned works [6], [7] deal with orthogonal series estimators. The aim of the present article is to complete the results obtained in these two publications with some results for the case of a random design observation model, where the observation points are uniformly distributed in the observation interval $[-T, T]$, $T > 0$.

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Hence, suppose the observations y_j, $j = 1, \dots, n$, follow the model

$$y_j = f(u_j) + \eta_j,$$

where $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ is an unknown real-valued function, u_j, $j = 1, \dots, n$, are realizations of independent and identically distributed random variables with uniform distribution ($u_j \sim U([-T, T])$), and η_j, $j = 1, \dots, n$, are realizations of uncorrelated random variables with zero mean $E_\eta \eta_j = 0$ and finite variance $E_\eta \eta_j^2 = \sigma_\eta^2$, which are independent of the observation points. Observe that we can generate realizations of independent random variables v_j, $j = 1, \dots, n$, with uniform distribution in $[-1, 1]$, i.e. $v_j \sim U([-1, 1])$, and then multiply them by $T > 0$ to obtain observation points in our model with observation interval $[-T, T]$.

Our estimator will be constructed from cardinal functions used for reconstruction of square integrable band-limited functions as cardinal series in the famous Whittaker–Shannon interpolation scheme [2], [3], [5]:

$$f(t) = \sum_{k=-\infty}^{\infty} f(k\tau) s_k(t), \tag{1}$$

where $\Omega \le \pi/\tau$. The cardinal functions are given by the formula (see [2])

$$s_k(t) = \operatorname{sinc}\left(\frac{\pi}{\tau}(t - k\tau)\right), \qquad k = 0, \pm 1, \pm 2, \dots, \tag{2}$$

where $\operatorname{sinc}(t) = \sin(t)/t$ for $t \neq 0$ and 1 for $t = 0$, and the series in (1) converges in $L^2(\mathbb{R})$ as well as uniformly on any compact interval [2]. The functions s_k, $k = 0, \pm 1, \pm 2, \dots$, form a complete orthogonal system in the space of square integrable and band-limited functions $L^2(\mathbb{R}) \cap BL(\Omega)$ provided that $\Omega \le \pi/\tau$. This is a simple consequence of the fact that they are inverse Fourier transforms of the functions $\tau \exp(il\tau\omega)$, $\omega \in [-\pi/\tau, \pi/\tau]$, $l = 0, \pm 1, \pm 2, \dots$, which form a complete orthogonal system in the space $L^2([-\pi/\tau, \pi/\tau])$ in the spectral domain. Thus, using the inverse Fourier transform formula, we also immediately have

$$|s_k(t)| \le 1 \quad \text{for } t \in \mathbb{R}, \qquad \|s_k\|^2 = \tau, \quad k = 0, \pm 1, \pm 2, \dots. \tag{3}$$

Let F denote the Fourier transform of f. Then by the Plancherel identity we obtain the following formula for the orthogonal series expansion coefficients of the regression function:

$$c_k = \frac{1}{\tau} \int_{\mathbb{R}} f(t) s_k(t)\, dt = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) \exp(ik\tau\omega)\, d\omega = f(k\tau) \tag{4}$$

for $k = 0, \pm 1, \pm 2, \dots$.

We shall investigate conditions for convergence of the integrated mean-square error $E\|\hat{f}_n - f\|^2$ and pointwise mean-square error $E(\hat{f}_n(t) - f(t))^2$

of the projection estimator

$$\hat{f}_n(t) = \sum_{|k|\le N} \hat{c}_k s_k(t), \quad \text{where} \quad \hat{c}_k = \frac{2T}{n\tau}\sum_{j=1}^{n} y_j s_k(u_j). \tag{5}$$

Observe that the estimator (5) is itself a band-limited function, i.e. $\hat{f}_n \in L^2(\mathbb{R}) \cap BL(\pi/\tau)$.

Obviously we have the following formulae for the bias and variance of the expansion coefficient estimators $\hat{c}_k$, $k = 0, \pm 1, \pm 2, \ldots$:

$$\begin{aligned}
E_u E_\eta \hat{c}_k &= \frac{1}{\tau}\int_{-T}^{T} f(t)s_k(t)\,dt, \\
B(\hat{c}_k) &= c_k - E_u E_\eta \hat{c}_k = \frac{1}{\tau}\int_{|t|>T} f(t)s_k(t)\,dt, \\
\operatorname{Var}(\hat{c}_k) &= E_u E_\eta(\hat{c}_k - E_u E_\eta \hat{c}_k)^2 \\
&= E_u\Big(\frac{2T}{n}\sum_{j=1}^{n}\Big[\frac{1}{\tau}f(u_j)s_k(u_j) - \frac{1}{2T}E_u E_\eta \hat{c}_k\Big]\Big)^2 \\
&\quad + \frac{(2T)^2\sigma_\eta^2}{(n\tau)^2}E_u\sum_{j=1}^{n} s_k^2(u_j) \\
&= \frac{2T}{n}\Big(\frac{1}{\tau^2}\int_{-T}^{T} f^2(t)s_k^2(t)\,dt - \frac{1}{2T}(E_u E_\eta \hat{c}_k)^2\Big) \\
&\quad + \frac{2T\sigma_\eta^2}{n\tau^2}\int_{-T}^{T} s_k^2(t)\,dt,
\end{aligned} \tag{6}$$

which implies

$$\operatorname{Var}(\hat{c}_k) \le \frac{2T}{n\tau^2}\Big(\int_{-T}^{T} f^2(t)s_k^2(t)\,dt + \sigma_\eta^2\int_{-T}^{T} s_k^2(t)\,dt\Big) \tag{7}$$

for $k = 0, \pm 1, \pm 2, \ldots$.

In the following lemma also an upper bound for the bias term is derived.

LEMMA 1.1. *If $f \in L^2(\mathbb{R})$ and $N < T/(2\tau)$, then for $k = 0, \pm 1, \ldots, \pm N$,*

$$B^2(\hat{c}_k) = \Big|\frac{1}{\tau}\int_{|t|>T} f(t)s_k(t)\,dt\Big|^2 \le \frac{5}{\pi^2 T}\int_{|t|>T} f^2(t)\,dt.$$

Proof. According to (2) the inequality

$$s_k^2(t) \le \frac{\tau^2}{\pi^2 t^2(1 - k\tau/t)^2}$$

holds for $t \neq 0$, and we have $k\tau/T \leq N\tau/T < 1/2$ for $k = 1, \ldots, N$, which gives after integration

$$\int\limits_{t>T} s_k^2(t)\,dt \leq \frac{4\tau^2}{\pi^2} \int\limits_{t>T} t^{-2}\,dt = \frac{4\tau^2}{\pi^2 T},$$

and similarly for $k = 0, 1, \ldots, N$,

$$\int\limits_{t<-T} s_k^2(t)\,dt \leq \frac{\tau^2}{\pi^2} \int\limits_{t<-T} t^{-2}\,dt = \frac{\tau^2}{\pi^2 T}.$$

Analogously, for $k = -N, \ldots, -1, 0$,

$$\int\limits_{t>T} s_k^2(t)\,dt \leq \frac{\tau^2}{\pi^2} \int\limits_{t>T} t^{-2}\,dt = \frac{\tau^2}{\pi^2 T},$$

and since $|k|\tau/T \leq N\tau/T < 1/2$ for $k = -N, \ldots, -2, -1$,

$$\int\limits_{t<-T} s_k^2(t)\,dt \leq \frac{4\tau^2}{\pi^2} \int\limits_{t<-T} t^{-2}\,dt = \frac{4\tau^2}{\pi^2 T}.$$

Applying the Cauchy–Schwarz inequality together with the above upper bounds on integrals of s_k^2 we obtain for $k = 0, \pm 1, \pm 2, \ldots$,

$$\Bigg|\frac{1}{\tau} \int\limits_{|t|>T} f(t) s_k(t)\,dt\Bigg|^2 \leq \frac{1}{\tau^2} \int\limits_{|t|>T} f^2(t)\,dt \int\limits_{|t|>T} s_k^2(t)\,dt \leq \frac{5}{\pi^2 T} \int\limits_{|t|>T} f^2(t)\,dt,$$

which completes the proof of the lemma. ∎

Another property of the system of cardinal functions is given in the next lemma.

LEMMA 1.2. *The complete system of orthogonal functions s_k, $k = 0, \pm 1, \pm 2, \ldots$, satisfies the identity*

$$\sum_{k=-\infty}^{\infty} s_k^2(x) = 1 \quad \textit{for } x \in \mathbb{R}.$$

Proof. For any $x \in \mathbb{R}$ the function $u(t) = \operatorname{sinc}\big(\frac{\pi}{\tau}(t - x)\big)$ is a translation of s_0 so that $u \in L^2(\mathbb{R}) \cap BL(\pi/\tau)$. Consequently, the Parseval identity applied to u yields, according to (3) and (4),

$$\tau = \|s_0\|^2 = \|u\|^2 = \tau \sum_{k=-\infty}^{\infty} \operatorname{sinc}^2\bigg(\frac{\pi}{\tau}(x - k\tau)\bigg) = \tau \sum_{k=-\infty}^{\infty} s_k^2(x),$$

which proves the assertion. ∎

In Section 2 asymptotic properties of the integrated mean-square error of the above estimator are examined, which is a global accuracy measure, while in Section 3 we investigate local properties of our estimator using pointwise mean-square error. Section 4 is devoted to extension of the results from the previous two sections to the case of an observation model with short and long range dependent observation errors. Strong convergence of the integrated error of the estimator is studied in Section 5.

2. Convergence of integrated mean-square error. The fact that the functions s_k, $k = 0, \pm 1, \pm 2, \ldots$, form a complete orthogonal system in $L^2(\mathbb{R}) \cap BL(\Omega)$ implies, in view of (3),

$$\begin{aligned} Q(\hat{f}_n) = E_u E_\eta \int_{\mathbb{R}} (f - \hat{f}_n)^2 &= \tau \sum_{|k| \le N} E_u E_\eta |\hat{c}_k - c_k|^2 + \tau \sum_{|k| > N} c_k^2 \\ &= \tau \sum_{|k| \le N} [\mathrm{Var}(\hat{c}_k) + B^2(\hat{c}_k)] + \tau \sum_{|k| > N} c_k^2. \end{aligned}$$

The above equality together with bounds for variance (7) and squared bias (Lemma 1.1) of the coefficient estimators $\hat{c}_k$ allow us to estimate the integrated mean-square error of the estimator $\hat{f}_n$. Namely, using Lemma 1.2 we easily obtain

$$Q(\hat{f}_n) \le \frac{2T}{n\tau} (\|f\|^2 + 2T\sigma_\eta^2) + (2N+1) \frac{5\tau}{\pi^2 T} \int_{|t| > T} f^2(t)\, dt + \tau \sum_{|k| > N} c_k^2$$

for $N < T/(2\tau)$. Putting $N = [T/(2\tau)]$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$, makes the last term on the right hand side the smallest possible. In view of the above upper bound on the IMSE, we can formulate the following theorem.

THEOREM 2.1. *If the sequence of positive real numbers $T(n)$, $n = 1, 2, \ldots$, satisfies $\lim_{n\to\infty} T(n) = \infty$ and $\lim_{n\to\infty} T(n)^2/n = 0$, and if $N(n) = [T(n)/(2\tau)]$, where $\tau \le \pi/\Omega$, then the projection estimator $\hat{f}_n$ of the regression function $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ is consistent in the sense of integrated mean-square error, i.e.* $\lim_{n\to\infty} E_u E_\eta \|f - \hat{f}_n\|^2 = 0$.

In order to obtain the convergence rate of integrated mean-square error we need the following lemma.

LEMMA 2.1. *Let $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ have Fourier transform $F \in L^2([-\Omega, \Omega])$ of bounded variation on $[-\Omega, \Omega]$. Then*

$$\int_{|t| > T} f^2(t)\, dt \le \frac{C^2(F)}{2\pi^2 T} \quad \text{for } T > 0 \quad \text{and} \quad \tau \sum_{|k| > N} c_k^2 \le \frac{C^2(F)}{2\pi^2 \tau N}$$

for $N = 1, 2, \ldots$, and $C(F) = V(F) + 2\sup_{\omega \in [-\Omega, \Omega]} |F(\omega)|$, where $V(F)$ denotes the total variation of F on $[-\Omega, \Omega]$.

Proof. Since $f \in BL(\Omega)$ we can write $f(t) = (2\pi)^{-1} \int_{-\Omega}^{\Omega} F(\omega)\exp(it\omega)\,d\omega$, and applying integration by parts [16] gives

$$f(t) = \frac{1}{2\pi it}[F(\Omega)\exp(i\Omega t) - F(-\Omega)\exp(-i\Omega t)] - \frac{1}{2\pi it}\int_{-\Omega}^{\Omega} \exp(it\omega)\,dF(\omega)$$

for $t \neq 0$, which yields $|f(t)| \le C(F)/(2\pi|t|)$ for $t \neq 0$ [16], and consequently implies the desired bound on the integral of $|f(t)|^2$.

According to (4), we have $c_k = f(k\tau)$, and therefore

$$\tau \sum_{|k|>N} c_k^2 \le 2\frac{C^2(F)}{(2\pi)^2\tau} \sum_{k=N+1}^{\infty} \frac{1}{k^2} \le 2\frac{C^2(F)}{(2\pi)^2\tau} \sum_{k=N+1}^{\infty} \frac{1}{k(k-1)} = \frac{C^2(F)}{2\pi^2\tau N}$$

for $N = 1, 2, \ldots$, and the proof is complete. ∎

Under the assumptions of Lemma 2.1 concerning the regression function our earlier upper bound for the IMSE takes the form

$$Q(\hat{f}_n) \le \frac{2T}{n\tau}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{2N+1}{T^2}\,\frac{5\tau C^2(F)}{2\pi^4} + \frac{1}{N}\,\frac{C^2(F)}{2\pi^2\tau},$$

where $0 < N < T/(2\tau)$. Assuming that $T \ge 4\tau$ and putting $N = [T/(2\tau)]$ we further obtain

$$Q(\hat{f}_n) \le \frac{2T}{n\tau}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{1}{T}\left[\frac{25C^2(F)}{8\pi^4} + \frac{2C^2(F)}{\pi^2}\right]. \tag{8}$$

Note also that to obtain the smallest bound in (8) we must use the largest possible τ, i.e. $\tau = \pi/\Omega$. Moreover, the following corollary is immediate.

COROLLARY 2.1. *If the regression function $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ satisfies the assumptions of Lemma* 2.1 *and $T(n) = n^{1/3}$, $N(n) = [T(n)/(2\tau)]$, $n = 1, 2, \ldots$, where $\tau \le \pi/\Omega$, then*

$$E_u E_\eta \|f - \hat{f}_n\|^2 = O(n^{-1/3}).$$

3. Pointwise mean-square consistency of the estimator. In this section we derive sufficient conditions for pointwise mean-square consistency of the projection estimator considered and examine its pointwise mean-square error convergence rate.

Since the cardinal series of the regression function f converges to $f(x)$ for $x \in \mathbb{R}$ [2], we immediately have

$$E_u E_\eta (f(x) - \hat{f}_n(x))^2 = E_u E_\eta \Big(\sum_{k=-N}^{N} (c_k - \hat{c}_k) s_k(x)\Big)^2 + r_N^2(x) + 2r_N(x) \sum_{k=-N}^{N} (c_k - E_u E_\eta \hat{c}_k) s_k(x),$$

where $r_N(x) = \sum_{|k|>N} c_k s_k(x)$. From the Cauchy–Schwarz inequality it further follows that

$$E_u E_\eta (f(x) - \hat{f}_n(x))^2 \le \sum_{k=-N}^{N} [\mathrm{Var}(\hat{c}_k) + B^2(\hat{c}_k)] \sum_{k=-N}^{N} s_k^2(x) + r_N^2(x) + 2|r_N(x)| \Big(\sum_{k=-N}^{N} B^2(\hat{c}_k)\Big)^{1/2} \Big(\sum_{k=-N}^{N} s_k^2(x)\Big)^{1/2},$$

and using again our bounds on variance (7) and squared bias (Lemma 1.1) of the coefficient estimators $\hat{c}_k$, together with the inequality $\sum_{k=-N}^{N} s_k^2(x) \le 1$ for $x \in \mathbb{R}$ (Lemma 1.2), we finally have

$$E_u E_\eta (f(x) - \hat{f}_n(x))^2 \le \frac{2T}{n\tau^2}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{(2N+1)5}{T\pi^2} \int_{|t|>T} f^2(t)\,dt + |r_N(x)| \frac{(2N+1)^{1/2} 20^{1/2}}{T^{1/2}\pi} \Big(\int_{|t|>T} f^2(t)\,dt\Big)^{1/2} + r_N^2(x) \tag{9}$$

for $N < T/(2\tau)$, $x \in \mathbb{R}$. For the regression function $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ its cardinal series converges uniformly on any compact interval $I \subset \mathbb{R}$ [3], so that $\lim_{n\to\infty} r_{N(n)}(x) = 0$ uniformly for $x \in I$, if only $\lim_{n\to\infty} N(n) = \infty$. Hence, the above bound on the pointwise mean-square error of our estimator implies the following theorem.

THEOREM 3.1. *If the sequence of positive real numbers $T(n)$, $n = 1, 2, \ldots$, satisfies $\lim_{n\to\infty} T(n) = \infty$ and $\lim_{n\to\infty} T(n)^2/n = 0$, and if $N(n) = [T(n)/(2\tau)]$, where $\tau \le \pi/\Omega$, then the projection estimator $\hat{f}_n$ of the regression function $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ is uniformly consistent in the sense of pointwise mean-square error in any compact interval $I \subset \mathbb{R}$, i.e.*

$$\lim_{n\to\infty} E_u E_\eta (f(x) - \hat{f}_n(x))^2 = 0 \quad \textit{uniformly for } x \in I.$$

The same assumptions on the regression function that were needed to obtain the IMSE convergence rate are also sufficient to determine the convergence rate of pointwise mean-square error. Indeed, we only have to estimate $|r_N(x)|$, which is the subject of the following lemma.

LEMMA 3.1. *Under the assumptions of Lemma 2.1 on f,*

$$|r_N(x)| = \Big|\sum_{|k|>N} c_k s_k(x)\Big| \le \frac{C(F)}{6^{1/2}\pi\tau N^{1/2}}$$

for $|x| \le N\tau$, $N = 1, 2, \ldots$.

Proof. By the Cauchy–Schwarz inequality for square summable series, the second assertion of Lemma 2.1, and definition (2),

$$|r_N(x)| = \Big|\sum_{|k|>N} c_k s_k(x)\Big| \le \Big(\sum_{|k|>N} c_k^2\Big)^{1/2}\Big(\sum_{|k|>N} s_k^2(x)\Big)^{1/2}$$
$$\le \left(\frac{C^2(F)}{2\pi^2\tau^2 N}\right)^{1/2}\left(\frac{\tau^2}{\pi^2}\sum_{|k|>N}\frac{1}{(k\tau-x)^2}\right)^{1/2}.$$

If $N > 0$ and $|x| \le N\tau$ we also have $(N+l)\tau - x \ge (N+l)\tau - N\tau = l\tau$, $l = 1, 2, \ldots$, and consequently

$$\frac{\tau^2}{\pi^2}\sum_{|k|>N}\frac{1}{(k\tau-x)^2} \le \frac{2\tau^2}{\pi^2}\sum_{l=1}^{\infty}\frac{1}{l^2\tau^2} = \frac{1}{3},$$

since $\sum_{l=1}^{\infty} 1/l^2 = \pi^2/6$. Hence, the proof is complete in view of the earlier inequality. ■

Now, taking into account (9) it is easy to see that for regression functions satisfying the assumptions of Lemma 2.1 we have the following bound for pointwise mean-square error:

$$E_u E_\eta (f(x) - \hat{f}_n(x))^2 \le \frac{2T}{n\tau^2}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{(2N+1)5C^2(F)}{T^2 2\pi^4}$$
$$+ \frac{(2N+1)^{1/2}5^{1/2}C^2(F)}{TN^{1/2}3^{1/2}\pi^3\tau} + \frac{C^2(F)}{6\pi^2\tau^2 N},$$

which is valid for $T \ge 2\tau$, $N = [T/(2\tau)]$, and $|x| \le N\tau$. It is again visible that to obtain the smallest bound we must use $\tau = \pi/\Omega$. Inserting in the last inequality $N = [T/(2\tau)]$ with $T \ge 4\tau$ yields

$$E_u E_\eta (f(x) - \hat{f}_n(x))^2 \le A\frac{T^2}{n} + D\frac{T}{n} + G\frac{1}{T}$$

for $|x| \le N\tau$, where $A, D, G > 0$ are suitably chosen constants. In order to ensure convergence to zero of the first term on the right hand side we must put $T(n) = n^\alpha$, where $0 < \alpha < 1/2$. Thus, the following corollary is valid.

COROLLARY 3.1. *If $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ satisfies the assumptions of Lemma 2.1 and $T(n) = n^{1/3}$, $N(n) = [T(n)/(2\tau)]$, $n = 1, 2, \ldots$, where $\tau \le \pi/\Omega$, then*

$$\sup_{|x|\le N(n)\tau} E_u E_\eta (f(x) - \hat{f}_n(x))^2 = O(n^{-1/3}).$$

4. Estimation under dependent observation errors. The results of the previous two sections can be readily extended to the case of stationary dependent observation errors η_j, $j = 1, \ldots, n$, with short and long range dependence. In that case we assume that $E_\eta \eta_j = 0$, $E_\eta \eta_j^2 = \sigma_\eta^2$, $j = 1, 2, \dots,$ and $E_\eta \eta_j \eta_{j+l} = r(l)$, $l = 1, 2, \dots$.

Non-zero covariances of the observation errors influence only the variances of the expansion coefficient estimators $\hat{c}_k$, $k = 0, \pm 1, \pm 2, \dots,$ and derivation analogous to the one of (7) shows that

$$\operatorname{Var}(\hat{c}_k) \le \frac{2T}{n\tau^2}\Big(\int_{-T}^{T} f^2(t) s_k^2(t)\,dt + \sigma_\eta^2 \int_{-T}^{T} s_k^2(t)\,dt\Big) + \frac{(2T)^2}{n^2\tau^2}\Big(\frac{1}{2T}\int_{-T}^{T} s_k(t)\,dt\Big)^2 \sum_{\substack{i,j=1\\ i\neq j}}^{n} |E_\eta \eta_i \eta_j|.$$

Further, the Cauchy–Schwarz inequality gives us finally

$$\operatorname{Var}(\hat{c}_k) \le \frac{2T}{n\tau^2}\Big(\int_{-T}^{T} f^2(t) s_k^2(t)\,dt + \sigma_\eta^2 \int_{-T}^{T} s_k^2(t)\,dt\Big) + \frac{2T}{n^2\tau^2}\int_{-T}^{T} s_k^2(t)\,dt \sum_{\substack{i,j=1\\ i\neq j}}^{n} |E_\eta \eta_i \eta_j|. \tag{10}$$

Now (10) and Lemma 1.2 yield

$$\sum_{|k|\le N} \operatorname{Var}(\hat{c}_k) \le \frac{2T}{n\tau^2}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{8T^2}{n^2\tau^2}\sum_{l=1}^{n-1}(n-l)|r(l)|. \tag{11}$$

For short range dependent (SRD) observation errors, $\sum_{l=1}^{\infty} |r(l)| < \infty$, and then

$$\sum_{|k|\le N} Var(\hat{c}_k) \le \frac{2T}{n\tau^2}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{8T^2}{n\tau^2}\sum_{l=1}^{\infty}|r(l)|.$$

This implies that Theorems 2.1 and 3.1 also hold in the case of SRD observation errors. Also Corollaries 2.1 and 3.1 concerning convergence rates are valid for such errors.

The class of long range dependent (LRD) observation errors is characterized by the requirement $\sum_{l=1}^{\infty} |r(l)| = \infty$. Assume that $r(l) = H/l^\gamma$, $l = 1, 2, \dots,$ where $H \neq 0$ and $0 < \gamma < 1$. Then the sum on the right hand side in (11), which is due to dependence of these errors, satisfies (see [15, Lemma 2.1])

$$\sum_{l=1}^{n-1}(n-l)|r(l)| \le \frac{|H| n^{2-\gamma}}{(1-\gamma)(2-\gamma)},$$

so that

$$\sum_{|k|\le N} \mathrm{Var}(\hat{c}_k) \le \frac{2T}{n\tau^2}(\|f\|^2 + 2T\sigma_\eta^2) + \frac{8T^2|H|}{n^\gamma\tau^2(1-\gamma)(2-\gamma)}.$$

Inspection of the bounds on the IMSE in Section 2 and the pointwise MSE in Section 3, which gave Theorems 2.1 and 3.1, shows that analogous theorems are also valid in the case of LRD observation errors if we replace the condition $\lim_{n\to\infty} T(n)^2/n = 0$ by $\lim_{n\to\infty} T(n)^2/n^\gamma = 0$. Similarly for the same observation errors and $T(n) = n^{\gamma/3}$, $n = 1, 2, \ldots$, corollaries on convergence rates analogous to Corollaries 2.1 and 3.1 are true with convergence rate replaced by $n^{-\gamma/3}$.

It is worth remarking that for $f \in L^2(\mathbb{R})$ we can calculate the estimators $\hat{c}_k$ according to (5), which estimate the coefficients c_k defined in (4), and since $s_k \in L^2(\mathbb{R}) \cap BL(\pi/\tau)$,

$$c_k = \frac{1}{\tau}\int_{\mathbb{R}} f(t)s_k(t)\,dt = \frac{1}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} F(\omega)\exp(ik\tau\omega)\,d\omega.$$

The orthogonal projection of $f \in L^2(\mathbb{R})$ on $L^2(\mathbb{R}) \cap BL(\pi/\tau)$ has Fourier transform $F_\tau(\omega) = F(\omega)$ for $\omega \in [-\pi/\tau, \pi/\tau]$ and $F_\tau(\omega) = 0$ for $\omega \notin [-\pi/\tau, \pi/\tau]$. Hence, we see immediately that the coefficients c_k are then the expansion coefficients of this projection f_τ with respect to the orthogonal system s_k, $k = 0, \pm1, \pm2, \ldots$. Accordingly, for $f \in L^2(\mathbb{R})$ the estimator $\hat{f}_n$ estimates the regression function projection f_τ and we also have

$$E_uE_\eta\|f - \hat{f}_n\|^2 = \|f - f_\tau\|^2 + E_uE_\eta\|f_\tau - \hat{f}_n\|^2.$$

5. Strong convergence. In this section we assume that the observation errors in our observation model are zero mean independent identically distributed random variables which are almost surely bounded, i.e. $|\eta_j| \le M_\eta < \infty$, $j = 1, \ldots, n$. In the following lemma we show that the regression function in this model is also bounded.

LEMMA 5.1. *Let $f \in L^2(\mathbb{R}) \cap BL(\Omega)$. Then*

$$M_f = \sup_{t\in\mathbb{R}} |f(t)| \le \sqrt{\Omega/\pi}\,\|f\|.$$

Proof. Since $f \in BL(\Omega)$ we can write $f(t) = (2\pi)^{-1}\int_{-\Omega}^{\Omega} F(\omega)\exp(it\omega)\,d\omega$ for $t \in \mathbb{R}$, and the Schwarz inequality gives the bound

$$|f(t)| \le \left(\frac{2\Omega}{2\pi}\right)^{1/2}\left(\frac{1}{2\pi}\int_{-\Omega}^{\Omega} |F(\omega)|^2\,d\omega\right)^{1/2} = \left(\frac{\Omega}{\pi}\right)^{1/2}\|f\|,$$

which proves the assertion. ∎

In view of the above lemma the observations of the regression function in our model satisfy $|y_j| = |f(u_j) + \eta_j| \le M_f + M_\eta$, $j = 1, \ldots, n$, so if we define the random variables $z_{kj} = y_j s_k(u_j)$, $j = 1, \ldots, n$, $k = 0, \pm 1, \pm 2, \ldots$, then (3) implies that they are also bounded and $|z_{kj}| \le M_z = M_f + M_\eta$.

According to (5) and (6),

$$\hat{c}_k - E_u E_\eta \hat{c}_k = \frac{2T}{n\tau} \sum_{j=1}^{n} z_{kj} - \frac{1}{\tau} \int_{-T}^{T} f(t) s_k(t)\, dt = \frac{2T}{n\tau} \sum_{j=1}^{n} (z_{kj} - E z_{kj}). \tag{12}$$

By the orthogonality of the cardinal functions s_k and the inequality $(a+b)^2 \le 2(a^2 + b^2)$ for $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned} I(\hat{f}_n) &= \int_{\mathbb{R}} (f - \hat{f}_n)^2 = \tau \sum_{|k| \le N} (\hat{c}_k - E_u E_\eta \hat{c}_k + E_u E_\eta \hat{c}_k - c_k)^2 + \tau \sum_{|k| > N} c_k^2 \\ &\le 2\tau \sum_{|k| \le N} (\hat{c}_k - E_u E_\eta \hat{c}_k)^2 + 2\tau \sum_{|k| \le N} B^2(\hat{c}_k) + \tau \sum_{|k| > N} c_k^2, \end{aligned}$$

so Lemma 1.1 further yields

$$I(\hat{f}_n) \le 2\tau \sum_{|k| \le N} (\hat{c}_k - E_u E_\eta \hat{c}_k)^2 + (2N+1) \frac{10\tau}{\pi^2 T} \int_{|t| > T} f^2(t) dt + \tau \sum_{|k| > N} c_k^2 \tag{13}$$

for $N < T/(2\tau)$. Now, we can prove the following theorem on strong convergence of the integrated error $I(\hat{f}_n)$.

THEOREM 5.1. *If $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ and $T(n) = n^\beta$, $N(n) = [T(n)/(2\tau)]$, $n = 1, 2, \ldots,$ where $0 < \beta < 1/3$, $\tau \le \pi/\Omega$, then the projection estimator $\hat{f}_n$ is strongly consistent in the sense of integrated square error, i.e.*

$$\lim_{n \to \infty} \int_{\mathbb{R}} (f - \hat{f}_n)^2 = 0 \quad a.s.$$

Proof. Set $T(n) = n^\beta$, where $0 < \beta < 1/3$. We see easily that for $\varepsilon > 0$ there is an n_0 such that for $n \ge n_0$ we have $T(n) \ge 2\tau$ and the sum of the second and third term on the right hand side of (13) is less than $2\varepsilon/3$. Hence, using (12) we obtain, for $n \ge n_0$,

$$\begin{aligned} P\{I(\hat{f}_n) > \varepsilon\} &\le P\Big\{2\tau \sum_{|k| \le N(n)} (\hat{c}_k - E_u E_\eta \hat{c}_k)^2 > \varepsilon/3\Big\} \\ &\le \sum_{|k| \le N(n)} P\bigg\{|\hat{c}_k - E_u E_\eta \hat{c}_k| > \sqrt{\frac{\varepsilon}{6\tau(2N(n)+1)}}\bigg\} \\ &= \sum_{|k| \le N(n)} P\bigg\{\bigg|\frac{1}{n} \sum_{j=1}^{n} (z_{kj} - E z_{kj})\bigg| > \frac{1}{2T(n)} \sqrt{\frac{\varepsilon\tau}{6(2N(n)+1)}}\bigg\}. \end{aligned}$$

Since the random variables z_{kj}, $j = 1, \ldots, n$, $k = 0, \pm 1, \pm 2, \ldots$, are almost surely bounded by the same constant M_z, we can use the well-known Hoeffding inequality to estimate the last probability:

$$P\{I(\hat{f}_n) > \varepsilon\} \le 2(2N(n)+1)\exp\left(-\frac{n\tau\varepsilon}{4T(n)^2 6(2N(n)+1)2M_z^2}\right),$$

and further since $N(n) = [T(n)/(2\tau)]$, where $T(n) \ge 2\tau$,

$$\begin{aligned} P\{I(\hat{f}_n) > \varepsilon\} &\le 6N(n)\exp\left(-\frac{n\tau\varepsilon}{4T(n)^2 18N(n)2M_z^2}\right) \\ &\le \frac{3T(n)}{\tau}\exp\left(-\frac{n\tau^2\varepsilon}{4T(n)^3 18M_z^2}\right) \le \frac{3}{\tau}\exp\left(-n\left[\frac{\tau^2\varepsilon}{72T(n)^3M_z^2} - \frac{\ln(T(n))}{n}\right]\right). \end{aligned}$$

Now, since $T(n) = n^\beta$, where $0 < \beta < 1/3$, the above bound takes the form

$$P\{I(\hat{f}_n) > \varepsilon\} \le \frac{3}{\tau}\exp\left(-n^{1-3\beta}\left[\frac{\tau^2\varepsilon}{72M_z^2} - \frac{\beta\ln(n)}{n^{1-3\beta}}\right]\right),$$

and the series $\sum_{n=1}^{\infty} P\{I(\hat{f}_n) > \varepsilon\}$ is summable. Invocation of the Borel–Cantelli lemma shows that $I(\hat{f}_n) \to 0$ a.s. ∎

In the next theorem the convergence rate of the integrated error $I(\hat{f}_n)$ is given. We say that a sequence of random variables ψ_n converges to zero a.s. *with rate* r_n, $n = 1, 2, \ldots$, if $a_n\psi_n/r_n \to 0$ a.s. for any sequence of real numbers $a_n \to 0$ as $n \to \infty$; we then write $\psi_n = O(r_n)$ a.s.

THEOREM 5.2. *If $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ satisfies the assumptions of Lemma 2.1, and $T(n) = n^{1/4}$, $N(n) = [T(n)/(2\tau)]$, $n = 1, 2, \ldots$, where $\tau \le \pi/\Omega$, then the integrated error $I(\hat{f}_n)$ converges to zero strongly with rate $n^{-\delta}$, where $0 < \delta < 1/4$, i.e.*

$$\lim_{n\to\infty} \int_{\mathbb{R}} (f - \hat{f}_n)^2 = O(n^{-\delta}) \quad a.s.$$

Proof. Taking into account (13) and Lemma 2.1 we see (analogously to (8)) that for $T \ge 4\tau$ and $N = [T/(2\tau)]$,

$$I(\hat{f}_n) \le 2\tau \sum_{|k|\le N} (\hat{c}_k - E_u E_\eta \hat{c}_k)^2 + \frac{1}{T}\left[\frac{50C^2(F)}{8\pi^4} + \frac{2C^2(F)}{\pi^2}\right]. \tag{14}$$

Assume that $a_n \to 0$ as $n \to \infty$, $0 < \delta < 1/4$, and set $T(n) = n^{1/4}$. We see immediately that for $\varepsilon > 0$ there is an n_0 such that for $n \ge n_0$ we have $T(n) \ge 4\tau$ and $Aa_n n^\delta/T(n) < \varepsilon/2$, where $A = (25\pi^{-2}/8 + 1)2C^2(F)/\pi^2$.

Hence, using (14) and (12) we obtain, for $n \geq n_0$,

$$P\{a_n n^\delta I(\hat{f}_n) > \varepsilon\} \leq P\Big\{2a_n n^\delta \tau \sum_{|k|\leq N(n)} (\hat{c}_k - E_u E_\eta \hat{c}_k)^2 > \varepsilon/2\Big\}$$

$$\leq \sum_{|k|\leq N(n)} P\bigg\{|\hat{c}_k - E_u E_\eta \hat{c}_k| > \sqrt{\frac{\varepsilon}{4\tau a_n n^\delta(2N(n)+1)}}\bigg\}$$

$$= \sum_{|k|\leq N(n)} P\bigg\{\bigg|\frac{1}{n}\sum_{j=1}^{n}(z_{kj} - Ez_{kj})\bigg| > \frac{1}{2T(n)}\sqrt{\frac{\varepsilon\tau}{4a_n n^\delta(2N(n)+1)}}\bigg\}.$$

Using the Hoeffding inequality gives

$$P\{a_n n^\delta I(\hat{f}_n) > \varepsilon\} \leq 2(2N(n)+1)\exp\bigg(-\frac{n\tau\varepsilon}{4T(n)^2 4a_n n^\delta(2N(n)+1)2M_z^2}\bigg),$$

and further since $N(n) = [T(n)/(2\tau)]$, where $T(n) \geq 2\tau$,

$$\begin{aligned} P\{a_n n^\delta I(\hat{f}_n) > \varepsilon\} &\leq 6N(n)\exp\bigg(-\frac{n^{1-\delta}\tau\varepsilon}{4T(n)^2 a_n 12N(n)2M_z^2}\bigg) \\ &\leq \frac{3T(n)}{\tau}\exp\bigg(-\frac{n^{1-\delta}\tau^2\varepsilon}{4T(n)^3 12a_n M_z^2}\bigg) \\ &\leq \frac{3}{\tau}\exp\bigg(-n^{1-\delta}\bigg[\frac{\tau^2\varepsilon}{48T(n)^3 a_n M_z^2} - \frac{\ln(T(n))}{n^{1-\delta}}\bigg]\bigg). \end{aligned}$$

Now, since $T(n) = n^{1/4}$, the above bound takes the form

$$P\{a_n n^\delta I(\hat{f}_n) > \varepsilon\} \leq \frac{3}{\tau}\exp\bigg(-n^{1-3/4-\delta}\bigg[\frac{\tau^2\varepsilon}{48a_n M_z^2} - \frac{\ln(n)}{4n^{1-3/4-\delta}}\bigg]\bigg),$$

and one can see that for $0 < \delta < 1/4$ the series $\sum_{n=1}^{\infty} P\{a_n n^\delta I(\hat{f}_n) > \varepsilon\}$ is summable. By the Borel–Cantelli lemma, $a_n n^\delta I(\hat{f}_n) \to 0$ a.s., and the proof is complete. ∎

6. Conclusions. This work is a continuation and extension of the author's previous article on asymptotic properties of orthogonal series regression estimation for a random uniform observation point design [14]. This time band-limited regression functions from the space $L^2(\mathbb{R})$ are considered and a more complex observation model is used, in which the observation interval expands as the number of observations grows. The projection estimator of band-limited regression functions, analogous to the one considered in this work, was earlier investigated by Pawlak and Rafajłowicz [6] in the case of the fixed design observation model $y_j = f(jh) + \delta_j$, $j = 0, \pm 1, \pm 2, \ldots, \pm n$, where $h > 0$ is the sampling rate and δ_j are uncorrelated stationary errors with zero mean and finite variance. They obtained the same IMSE convergence rate $n^{-1/3}$ for $h(n) = an^{-2/3}$ and $N(n) = bn^{1/3}$, where $a, b > 0$, $a > b\tau$, in the regression function class considered in Corollary 2.1.

Conditions for consistency in the sense of pointwise mean-square error of our projection estimator and its convergence rates were not examined earlier, but the rate $n^{-1/3}$ obtained by the author does not seem to be favorable in comparison to other estimators for band-limited regression functions. Pawlak and Stadtmüller [10], [11] examined asymptotic properties of estimators based on modifications of the Shannon–Whittaker interpolation scheme (1), which are appropriate for the above mentioned fixed design observation model. They showed that their estimators attain pointwise mean-square error convergence rate $n^{-2/3}$ under the assumption that the Fourier transform F of the regression function $f \in L^2(\mathbb{R}) \cap BL(\Omega)$ is Lipschitz continuous at $\pm\Omega$, which is significantly different from the assumption considered here.

In a later work Pawlak and Stadtmüller [13] investigated properties of one of their estimators for the fixed design observation model assuming short range and long range dependence of observation errors. They obtained convergence rates of the pointwise MSE respectively $n^{-2/3}$ and $n^{-2\gamma/(2+\gamma)}$ for SRD and LRD observation errors, in a class of regression functions which is wider than the one considered in Corollary 3.1. Namely, they assumed that the regression function satisfies $|f(t)| \leq c|t|^{-1}$ for $|t| > 0$, where $c > 0$. Convergence rates of our projection estimator for such errors are slower, $n^{-1/3}$ and $n^{-\gamma/3}$, respectively. This means that estimators based on modifications of the interpolation scheme may be more appropriate for estimating point values of the regression function, than those based on orthogonal series. Nevertheless, we obtained IMSE convergence rates of the projection estimator in the case of SRD and LRD observation errors in a class of regression functions that is different from the one in [13], where band-limited regression functions satisfying $|f(t)| \leq c|t|^{-(s+1)}$ for $|t| > 0$ were considered, with $c > 0$ and $s > 0$.

Strong convergence and convergence rates of the integrated square error of estimators appropriate for a fixed design observation model with band-limited regression functions and bounded observation errors were analysed in [4] and [9]. However, the strong convergence rates were obtained under significantly different conditions on regression functions, and under other conditions which are not comparable to the ones in the present work.

Another approach to regression function estimation which is appropriate for random uniform design is presented in [1], where the concept of wavelet transform and wavelet shrinkage is applied to the problem of recovering a function observed on a compact interval.

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