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NEWTON-TYPE ITERATIVE METHODS FOR NONLINEAR ILL-POSED HAMMERSTEIN-TYPE EQUATIONS

Abstract. We use a combination of modified Newton method and Tikhonov regularization to obtain a stable approximate solution for nonlinear ill-posed Hammerstein-type operator equations $KF(x) = y$. It is assumed that the available data is y^δ with $\|y - y^\delta\| \le \delta$, $K : Z \to Y$ is a bounded linear operator and $F : X \to Z$ is a nonlinear operator where X, Y, Z are Hilbert spaces. Two cases of F are considered: where $F'(x_0)^{-1}$ exists ($F'(x_0)$ is the Fréchet derivative of F at an initial guess x_0) and where F is a monotone operator. The parameter choice using an a priori and an adaptive choice under a general source condition are of optimal order. The computational results provided confirm the reliability and effectiveness of our method.

1. Introduction. This paper is devoted to nonlinear ill-posed Hammerstein-type operator equations. Recall that [13, 14, 16, 17] an equation

$$(KF)x = y \tag{1.1}$$

is called a *nonlinear ill-posed Hammerstein-type operator equation*. Here $F : D(F) \subseteq X \to Z$, is a nonlinear operator, $K : Z \to Y$ is a bounded linear operator and X, Z, Y are Hilbert spaces with corresponding inner product $\langle \cdot , \cdot \rangle_X$, $\langle \cdot , \cdot \rangle_Z$, $\langle \cdot , \cdot \rangle_Y$, and norm $\| \cdot \|_X$, $\| \cdot \|_Z$, $\| \cdot \|_Y$ respectively. A typical example of a Hammerstein-type operator is the nonlinear integral operator

$$(Ax)(t) := \int_0^1 k(s,t) f(s, x(s))\, ds$$

where $k(\cdot , \cdot) \in L^2([0,1] \times [0,1])$, $x \in L^2[0,1]$ and $t \in [0,1]$.

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The above integral operator A admits a representation of the form $A = KF$ where $K : L^2[0,1] \to L^2[0,1]$ is a linear integral operator with kernel $k(t,s)$ defined as

$$Kx(t) = \int_0^1 k(t,s)x(s)\,ds$$

and $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ is a nonlinear superposition operator (cf. [24]) defined as

$$Fx(s) = f(s, x(s)). \tag{1.2}$$

The third author and his collaborators [13,14,16,17] studied ill-posed Hammerstein-type equations extensively under some assumptions on the Fréchet derivative of F. Precisely, in [13, 17], it is assumed that $F'(x_0)^{-1}$ exists and in [16] it is assumed that $F'(x)^{-1}$ exists for all x in a ball of radius r around x_0.

Note that if the function f in (1.2) is differentiable with respect to the second variable and $\partial_2 f(t, x(t)) \geq \kappa_1$ for all $x \in B_r(x_0)$ and $t \in [0,1]$, then $F'(u)^{-1}$ exists and is a bounded operator for all $u \in B_r(x_0)$ (see [17, Remark 2.1]); here $\partial_2 f(t,s)$ represents the partial derivative of f with respect to the second variable.

Throughout this paper it is assumed that the available data is y^δ with

$$\|y - y^\delta\|_Y \leq \delta,$$

and hence one has to consider the equation

$$(KF)x = y^\delta \tag{1.3}$$

instead of (1.1). Observe that the solution x of (1.3) can be obtained by solving

$$Kz = y^\delta \tag{1.4}$$

for z and then solving the nonlinear problem

$$F(x) = z. \tag{1.5}$$

In [16], to solve (1.5), George and Kunhanandan considered the sequence defined iteratively by

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta)$$

where $x_{0,\alpha}^\delta := x_0$ and

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(y^\delta - KF(x_0)) + F(x_0), \tag{1.6}$$

and obtained local quadratic convergence.

Recall that a sequence (x_n) in X with $\lim x_n = x^*$ is said to be *convergent of order* $p > 1$ if there exist positive reals c_1, c_2 such that for all $n \in \mathbb{N}$

$$\|x_n - x^*\|_X \leq c_1 e^{-c_2 p^n}.$$

If the sequence (x_n) has the property that $\|x_n - x^*\|_X \le c_1 q^n$, $0 < q < 1$, then (x_n) is said to be *linearly convergent*. For an extensive discussion of convergence rate see Kelley [23].

In [17], George and Nair studied the modified Lavrent'ev regularization

$$z_\alpha^\delta = (K + \alpha I)^{-1}(y^\delta - KF(x_0))$$

to obtain an approximate solution of (1.4), and introduced modified Newton's iteration

$$x_{n,\alpha}^\delta = x_{n-1,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n-1,\alpha}^\delta) - F(x_0) - z_\alpha^\delta)$$

to solve (1.5) and obtained local linear convergence. In fact in [16] and [17], a solution $\hat{x}$ of (1.1) is called an x_0-*minimum norm solution* if

$$\|F(\hat{x}) - F(x_0)\|_Z := \min\{\|F(x) - F(x_0)\|_Z : KF(x) = y,\, x \in D(F)\}. \tag{1.7}$$

We assume throughout that the solution $\hat{x}$ satisfies (1.7). In [13, 14, 16, 17], it is assumed that the ill-posedness of (1.1) is due to the nonclosedness of the operator K. In this paper we consider two cases:

Case (1): $F'(x_0)^{-1}$ exists and is a bounded operator, i.e., (1.5) is regular.

Case (2): F is monotone [26, 31], $Z = X$ is a real Hilbert space and $F'(x_0)^{-1}$ does not exist, i.e., (1.5) is ill-posed.

The case when F is not monotone and $F'(x_0)^{-1}$ does not exist is the subject matter of the forthcoming paper.

One of the advantages of (approximately) solving (1.4) and (1.5) to obtain an approximate solution for (1.3) is that one can use any regularization method [8, 22] for linear ill-posed equations for solving (1.4), and any iterative method [10, 12] for solving (1.5). In fact in this paper we consider Tikhonov regularization [11,13,15,16,20] to approximately solve (1.4) and we consider a modified two-step Newton method [1, 6, 7, 9, 21, 25] to solve (1.5). Note that the regularization parameter α is chosen according to the adaptive method considered by Pereverzev and Schock [28] for linear ill-posed operator equations and the same parameter α is used to solve the nonlinear operator equation (1.5), so the choice of the regularization parameter does not depend on the nonlinear operator F; this is another advantage over treating (1.3) as a single nonlinear operator equation.

This paper is organized as follows. Preparatory results are given in Section 2. Section 3 contains the proposed iterative method for Case (1) and Case (2). Section 4 deals with the algorithm implementing the proposed method. Numerical examples are given in Section 5. Finally the paper ends with some conclusions in Section 6.

2. Preparatory results. In this section we consider the Tikhonov regularized solution z_α^δ defined in (1.6) and obtain a priori and a posteriori error estimates for $\|F(\hat{x}) - z_\alpha^\delta\|_Z$. The following assumption is required to obtain the error estimate.

Assumption 2.1. There exists a continuous, strictly increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|K^*K\|_{Y\to X}$ satisfying

- $\lim_{\lambda\to 0} \varphi(\lambda) = 0$,
- $\sup_{\lambda\geq 0} \dfrac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha)$ for all $\lambda \in (0, a]$, and
- there exists $v \in X$ with $\|v\|_X \leq 1$ such that

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)v.$$

Theorem 2.2 (see [16, (4.3)]). *Let* z_α^δ *be as in* (1.6) *and suppose Assumption* 2.1 *holds. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\|_Z \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{2.1}$$

2.1. A priori choice of the parameter. Note that the bound $\varphi(\alpha) + \delta/\sqrt{\alpha}$ in (2.1) is of optimal order for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \delta/\sqrt{\alpha_\delta}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \leq \|K\|_Y^2$. Then $\delta = \sqrt{\alpha_\delta}\,\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$ and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)).$$

So the relation (2.1) leads to $\|F(\hat{x}) - z_\alpha^\delta\|_Z \leq 2\psi^{-1}(\delta)$.

2.2. An adaptive choice of the parameter. In this paper, we propose choosing the parameter α according to the adaptive choice established by Pereverzev and Shock [28] for ill-posed problems. We denote by D_M the set of possible values of the parameter α,

$$D_M = \{\alpha_i = \alpha_0\mu^{2i} : i = 0, 1, \ldots, M\}, \qquad \mu > 1.$$

Then the adaptive choice of a numerical value k for the parameter α uses the rule

$$k := \max\{i : \alpha_i \in D_M^+\} \tag{2.2}$$

where $D_M^+ = \{\alpha_i \in D_M : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\|_Z \leq 4\delta/\sqrt{\alpha_j},\ j = 0, 1, \ldots, i-1\}$. Let

$$l := \max\{i : \varphi(\alpha_i) \leq \delta/\sqrt{\alpha_i}\}. \tag{2.3}$$

We will use the following theorem from [16] for our error analysis.

Theorem 2.3 (cf. [16, Theorem 4.3]). *Let* l *be as in* (2.3), k *be as in* (2.2) *and* $z_{\alpha_k}^\delta$ *be as in* (1.6) *with* $\alpha = \alpha_k$. *Then* $l \leq k$ *and*

$$\|F(\hat{x}) - z_{\alpha_k}^\delta\|_Z \leq \left(2 + \frac{4\mu}{\mu - 1}\right)\mu\psi^{-1}(\delta).$$

3. Convergence analysis. Throughout this paper we assume that the operator F has a uniformly bounded Fréchet derivative $F'(\cdot)$ for all $x \in D(F)$. In the earlier papers [16, 18, 19] the authors used the following assumption:

Assumption 3.1 (cf. [30, Assumption 3]). There exists a constant $K_0 \geq 0$ such that for every $x, u \in B_r(x_0) \cup B_r(\hat{x}) \subseteq D(F)$ and $v \in X$ there exists an element $\Phi(x, u, v) \in X$ such that

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \qquad \|\Phi(x, u, v)\|_X \leq K_0\|v\|_X\|x - u\|_X.$$

The hypotheses of Assumption 3.1 may not hold or may be very time-consuming or impossible to verify in general. In particular, just as for well-posed nonlinear equations, the computation of the Lipschitz constant K_0, even if this constant exists, is very difficult. Moreover, there are classes of operators for which Assumption 3.1 is not satisfied but the iterative method converges.

In the present paper, we extend the applicability of the Newton-type iterative method under less computational cost. We achieve this under the following weaker assumption:

Assumption 3.2. Let $x_0 \in X$. There exists a constant k_0 such that for every $u \in B_r(x_0) \subseteq D(F)$ and $v \in X$, there exists $\Phi_0(x_0, u, v) \in X$ satisfying

$$[F'(x_0) - F'(u)]v = F'(x_0)\Phi_0(x_0, u, v),$$
$$\|\Phi(x_0, u, v)\|_X \leq k_0\|v\|_X\|x_0 - u\|_X.$$

Note that

$$k_0 \leq K_0$$

in general and K_0/k_0 can be arbitrarily large. The advantages of the new approach are:

(1) Assumption 3.2 is weaker than Assumption 3.1.
(2) The computational cost of finding the constant k_0 is less than that for the constant K_0, even when $K_0 = k_0$.
(3) The sufficient convergence criteria are weaker.
(4) The computable error bounds on the distances involved (including k_0) are less costly and more precise than the old ones (including K_0).
(5) The information on the location of the solution is more precise.
(6) The convergence domain of the iterative method is larger.

These advantages are also important in computations since they provide under less computational cost a wider choice of initial guesses for the iterative method and the computation of fewer iterates to achieve a desired error tolerance. Numerical examples for (1)–(6) are presented in Section 4.

3.1. Iterative method for Case (1). In this subsection for an initial guess $x_0 \in X$, we consider the sequence v^δ_{n,α_k}, defined iteratively by

$$v^\delta_{n,\alpha_k} = v^\delta_{n,\alpha_k} - F'(x_0)^{-1}(F(v^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k})$$

where $v^\delta_{0,\alpha_k} = x_0$, to obtain an approximation $x^\delta_{\alpha_k}$ of x such that $F(x) = z^\delta_{\alpha_k}$.

Let

$$y^\delta_{n,\alpha_k} = v^\delta_{2n-1,\alpha_k}, \tag{3.1}$$

$$x^\delta_{n+1,\alpha_k} = v^\delta_{2n,\alpha_k}, \tag{3.2}$$

for $n > 0$. We will use the following notations:

$$\begin{aligned} M &\geq \|F'(x_0)\|_{X\to Z}, \\ \beta &:= \|F'(x_0)^{-1}\|_{Z\to X}, \\ k_0 &< \frac{1}{4}\min\Big\{1, \frac{1}{\beta}\Big\}, \\ \delta_0 &< \frac{\sqrt{\alpha_0}}{4k_0\beta}, \\ \rho &:= \frac{1}{M}\left(\frac{1}{4k_0\beta} - \frac{\delta_0}{\sqrt{\alpha_0}}\right), \\ \gamma_\rho &:= \beta\left[M\rho + \frac{\delta_0}{\sqrt{\alpha_0}}\right], \end{aligned}$$

and

$$e^\delta_{n,\alpha_k} := \|y^\delta_{n,\alpha_k} - x^\delta_{n,\alpha_k}\|_X, \qquad \forall n \geq 0. \tag{3.3}$$

For convenience, we write x_n, y_n and e_n for x^δ_{n,α_k}, y^δ_{n,α_k} and e^δ_{n,α_k} respectively.

Further we define

$$q := k_0 r, \qquad r \in (r_1, r_2) \tag{3.4}$$

where

$$r_1 = \frac{1 - \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}, \qquad r_2 = \min\left\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\gamma_\rho}}{2k_0}\right\}.$$

Note that r is well defined because $\gamma_\rho \leq 1/(4k_0)$. We will use the relation $e_0 \leq \gamma_\rho$, which can be seen as follows:

$$\begin{aligned} e_0 = \|y_0 - x_0\|_X &= \|F'(x_0)^{-1}(F(x_0) - z^\delta_{\alpha_k})\|_X \\ &\leq \|F'(x_0)^{-1}\|_{Z\to X}\|(F(x_0) - z^\delta_{\alpha_k})\|_Z \\ &\leq \beta\|F(x_0) - z_{\alpha_k} + z_{\alpha_k} - z^\delta_{\alpha_k}\|_Z \\ &\leq \beta[\|F(x_0) - F(\hat{x})\|_Z + \|z_{\alpha_k} - z^\delta_{\alpha_k}\|_Z] \\ &\leq \beta[M\rho + \delta/\sqrt{\alpha}] \leq \beta[M\rho + \delta_0/\sqrt{\alpha_0}] = \gamma_\rho. \end{aligned}$$

THEOREM 3.3. *Let* e_n, q *be as in* (3.3), (3.4), *and* x_n, y_n *be as in* (3.2), (3.1) *with* $\delta \in (0, \delta_0]$. *Then by Assumption* 3.2 *and Theorem* 2.3, $x_n, y_n \in B_r(x_0)$ *and the following estimates hold for all* $n \geq 0$*:*

(a) $\|x_{n+1} - y_n\|_X \leq q\|y_n - x_n\|_X$,
(b) $\|y_{n+1} - x_{n+1}\|_X \leq q^2\|y_n - x_n\|_X$,
(c) $e_n \leq q^{2n}\gamma_\rho$.

Proof. Suppose $x_n, y_n \in B_r(x_0)$. Then

$$\begin{aligned} x_{n+1} - y_n &= y_n - x_n - F'(x_0)^{-1}(F(y_n) - F(x_n)) \\ &= F'(x_0)^{-1}[F'(x_0)(y_n - x_n) - (F(y_n) - F(x_n))] \\ &= F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(x_n + t(y_n - x_n))](y_n - x_n)\, dt, \end{aligned}$$

and hence by Assumption 3.2, we have

$$\|x_{n+1} - y_n\|_X \leq k_0 r\|y_n - x_n\|_X \leq q\|y_n - x_n\|_X.$$

This proves (a).

To prove (b) we observe that

$$\begin{aligned} e_{n+1} = \|y_{n+1} - x_{n+1}\|_X &= \|x_{n+1} - y_n - F'(x_0)^{-1}(F(x_{n+1}) - F(y_n))\|_X \\ &= \left\| F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(y_n + t(x_{n+1} - y_n))]\, dt\, (x_{n+1} - y_n)\right\|_X \\ &\leq k_0 r\|y_n - x_{n+1}\|_X \leq q^2\|x_n - y_n\|_X. \end{aligned}$$

The last but one step follows from Assumption 3.2, and the last step follows from (a). This completes the proof of (b), and (c) follows from (b).

Now we shall show by induction that $x_n, y_n \in B_r(x_0)$. For $n = 1$,

$$\begin{aligned} x_1 - y_0 &= y_0 - x_0 - F'(x_0)^{-1}(F(y_0) - F(x_0)) \\ &= F'(x_0)^{-1}[F'(x_0)(y_0 - x_0) - (F(y_0) - F(x_0))] \\ &= F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(x_0 + t(y_0 - x_0))](y_0 - x_0)\, dt, \end{aligned}$$

and hence by Assumption 3.2, we have

$$\|x_1 - y_0\|_X \leq \frac{k_0}{2}\|y_0 - x_0\|_X^2 \leq k_0 r e_0. \tag{3.5}$$

So by the triangle inequality and (3.5)

$$\begin{aligned} \|x_1 - x_0\|_X &\leq \|x_1 - y_0\|_X + \|y_0 - x_0\|_X \\ &\leq (1+q)e_0 \leq \frac{e_0}{1-q} \leq \frac{\gamma_\rho}{1-q} \leq r, \end{aligned}$$

i.e., $x_1 \in B_r(x_0)$. Observe that

$$\begin{aligned}\|y_1 - x_1\|_X &= \|x_1 - y_0 - F'(x_0)^{-1}(F(x_1) - F(y_0))\|_X \\ &\le k_0 r \|x_1 - y_0\|_X,\end{aligned}$$

and hence by (3.5),

$$\|y_1 - x_1\|_X \le q^2 e_0. \tag{3.6}$$

Therefore by (3.4), (3.6) and the triangle inequality,

$$\begin{aligned}\|y_1 - x_0\|_X &\le \|y_1 - x_1\|_X + \|x_1 - x_0\|_X \\ &\le (1 + q + q^2)e_0 \\ &\le \frac{e_0}{1-q} \le \frac{\gamma_\rho}{1-q} \le r,\end{aligned}$$

i.e., $y_1 \in B_r(x_0)$. Suppose $x_m, y_m \in B_r(x_0)$. Then

$$\begin{aligned}\|x_{m+1} - x_0\|_X &\le \|x_{m+1} - x_m\|_X + \|x_m - x_{m-1}\|_X + \cdots + \|x_1 - x_0\|_X \\ &\le (q+1)e_m + (q+1)e_{m-1} + \cdots + (q+1)e_0 \\ &\le (q+1)(e_m + e_{m-1} + \cdots + e_0) \\ &\le (q+1)(q^{2m} + q^{2(m-1)} + \cdots + 1)e_0 \\ &\le (q+1)\frac{1-q^{2m+1}}{1-q^2} e_0 \\ &\le \frac{e_0}{1-q} \le \frac{\gamma_\rho}{1-q} \le r,\end{aligned}$$

i.e., $x_{m+1} \in B_r(x_0)$, and

$$\begin{aligned}\|y_{m+1} - x_0\|_X &\le \|y_{m+1} - x_{m+1}\|_X + \|x_{m+1} - x_0\|_X \\ &\le q^2 e_m + (q+1)e_m + (q+1)e_{m-1} + \cdots + (q+1)e_0 \\ &\le (q^2 + q + 1)e_m + (q+1)e_{m-1} + \cdots + (q+1)e_0 \\ &\le (q^{2(m+1)} + \cdots + q^3 + q^2 + q + 1)e_0 \\ &\le \frac{e_0}{1-q} \le \frac{\gamma_\rho}{1-q} \le r,\end{aligned}$$

i.e., $y_{m+1} \in B_r(x_0)$. Thus by induction, $x_n, y_n \in B_r(x_0)$. This completes the proof of the theorem.

The main result of this section is the following theorem:

Theorem 3.4. *Let x_n and y_n be as in* (3.2) *and* (3.1), *and suppose the assumptions of Theorem 3.3 hold. Then (x_n) is a Cauchy sequence in $B_r(x_0)$ and converges to $x^\delta_{\alpha_k} \in \overline{B_r(x_0)}$. Further $F(x^\delta_{\alpha_k}) = z^\delta_{\alpha_k}$ and*

$$\|x_n - x^\delta_{\alpha_k}\|_X \le Cq^{2n} \quad \textit{where} \quad C = \frac{\gamma_\rho}{1-q}.$$

Proof. Using the relations (b) and (c) of Theorem 3.3, we obtain

$$\begin{aligned}\|x_{n+m}-x_n\|_X &\le \sum_{i=0}^{m-1}\|x_{n+i+1}-x_{n+i}\|_X \le \sum_{i=0}^{m-1}(1+q)e_{n+i}\\ &\le \sum_{i=0}^{m-1}(1+q)q^{2(n+i)}e_0\\ &= (1+q)q^{2n}e_0+(1+q)q^{2(n+1)}e_0+\cdots+(1+q)q^{2(n+m)}e_0\\ &\le (1+q)q^{2n}(1+q^2+q^{2(2)}+\cdots+q^{2m})e_0\\ &\le q^{2n}\frac{1-(q^2)^{m+1}}{1-q}\gamma_\rho \le Cq^{2n}.\end{aligned}$$

Thus x_n is a Cauchy sequence in $B_r(x_0)$, and hence it converges, say to $x^\delta_{\alpha_k} \in \overline{B_r(x_0)}$. Observe that

$$\begin{aligned}\|F(x_n)-z^\delta_{\alpha_k}\|_Z &= \|F'(x_0)(x_n-y_n)\|_Z \le \|F'(x_0)\|_{X\to Z}\|(x_n-y_n)\|_Z\\ &\le Me_n \le Mq^{2n}\gamma_\rho.\end{aligned} \tag{3.7}$$

Now by letting $n\to\infty$ in (3.7) we obtain $F(x^\delta_{\alpha_k}) = z^\delta_{\alpha_k}$. This completes the proof.

Hereafter we assume that

$$\|\hat{x}-x_0\|_X < \rho \le r.$$

THEOREM 3.5. *Suppose that Assumption* 3.2 *holds. Then*

$$\|\hat{x}-x^\delta_{\alpha_k}\|_X \le \frac{\beta}{1-k_0r}\|F(\hat{x})-z^\delta_{\alpha_k}\|_Z.$$

Proof. Note that $k_0r<1$, and by Assumption 3.2, we have

$$\begin{aligned}\|\hat{x}-x^\delta_{\alpha_k}\|_X &\le \|\hat{x}-x^\delta_{\alpha_k}+F'(x_0)^{-1}[F(x^\delta_{\alpha_k})-F(\hat{x})+F(\hat{x})-z^\delta_{\alpha_k}]\|_X\\ &\le \|F'(x_0)^{-1}[F'(x_0)(\hat{x}-x^\delta_{\alpha_k})+F(x^\delta_{\alpha_k})-F(\hat{x})]\|_X\\ &\quad+\|F'(x_0)^{-1}(F(\hat{x})-z^\delta_{\alpha_k})\|_X\\ &\le k_0\|x_0-\hat{x}-t(x^\delta_{\alpha_k}-\hat{x})\|_X\|\hat{x}-x^\delta_{\alpha_k}\|_X+\beta\|F(\hat{x})-z^\delta_{\alpha_k}\|_Z\\ &\le k_0r\|\hat{x}-x^\delta_{\alpha_k}\|_Z+\beta\|F(\hat{x})-z^\delta_{\alpha_k}\|_Z.\end{aligned}$$

This completes the proof.

The following theorem is a consequence of Theorems 3.4 and 3.5.

THEOREM 3.6. *Let x_n be as in* (3.2), *and suppose that the assumptions of Theorems* 3.4 *and* 3.5 *hold. Then*

$$\|\hat{x}-x_n\|_X \le Cq^{2n}+\frac{\beta}{1-k_0r}\|F(\hat{x})-z^\delta_{\alpha_k}\|_Z$$

where C is as in Theorem 3.4.

Observe that from Section 2.2, $l \le k$ and $\alpha_\delta \le \alpha_{l+1} \le \mu\alpha_l$, we have

$$\frac{\delta}{\sqrt{\alpha_k}} \le \frac{\delta}{\sqrt{\alpha_l}} \le \mu\frac{\delta}{\sqrt{\alpha_\delta}} = \mu\varphi(\alpha_\delta) = \mu\psi^{-1}(\delta).$$

This leads to the following theorem:

THEOREM 3.7. *Let* x_n *be as in* (3.2), *and suppose that the assumptions of Theorems* 2.3, 3.4 *and* 3.5 *hold. Let*

$$n_k := \min\{n : q^{2n} \le \delta/\sqrt{\alpha_k}\}.$$

Then

$$\|\hat{x} - x_{n_k}\|_X = O(\psi^{-1}(\delta)).$$

3.2. Iterative method for Case (2). F is a monotone operator (i.e., $\langle F(x) - F(y), x - y\rangle \ge 0$ for all $x, y \in D(F)$), $Z = X$ is a real Hilbert space and $F'(x_0)^{-1}$ does not exist. Thus the ill-posedness of (1.1) in this case is due to the ill-posedness of F as well as the nonclosedness of the range of the linear operator K. The following assumptions are needed in addition to the earlier assumptions for our convergence analysis.

ASSUMPTION 3.8. There exists a continuous, strictly increasing function $\varphi_1 : (0, b] \to (0, \infty)$ with $b \ge \|F'(x_0)\|_{X\to X}$ satisfying

- $\lim_{\lambda\to 0} \varphi_1(\lambda) = 0$,
- $\sup\limits_{\lambda\ge 0} \dfrac{\alpha\varphi_1(\lambda)}{\lambda + \alpha} \le \varphi_1(\alpha)$ for all $\lambda \in (0, b]$,
- there exists $v \in X$ with $\|v\|_X \le 1$ (cf. [26]) such that

$$x_0 - \hat{x} = \varphi_1(F'(x_0))v.$$

ASSUMPTION 3.9. For each $x \in B_{\tilde{r}}(x_0)$ there exists a bounded linear operator $G(x, x_0)$ (see [29]) such that

$$F'(x) = F'(x_0)G(x, x_0)$$

with $\|G(x, x_0)\|_{X\to X} \le k_2$.

The iterative method for this case is

$$\tilde{v}^\delta_{n,\alpha_k} = \tilde{v}^\delta_{n,\alpha_k} - R(x_0)^{-1}\left[F(\tilde{v}^\delta_{n,\alpha_k}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{v}^\delta_{n,\alpha_k} - x_0)\right]$$

where $\tilde{v}^\delta_{0,\alpha_k} := x_0$ is the initial guess and $R(x_0) := F'(x_0) + (\alpha_k/c)I$, with $c \le \alpha_k$. Let

$$\tilde{y}^\delta_{n,\alpha_k} = \tilde{v}^\delta_{2n-1,\alpha_k}, \tag{3.8}$$

$$\tilde{x}^\delta_{n+1,\alpha_k} = \tilde{v}^\delta_{2n,\alpha_k}, \tag{3.9}$$

for $n > 0$.

First we prove that $\tilde{x}_{n,\alpha_k}$ converges to the zero x^{δ}_{c,α_k} of

$$F(x) + \frac{\alpha_k}{c}(x - x_0) = z^{\delta}_{\alpha_k}, \tag{3.10}$$

and then we prove that x^{δ}_{c,α_k} is an approximation for $\hat{x}$.

Let

$$\tilde{e}^{\delta}_{n,\alpha_k} := \|\tilde{y}^{\delta}_{n,\alpha_k} - \tilde{x}^{\delta}_{n,\alpha_k}\|_X, \qquad \forall n \geq 0. \tag{3.11}$$

For the sake of simplicity, we use the notation $\tilde{x}_n$, $\tilde{y}_n$ and $\tilde{e}_n$ for $\tilde{x}^{\delta}_{n,\alpha_k}$, $\tilde{y}^{\delta}_{n,\alpha_k}$ and $\tilde{e}^{\delta}_{n,\alpha_k}$ respectively.

Hereafter we assume that $\|\hat{x} - x_0\|_X < \rho \leq \tilde{r}$ where

$$\rho < \frac{1}{M}\left(1 - \frac{\delta_0}{\sqrt{\alpha_0}}\right)$$

with $\delta_0 < \sqrt{\alpha_0}$. Let

$$\tilde{\gamma}_\rho := M\rho + \frac{\delta_0}{\sqrt{\alpha_0}},$$

and define

$$q_1 = k_0\tilde{r}, \qquad \tilde{r} \in (\tilde{r_1}, \tilde{r_2}), \tag{3.12}$$

where

$$\tilde{r}_1 = \frac{1 - \sqrt{1 - 4k_0\tilde{\gamma}_\rho}}{2k_0}, \qquad \tilde{r}_2 = \min\left\{\frac{1}{k_0}, \frac{1 + \sqrt{1 - 4k_0\tilde{\gamma}_\rho}}{2k_0}\right\}.$$

THEOREM 3.10. *Let $\tilde{e}_n$ and q_1 be as in* (3.11) *and* (3.12)*, $\tilde{x}_n$ and $\tilde{y}_n$ be as in* (3.9) *and* (3.8) *with $\delta \in (0, \delta_0]$, and suppose Assumption* 3.2 *holds. Then, for all $n \geq 0$:*

(a) $\|\tilde{x}_n - \tilde{y}_{n-1}\|_X \leq q_1\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|_X$,
(b) $\|\tilde{y}_n - \tilde{x}_n\|_X \leq q_1^2\|\tilde{y}_{n-1} - \tilde{x}_{n-1}\|_X$,
(c) $\tilde{e}_n \leq q_1^{2n}\tilde{\gamma}_\rho$.

Proof. Suppose $\tilde{x}_n, \tilde{y}_n \in B_{\tilde{r}}(x_0)$. Then

$$\begin{aligned}
\tilde{x}_n - \tilde{y}_{n-1} &= \tilde{y}_{n-1} - \tilde{x}_{n-1} \\
&\qquad - R(x_0)^{-1}\left(F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}) + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - \tilde{x}_{n-1})\right) \\
&= R(x_0)^{-1}\Big[R(x_0)(\tilde{y}_{n-1} - \tilde{x}_{n-1}) \\
&\qquad\qquad - (F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1})) - \frac{\alpha_k}{c}(\tilde{y}_{n-1} - \tilde{x}_{n-1})\Big] \\
&= R(x_0)^{-1}\int_0^1 [F'(x_0) - (F(\tilde{y}_{n-1}) - F(\tilde{x}_{n-1}))](\tilde{y}_{n-1} - \tilde{x}_{n-1})\,dt.
\end{aligned}$$

Now since $\|R(x_0)^{-1}F'(x_0)\|_{X\to X} \le 1$, (a) follows as in Theorem 3.3. Again observe that

$$\begin{aligned}\tilde{e}_n &\le \left\|\tilde{x}_n - \tilde{y}_{n-1} - R(x_0)^{-1}\Big(F(\tilde{x}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}_n - x_0)\Big)\right\|_X \\ &\quad + \left\|R(x_0)^{-1}\Big(F(\tilde{y}_{n-1}) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{y}_{n-1} - x_0)\Big)\right\|_X \\ &\le \left\|R(x_0)^{-1}\Big[R(x_0)(\tilde{x}_n - \tilde{y}_{n-1}) - (F(\tilde{x}_n) - F(\tilde{y}_{n-1})) - \frac{\alpha_k}{c}(\tilde{x}_n - \tilde{y}_{n-1})\Big]\right\|_X \\ &\le \left\|R(x_0)^{-1}\int_0^1 [F'(x_0) - (F(\tilde{x}_n) - F(\tilde{y}_{n-1}))]\,dt\,(\tilde{x}_n - \tilde{y}_{n-1})\right\|_X.\end{aligned}$$

So the remaining part of the proof is analogous to the proof of Theorem 3.3.

THEOREM 3.11. *Let $\tilde{y}_n$ and $\tilde{x}_n$ be as in* (3.8) *and* (3.9), *and suppose the assumptions of Theorem* 3.10 *hold. Then $(\tilde{x}_n)$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$ and converges to $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$. Further*

$$F(x^\delta_{c,\alpha_k}) + \frac{\alpha_k}{c}(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$$

and

$$\|\tilde{x}_n - x^\delta_{c,\alpha_k}\|_X \le \tilde{C}q_1^{2n} \quad \textit{where} \quad \tilde{C} = \frac{\tilde{\gamma}_\rho}{1 - q_1}.$$

Proof. Analogously to the proof of Theorem 3.4, one can prove that $\tilde{x}_n$ is a Cauchy sequence in $B_{\tilde{r}}(x_0)$, and hence it converges, say to $x^\delta_{c,\alpha_k} \in \overline{B_{\tilde{r}}(x_0)}$ and

$$\begin{aligned}(3.13)\qquad \left\|F(\tilde{x}_n) - z^\delta_{\alpha_k} + \frac{\alpha_k}{c}(\tilde{x}_n - x_0)\right\|_X &= \|R(x_0)(\tilde{x}_n - \tilde{y}_n)\|_X \\ &\le \|R(x_0)\|_{X\to X}\|(\tilde{x}_n - \tilde{y}_n)\|_X \le (\|F'(x_0)\|_{X\to X} + \alpha_k/c)\tilde{e}_n \\ &\le (\|F'(x_0)\|_{X\to X} + \alpha_k/c)q_1^{2n}\tilde{e}_0 \le (\|F'(x_0)\|_{X\to X} + \alpha_k/c)q_1^{2n}\tilde{\gamma}_\rho.\end{aligned}$$

Now by letting $n \to \infty$ in (3.13) we obtain $F(x^\delta_{c,\alpha_k}) + (\alpha_k/c)(x^\delta_{c,\alpha_k} - x_0) = z^\delta_{\alpha_k}$. This completes the proof.

Assume that $k_2 < \frac{1-k_0\tilde{r}}{1-c}$ and for simplicity that $\varphi_1(\alpha) \le \varphi(\alpha)$ for $\alpha > 0$.

THEOREM 3.12. *Suppose x^δ_{c,α_k} is the solution of* (3.10) *and Assumptions* 3.2, 3.8 *and* 3.9 *hold. Then*

$$\|\hat{x} - x^\delta_{c,\alpha_k}\|_X = O(\psi^{-1}(\delta)).$$

Proof. Note that $c(F(x^\delta_{c,\alpha_k}) - z^\delta_{\alpha_k}) + \alpha_k(x^\delta_{c,\alpha_k} - x_0) = 0$, so

$$\begin{aligned}
&(F'(x_0) + \alpha_k I)(x^\delta_{c,\alpha_k} - \hat{x})\\
&\quad = (F'(x_0) + \alpha_k I)(x^\delta_{c,\alpha_k} - \hat{x}) - c(F(x^\delta_{c,\alpha_k}) - z^\delta_{\alpha_k}) - \alpha_k(x^\delta_{c\alpha} - x_0)\\
&\quad = \alpha_k(x_0 - \hat{x}) + F'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) - c[F(x^\delta_{c,\alpha_k}) - z^\delta_{\alpha_k}]\\
&\quad = \alpha_k(x_0 - \hat{x}) + F'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) - c[F(x^\delta_{c,\alpha_k}) - F(\hat{x}) + F(\hat{x}) - z^\delta_{\alpha_k}]\\
&\quad = \alpha_k(x_0 - \hat{x}) - c(F(\hat{x}) - z^\delta_{\alpha_k}) + F'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) - c[F(x^\delta_{c,\alpha_k}) - F(\hat{x})].
\end{aligned}$$

Thus

$$\begin{aligned}
(3.14)\qquad &\|x^\delta_{c,\alpha_k} - \hat{x}\|_X\\
&\le \|\alpha_k(F'(x_0 + \alpha_k I)^{-1}(x_0 - \hat{x})\|_X\\
&\quad + \|(F'(x_0) + \alpha_k I)^{-1} c(F(\hat{x}) - z^\delta_{\alpha_k})\|_X\\
&\quad + \|(F'(x_0) + \alpha_k I)^{-1}[F'(x_0)(x^\delta_{c,\alpha_k} - \hat{x}) - c(F(x^\delta_{c,\alpha_k}) - F(\hat{x}))]\|_X\\
&\le \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\|_X + \|F(\hat{x}) - z^\delta_{\alpha_k}\|_X\\
&\quad + \Big\|(F'(x_0) + \alpha_k I)^{-1}\int_0^1 [F'(x_0) - cF'(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x}))](x^\delta_{c,\alpha_k} - \hat{x})\,dt\Big\|_X\\
&=: \|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\|_X + \|F(\hat{x}) - z^\delta_{\alpha_k}\|_X + \Gamma.
\end{aligned}$$

So by Assumption 3.9, we obtain

$$\begin{aligned}
(3.15)\qquad &\\
\Gamma &\le \Big\|(F'(x_0) + \alpha_k I)^{-1}\int_0^1 [F'(x_0) - F'(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x}))](x^\delta_{c,\alpha_k} - \hat{x})\,dt\Big\|_X\\
&\quad + (1-c)\Big\|(F'(x_0) + \alpha I)^{-1}F'(x_0)\int_0^1 G(\hat{x} + t(x^\delta_{c,\alpha_k} - \hat{x}), x_0)(x^\delta_{c,\alpha_k} - \hat{x})\,dt\Big\|_X\\
&\le k_0\tilde{r}\|x^\delta_{c,\alpha_k} - \hat{x}\|_X + (1-c)k_2\|x^\delta_{c,\alpha_k} - \hat{x}\|_X,
\end{aligned}$$

and hence by (3.14) and (3.15), we have

$$\begin{aligned}
\|x^\delta_{c,\alpha_k} - \hat{x}\|_X &\le \frac{\|\alpha_k(F'(x_0) + \alpha_k I)^{-1}(x_0 - \hat{x})\|_X + \|F(\hat{x}) - z^\delta_{\alpha_k}\|_X}{1 - (1-c)k_2 - k_0\tilde{r}}\\
&\le \frac{\varphi_1(\alpha_k) + \big(2 + \frac{4\mu}{\mu-1}\big)\mu\psi^{-1}(\delta)}{1 - (1-c)k_2 - k_0\tilde{r}}\\
&= O(\psi^{-1}(\delta)).
\end{aligned}$$

This completes the proof of the theorem.

The following theorem is a consequence of Theorems 3.11 and 3.12.

THEOREM 3.13. *Let* $\tilde{x}_n$ *be as in* (3.9), *and suppose that the assumptions of Theorems* 3.11 *and* 3.12 *hold. Then*

$$\|\hat{x} - \tilde{x}_n\|_X \le \tilde{C} q_1^{2n} + O(\psi^{-1}(\delta))$$

where $\tilde{C}$ *is as in Theorem* 3.11.

THEOREM 3.14. *Let* $\tilde{x}_n$ *be as in* (3.9), *and suppose that the assumptions of Theorems* 2.3, 3.11 *and* 3.12 *hold. Let*

$$n_k := \min\{n : q_1^{2n} \le \delta/\sqrt{\alpha_k}\}.$$

Then

$$\|\hat{x} - \tilde{x}_{n_k}\|_X = O(\psi^{-1}(\delta)).$$

REMARK 3.15. Let us denote by $\bar{r}_1$, $\bar{\gamma}_\rho$, $\bar{q}$, $\bar{\delta}_0$ the parameters using K_0 instead of k_0 for Case 1 (and similarly for Case 2). Then we have

$$r_1 \le \bar{r}_1, \quad \bar{\delta}_0 \le \delta_0, \quad \bar{\gamma}_\rho \le \gamma_\rho, \quad q \le \bar{q}.$$

Moreover, strict inequalities hold in the preceding estimates if $k_0 < K_0$. Let $h_0 = 4k_0\gamma_\rho$ and $h = 4K_0\bar{\gamma}_\rho$. We can certainly choose γ_ρ sufficiently close to $\bar{\gamma}_\rho$. Then we have $h \le 1 \Rightarrow h_0 \le 1$ but not necessarily vice versa unless $k_0 = K_0$ and $\gamma_\rho = \bar{\gamma}_\rho$. Finally, $h_0/h \to 0$ as $k_0/K_0 \to 0$. The last estimate shows by how many times our new approach using k_0 can expand the applicability of the old approach using K_0 for these methods. Hence, all the above justifies the claims made in the introduction of the paper. Finally we note that the results obtained here are useful even if Assumption 3.1 is satisfied but the sufficient convergence condition $h \le 1$ is not satisfied but $h_0 \le 1$ is satisfied. Indeed, we can proceed with the iterative method described in Case (1) (or Case (2)) until a finite step N such that $h \le 1$ with x^δ_{N+1,α_N} as a starting point for faster methods such as (1.6). Such an approach has already been employed in [2], [5] and [4] where the modified Newton's method is used as a predictor for Newton's method.

4. Algorithm.

Note that for $i, j \in \{0, 1, \ldots, M\}$,

$$z^\delta_{\alpha_i} - z^\delta_{\alpha_j} = (\alpha_j - \alpha_i)(K^*K + \alpha_j I)^{-1}(K^*K + \alpha_i I)^{-1}[K^*(y^\delta - KF(x_0))].$$

The algorithm for implementing the iterative methods considered in Section 3 involves the following steps:

- $\alpha_0 = \delta^2$;
- $\alpha_i = \mu^{2i}\alpha_0, \ \mu > 1$;
- solve $(K^*K + \alpha_i I)w_i = K^*(y^\delta - KF(x_0))$ for w_i;
- solve $(K^*K + \alpha_j I)z_{ij} = (\alpha_j - \alpha_i)w_i$ for z_{ij}, $j < i$;
- if $\|z_{ij}\|_X > 4/\mu^j$, then take $k = i - 1$;
- otherwise, repeat with $i + 1$ in place of i;

- choose $n_k = \min\{n : q^{2n} \le \delta/\sqrt{\alpha_k}\}$ in Case (1) and $n_k = \min\{n : q_1^{2n} \le \delta/\sqrt{\alpha_k}\}$ in Case (2);
- solve x_{n_k} using the iteration (3.2) or $\tilde{x}_{n_k}$ using the iteration (3.9).

5. Numerical examples. We present five numerical examples in this section. First, we consider two examples to illustrate the algorithm considered in the above sections. We apply the algorithm by choosing a sequence (V_N) of finite-dimensional subspaces of X with $\dim V_N = N+1$. Precisely V_N is the space of linear splines in a uniform grid of $N+1$ points in $[0,1]$. Then we present two examples where Assumption 3.2 is satisfied but Assumption 3.1 is not. In the last example we show that k_0/K_0 can be arbitrarily small.

EXAMPLE 5.1. In this example for Case (1), we consider the operator $KF : D(KF) \subseteq L^2(0,1) \to L^2(0,1)$ with $K : L^2(0,1) \to L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)\,ds$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1, \end{cases}$$

and

$$F : D(F) \subseteq L^2(0,1) \to L^2(0,1)$$

defined by $F(u) := u^3$. Then the Fréchet derivative of F is given by $F'(u)w = 3(u)^2 w$.

In our computation, we take

$$y(t) = \frac{837t}{6160} - \frac{t^2}{16} - \frac{t^{11}}{110} - \frac{3t^5}{80} - \frac{3t^8}{112} \quad \text{and} \quad y^\delta = y + \delta.$$

Then the exact solution is

$$\hat{x}(t) = 0.5 + t^3.$$

We use

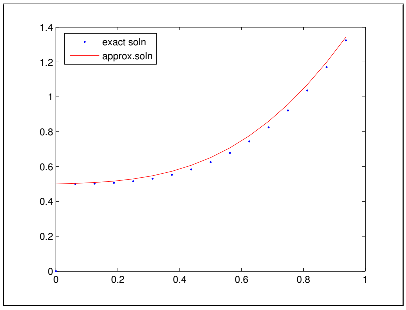
$$x_0(t) = 0.5 + t^3 - \frac{3}{56}(t - t^8)$$

as our initial guess.

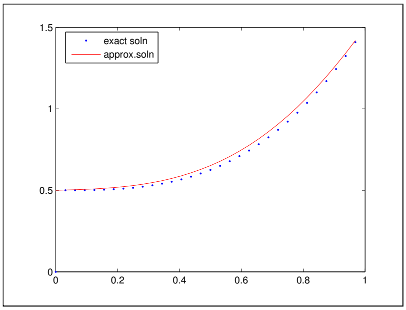
We choose $\alpha_0 = (1.3)^2\delta^2$, $\mu = 1.2$, $\delta = 0.0667$, the Lipschitz constant k_0 equals approximately 0.23 and $r = 1$, so that $q = k_0 r = 0.23$. The iterations and corresponding error estimates are given in Table 1. The plots of the exact solution and the approximate solution obtained are given in Figures 1 and 2. The last column of Table 1 shows that the error $\|x_{n_k} - \hat{x}\|_X$ is of order $O(\delta^{1/2})$.

Table 1

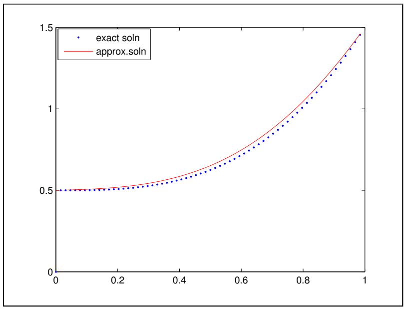
N	k	α_k	$\|x_{n_k} - \hat{x}\|_X$	$\frac{\|x_{n_k} - \hat{x}\|_X}{\delta^{1/2}}$
16	4	0.0231	0.5376	2.0791
32	4	0.0230	0.5301	2.0523
64	4	0.0229	0.5257	2.0359
128	4	0.0229	0.5234	2.0270
256	4	0.0229	0.5222	2.0224
512	4	0.0229	0.5216	2.0200
1024	4	0.0229	0.5213	2.0188



$N = 16$



$N = 32$



$N = 64$

Fig. 1. Curves of the exact and approximate solutions

Example 5.2. In this example for Case (2), we consider the operator $KF : D(KF) \subseteq L^2(0,1) \to L^2(0,1)$ with $K : L^2(0,1) \to L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)\,ds$$

and $F : D(F) \subseteq L^2(0,1) \to L^2(0,1)$ defined by

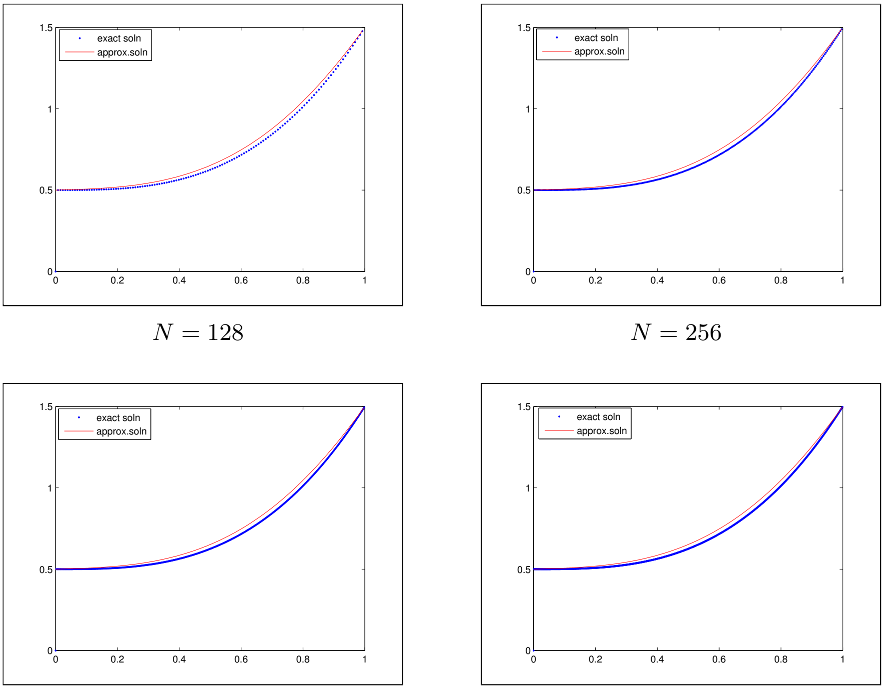
$$F(u) := \int_0^1 k(t,s)u^3(s)\,ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases}$$

Then for all $x(t), y(t)$ with $x(t) > y(t)$ (see [30, Section 4.3]),

$$\langle F(x) - F(y), x - y\rangle = \int_0^1 \Big[\int_0^1 k(t,s)(x^3 - y^3)(s)\,ds\Big](x-y)(t)\,dt \ge 0.$$



$N = 128$ $N = 256$

$N = 512$ $N = 1024$

Fig. 2. Curves of the exact and approximate solutions

Thus the operator F is monotone. Its Fréchet derivative is given by

$$F'(u)w = 3\int_0^1 k(t,s)u(s)^2 w(s)\,ds.$$

So for any $u \in B_r(x_0)$, where $x_0(s) \geq k_3 > 0$ for all $s \in (0,1)$, we have

$$F'(u)w = F'(x_0)G(u,x_0)w,$$

where $G(u,x_0) = (u/x_0)^2$.

In our computation, we take

$$y(t) = \frac{1}{110}\left(\frac{t^{13}}{156} - \frac{t^3}{6} + \frac{25t}{156}\right) \quad \text{and} \quad y^\delta = y + \delta.$$

Then the exact solution is

$$\hat{x}(t) = t^3.$$

We use

$$x_0(t) = t^3 + \frac{3}{56}(t - t^8)$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

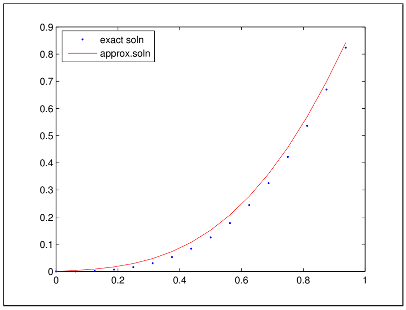
$$x_0 - \hat{x} = \frac{3}{56}(t - t^8) = F'(x_0)\left(\frac{t^6}{x_0(t)^2}\right) = \varphi_1(F'(x_0))\left(\frac{t^6}{x_0(t)^2}\right)$$

where $\varphi_1(\lambda) = \lambda$. Thus we expect to have an accuracy of order at least $O(\delta^{1/2})$.

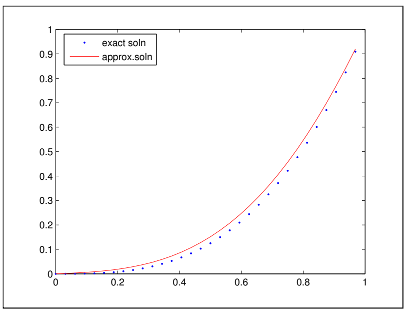
We choose $\alpha_0 = (1.3)\delta$, $\delta = 0.0667 =: c$, the Lipschitz constant k_0 equals approximately 0.21 as in [30] and $\tilde{r} = 1$, so that $q_1 = k_0\tilde{r} = 0.21$. The results of the computation are presented in Table 2. The plots of the exact solution and the approximate solution obtained are given in Figures 3 and 4.

Table 2

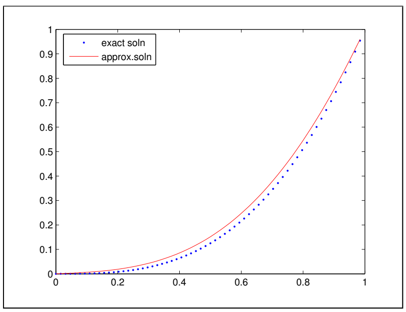
N	k	α_k	$\|\tilde{x}_{n_k} - \hat{x}\|_X$	$\frac{\|\tilde{x}_{n_k} - \hat{x}\|_X}{\delta^{1/2}}$
8	4	0.0494	0.1881	0.7200
16	4	0.0477	0.1432	0.5531
32	4	0.0473	0.1036	0.4010
64	4	0.0472	0.0726	0.2812
128	4	0.0471	0.0491	0.1900
256	4	0.0471	0.0306	0.1187
512	4	0.0471	0.0140	0.0543
1024	4	0.0471	0.0133	0.0515



$N = 16$



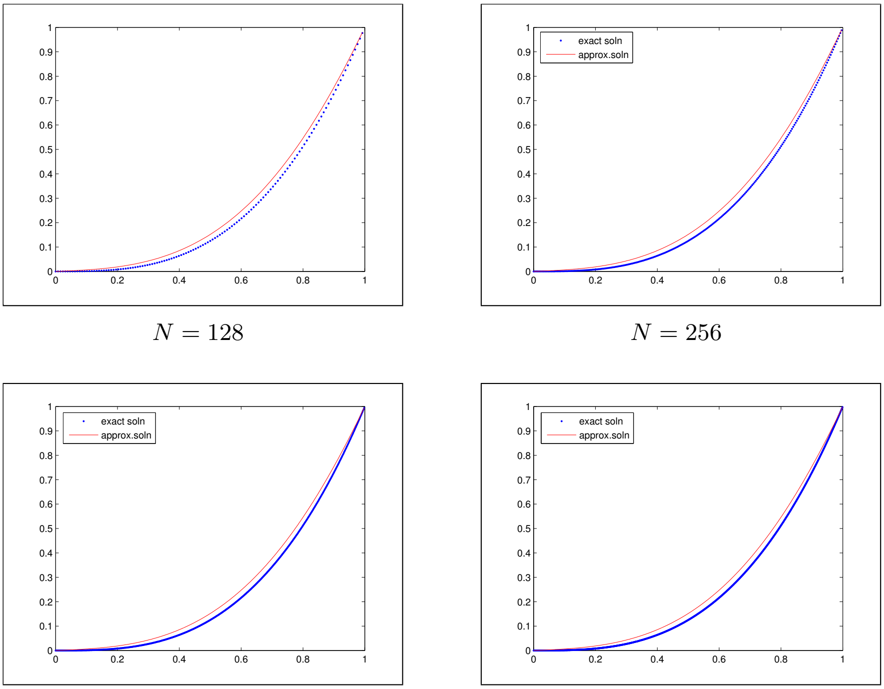
$N = 32$



$N = 64$

Fig. 3. Curves of the exact and approximate solutions

In the next two cases, we present examples for nonlinear equations where Assumption 3.2 is satisfied but Assumption 3.1 is not.



$N = 128$ $N = 256$

$N = 512$ $N = 1024$

Fig. 4. Curves of the exact and approximate solutions

EXAMPLE 5.3. Let $X = Y = \mathbb{R}$, $D = [0, \infty)$, $x_0 = 1$ and define a function F on D by

$$F(x) = \frac{x^{1+1/i}}{1+1/i} + c_1 x + c_2, \tag{5.1}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. Hence, Assumption 3.1 is not satisfied. However the central Lipschitz condition (Assumption 3.2) holds for $k_0 = 1$.

Indeed, we have

$$\|F'(x) - F'(x_0)\|_X = |x^{1/i} - x_0^{1/i}| = \frac{|x - x_0|}{x_0^{(i-1)/i} + \cdots + x^{(i-1)/i}},$$

so

$$\|F'(x) - F'(x_0)\|_X \leq k_0 |x - x_0|.$$

EXAMPLE 5.4. We consider the integral equations

$$u(s) = f(s) + \lambda \int_a^b G(s,t) u(t)^{1+1/n}\, dt, \qquad n \in \mathbb{N}. \tag{5.2}$$

Here, f is a given continuous function satisfying $f(s) > 0$ for $s \in [a, b]$, λ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \lambda u^{1+1/n}, \qquad u(a) = f(a),\ u(b) = f(b).$$

Such problems have been considered in [1–5].

Equations (5.2) generalize equations of the form

$$u(s) = \int_a^b G(s,t)u(t)^n\,dt \tag{5.3}$$

studied in [1–5]. Instead of (5.2) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \qquad \Omega = \{u \in C[a,b] : u(s) \geq 0,\ s \in [a,b]\}$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t)u(t)^{1+1/n}\,dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda\left(1 + \frac{1}{n}\right)\int_a^b G(s,t)u(t)^{1/n}v(t)\,dt, \qquad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1]$, $G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\|_{C[a,b]\to C[a,b]} = |\lambda|\left(1 + \frac{1}{n}\right)\int_a^b x(t)^{1/n}\,dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\|_{C[a,b]\to C[a,b]} \leq L_1\|x - y\|_{C[a,b]},$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n}\,dt \leq L_2 \max_{x\in[0,1]} x(s) \tag{5.4}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = t/j, \qquad j \geq 1,\ t \in [0,1].$$

If these are substituted into (5.4), we have

$$\frac{1}{j^{1/n}(1 + 1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1 + 1/n), \qquad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (5.4) is not satisfied in this case. Hence Assumption 3.1 is not satisfied. However, Assumption 3.2 holds. To show this, let

$$x_0(t) = f(t), \qquad \gamma = \min_{s\in[a,b]} f(s), \qquad \alpha > 0.$$

Then for $v \in \Omega$,

$$\begin{aligned}
&\|[F'(x) - F'(x_0)]v\|_{C[a,b]} \\
&\qquad = |\lambda|\left(1+\frac{1}{n}\right) \max_{s\in[a,b]} \left|\int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)\,dt\right| \\
&\qquad \le |\lambda|\left(1+\frac{1}{n}\right) \max_{s\in[a,b]} G_n(s,t)
\end{aligned}$$

where

$$G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n}f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}}\|v\|_{C[a,b]}.$$

Hence,

$$\begin{aligned}
\|[F'(x) - F'(x_0)]v\|_{C[a,b]} &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)\,dt\,\|x - x_0\|_{C[a,b]} \\
&\le k_0\|x - x_0\|_{C[a,b]},
\end{aligned}$$

where

$$k_0 = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} N$$

and

$$N = \max_{s\in[a,b]} \int_a^b G(s,t)\,dt.$$

Then Assumption 3.2 holds for sufficiently small λ.

Example 5.5. Define a scalar function F by

$$F(x) = d_0 x + d_1 + d_2 \sin e^{d_3 x}, \qquad x_0 = 0,$$

where $d_i,\ i = 0,1,2,3$, are given parameters. Then it can easily be seen that for d_3 large and d_2 sufficiently small, k_0/K_0 can be arbitrarily small.

6. Conclusion. We presented an iterative method which is a combination of a modified Newton method and Tikhonov regularization to obtain an approximate solution for a nonlinear ill-posed Hammerstein-type operator equation $KF(x) = y$, with the available noisy data y^δ in place of the exact data y. In fact we considered two cases, where $F'(x_0)^{-1}$ exists and where F is monotone but $F'(x_0)^{-1}$ does not exist. In both cases, the derived error estimates using an a priori and balancing principle are of optimal order with respect to the general source condition. The results of computational experiments confirm the reliability of our approach.

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