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STABILITY OF FINITE DIFFERENCE SCHEMES FOR CERTAIN PROBLEMS IN BIOLOGY

Abstract. We consider a generalized 1-D von Foerster equation. We present two discretization methods for the initial value problem and study stability of finite difference schemes on regular meshes.

1. Introduction. Suppose that $c\colon E \times \mathbb{R}_+ \to \mathbb{R}$ and $\lambda\colon E \times \mathbb{R}_+^2 \to \mathbb{R}$, where $E = [0,a] \times \mathbb{R}_+$, $a > 0$. We study the initial-boundary value problem

$$\partial_t u(t,x) + c(t,x,z(t))\partial_x u(t,x) = \lambda(t,x,u(t,x),z(t)), \tag{1}$$

where

$$z(t) = z[u(t,\cdot)] = \int_0^\infty u(t,x)\,dx, \qquad t \in [0,a]. \tag{2}$$

Equation (1), which generalizes the classical von Foerster biological model [3, 4, 13], will be considered with the initial condition

$$u(0,x) = v(x), \qquad x \in \mathbb{R}_+, \tag{3}$$

where $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a given continuous integrable function. If $c(t,x,z(t)) \geq 0$, then the well-posedness of problem (1)–(3) demands the condition $c(t,0,z(t)) \leq 0$, that is: the characteristics either go out of the set through the lateral boundary or meet the boundary and remain there.

Some existence, uniqueness and qualitative theorems for the von Foerster problem and other problems of mathematical biology have been established in [1]–[7], [9] and [12]–[13]. We are interested in discretization of problem (1)–(3). The Lax equivalence theorem splits this task into investigating stability and consistency. We analyze convergence of finite difference schemes for problem (1)–(3) on rectangular meshes. Some results connected with

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stability and convergence of finite difference schemes for differential and functional differential equalities can be found in [8, 11, 15].

The von Foerster equation is a classical model for description of age-dependent population dynamics. In the biological interpretation of this equation, $t \geq 0$ denotes time, and $x \geq 0$ the maturity of the population. The unknown function $u(t,x) \geq 0$ stands for the density of the population, hence it has to be integrable with respect to x for all $t \geq 0$. The function $z(t)$ given by (2) denotes the global number of individuals at time $t \geq 0$. Although it is clear that the maturity of the population is bounded, we consider the case where the maturity is arbitrarily large.

For some historical background of mathematical biology models we refer to [1, 5, 12, 16, 17] and [6, 7, 10, 14].

The paper is organized as follows. First we introduce the mesh in the domain E and present two discretization methods for problem (1)–(3): by using forward and backward spatial difference quotients. The nonlocal term is approximated by the extended trapezoidal rule and its finite version. We introduce appropriate normed spaces and give main assumptions. Next, we prove stability lemmas for both schemes by considering perturbations with respect to the right side and the initial condition. We prove the stability of these schemes with respect to the quadrature, that is, check how sensitive the scheme is with regard to restricting the extended trapezoidal rule to certain finite subregions. The main stability theorem is a simple consequence of these lemmas. Finally, we prove a consistency theorem of backward schemes and give illustrative numerical examples.

2. Discretization of the differential problem. We introduce in $E = [0,a] \times \mathbb{R}_+$ a rectangular mesh as follows. For two given steps $h_0 \in (0,a)$ and $h_1 \in (0,\infty)$ we denote by $(t^{(i)}, x^{(j)})$ the knots $(h_0 i, h_1 j)$. Define $\mathcal{N}_h = \{(t^{(i)}, x^{(j)}) : i,j \in \mathbb{N}\}$ and $E_h = E \cap \mathcal{N}_h$. The value of any discrete function u: $E_h \to \mathbb{R}_+$ at the knot $(t^{(i)}, x^{(j)})$ will be denoted by $u^{(i,j)} = u(t^{(i)}, x^{(j)})$.

Define the discrete operators $\delta_0, \delta_+, \delta_-, Q_h$ by

$$\delta_0 u^{(i,j)} = \frac{u^{(i+1,j)} - u^{(i,j)}}{h_0},$$

$$\delta_+ u^{(i,j)} = \frac{u^{(i,j+1)} - u^{(i,j)}}{h_1}, \qquad \delta_- u^{(i,j)} = \frac{u^{(i,j)} - u^{(i,j-1)}}{h_1},$$

$$(Q_h u)_i = h_1 \sum_{j=0}^{\infty} \frac{u^{(i,j)} + u^{(i,j+1)}}{2} \qquad \text{(extended trapezoidal rule).}$$

The operator $\delta_0 u^{(i,j)}$ approximates the derivative $\partial_t u(t^{(i)}, x^{(j)})$, whereas δ_+ and δ_- approximate $\partial_x u(t^{(i)}, x^{(j)})$. The quadrature $(Q_h u)_i$ is a second-order approximation of the integral (2) at $t = t^{(i)}$. When performing practical computations we replace $(Q_h u)_i$ by

$$(Q_h^{N_h} u)_i = h_1 \sum_{j=0}^{N_h} \frac{u^{(i,j)} + u^{(i,j+1)}}{2},$$

where N_h is a sufficiently large number (usually proportional to $\frac{1}{h_1} \log \frac{1}{h_1}$). It is important that $h_1 N_h \to \infty$ as $h_1 \to 0$.

We consider two finite difference problems for (1)–(3). The first *forward scheme* consists of the difference equation

$$\delta_0 u^{(i,j)} + c^{(i,j)}[z]\delta_+ u^{(i,j)} = \lambda^{(i,j)}[u,z] \quad \text{on } E_h^+ \tag{4}$$

with

$$z^{(i)} = (Q_h u)_i, \tag{5}$$

where

$$c^{(i,j)}[z] = c(t^{(i)}, x^{(j)}, z^{(i)}), \qquad \lambda^{(i,j)}[u,z] = \lambda(t^{(i)}, x^{(j)}, u^{(i,j)}, z^{(i)}),$$
$$E_h^+ = E \cap \mathcal{N}_h^+, \qquad \mathcal{N}_h^+ = \{(t^{(i)}, x^{(j)}) : i, j \geq 0\},$$

and the initial condition

$$u^{(0,j)} = v^{(j)} \quad \text{for } j \in \mathbb{N}. \tag{6}$$

The *backward scheme* consists of the difference equation

$$\delta_0 u^{(i,j)} + c^{(i,j)}[z]\delta_- u^{(i,j)} = \lambda^{(i,j)}[u,z] \quad \text{on } E_h^-, \tag{7}$$

where $E_h^- = E \cap \mathcal{N}_h^-$, $\mathcal{N}_h^- = \{(t^{(i)}, x^{(j)}) : i \geq 0,\ j > 0\}$, and

$$\delta_0 u^{(i,0)} = \lambda^{(i,0)}[u,z], \qquad u^{(0,0)} = v^{(0)} \qquad (i = 0, 1, \ldots, N_0 - 1), \tag{8}$$

with the quadrature (5) and the initial condition

$$u^{(0,j)} = v^{(j)} \quad \text{for } j \in \mathbb{N}. \tag{9}$$

The analysis of the finite difference problems (4)–(6) and (7)–(9) can be divided into two steps: stability and consistency. We are mainly interested in stability. The treatment of consistency is standard.

Denote by $L^\infty(\mathbb{R}_+)$ and $L^1(\mathbb{R}_+)$ the spaces of essentially bounded measurable functions and Lebesgue integrable functions defined on $\mathbb{R}_+$, and let $C(X, \mathbb{R})$ denote the class of all continuous functions u: $X \to \mathbb{R}$.

The problem considered is nonlocal, hence we have to introduce the following normed spaces. In the space l^∞ of bounded sequences $\psi = (\psi_j)_{j \in \mathbb{N}}$, we have the norm

$$\|\psi\|_\infty = \sup_{j \in \mathbb{N}} |\psi_j| \quad \text{for } (\psi_j) \in l^\infty.$$

The space l^1 of summable sequences $\varphi = (\varphi_j)_{j\in\mathbb{N}}$ is equipped with the norm

$$\|\varphi\|_1 = h_1 \sum_{j=0}^{\infty} |\varphi_j| \quad \text{for } (\varphi_j) \in l^1.$$

DEFINITION 1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is l^1-*bounded* if

$$\exists_{C>0}\ \forall_{h_1>0} \quad \|f_{|E_{h_1}}\|_1 \le C,$$

where E_{h_1} is the mesh generated by the step h_1.

The l^1-boundedness is slightly stronger than the usual Riemann integrability condition.

We state the main assumptions on the given functions:

ASSUMPTION $[V]$. The initial function $v\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, bounded and l^1-bounded.

ASSUMPTION $[C]$. The function $c\colon E \times \mathbb{R}_+ \to \mathbb{R}$ is bounded, continuous, and there exists $L_c > 0$ such that

$$|c(t,x,q) - c(t,\overline{x},\overline{q})| \le L_c(|x-\overline{x}| + |q-\overline{q}|)$$

for $(t,x,q),(t,\overline{x},\overline{q}) \in E \times \mathbb{R}_+$. Moreover,

$$c(t,0,q) \le 0 \quad \text{for } t \in [0,a],\ q \in \mathbb{R}_+.$$

ASSUMPTION $[\Lambda]$. The function $\lambda\colon E \times \mathbb{R}_+^2 \to \mathbb{R}$ is continuous and there is a constant $L_\lambda > 0$ and an l^1-bounded function $L_z(\cdot) \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ such that

$$|\lambda(t,x,\overline{p},\overline{q}) - \lambda(t,x,p,q)| \le L_\lambda|\overline{p}-p| + L_z(x)|\overline{q}-q|$$

for $(t,x) \in E$ and $p,q,\overline{p},\overline{q} \in \mathbb{R}_+$.

3. Stability of forward and backward schemes. To prove the stability of problems (4)–(6) and (7)–(9) we consider a perturbed scheme, which for the forward scheme looks as follows:

$$\delta_0\overline{u}^{(i,j)} + c^{(i,j)}[\overline{z}]\delta_+\overline{u}^{(i,j)} = \lambda^{(i,j)}[\overline{u},\overline{z}] + \xi^{(i,j)} \quad \text{on } E_h^+ \tag{10}$$

with

$$\overline{z}^{(i)} = (Q_h\overline{u})_i, \tag{11}$$

and perturbed initial conditions

$$\overline{u}^{(0,j)} = v^{(j)} + \widehat{\xi}^{(0,j)} \quad \text{for } j \in \mathbb{N}, \tag{12}$$

where the perturbations $\xi^{(i,j)}$ are the discretization errors for (1) and $\widehat{\xi}^{(0,j)}$ are perturbations of (3).

We also consider the perturbed difference equations for the backward scheme on the mesh E_h^-:

$$\delta_0\overline{u}^{(i,j)} + c^{(i,j)}[\overline{z}]\delta_-\overline{u}^{(i,j)} = \lambda^{(i,j)}[\overline{u},\overline{z}] + \xi^{(i,j)} \tag{13}$$

with the perturbed quadrature (11), initial condition (12) and boundary condition

$$\delta_0 \overline{u}^{(i,0)} = \lambda^{(i,0)}[\overline{u}, \overline{z}] + \xi^{(i,0)} \quad \text{for } i = 0, 1, \ldots, N_0 - 1. \tag{14}$$

We now prove the stability lemma for scheme (4)–(6).

LEMMA 2. *Suppose that* $u, \overline{u} : E_h \to \mathbb{R}_+$ *and*

(i) u *is a bounded and* l^1*-bounded solution of problem* (4)–(6), *satisfying the discrete Lipschitz condition*

$$|u^{(i,j+1)} - u^{(i,j)}| \le L_u h_1 \quad \text{with some } L_u > 0,$$

(ii) $\overline{u}$ *is a bounded and* l^1*-bounded solution of problem* (10)–(12) *with*

$$\|\xi^{(i)}\|_\infty \le C_h, \quad \|\widehat{\xi}^{(0)}\|_\infty \le C_{0,h}, \quad \|\xi^{(i)}\|_1 \le \overline{C}_h, \quad \|\widehat{\xi}^{(0)}\|_1 \le \overline{C}_{0,h},$$

$i = 0, 1, \ldots, N_0$, *where* $C_{0,h}, C_h, \overline{C}_{0,h}, \overline{C}_h \to 0$ *as* $\|h\| = \max\{h_0, h_1\} \to 0$,

(iii) c *satisfies the stability conditions*

$$1 + \frac{h_0}{h_1} c(t, x, q) \ge 0, \quad c(t, x, q) \le 0 \quad \text{for } (t, x, q) \in E \times \mathbb{R}_+,$$

(iv) *Assumptions* $[C]$ *and* $[\Lambda]$ *hold.*

Then $|\overline{u}^{(i,j)} - u^{(i,j)}|$ *converges uniformly to* 0 *as* $\|h\| \to 0$ *in both the supremum norm and* l^1 *norm.*

REMARK 3. If c does not depend on the last variable, then the discrete Lipschitz condition in (i) can be omitted.

Proof. Set $\omega^{(i,j)} = \overline{u}^{(i,j)} - u^{(i,j)}$ (the error of the scheme). Subtracting (10) and (4) we obtain the explicit recurrence error equation

$$\begin{aligned} \omega^{(i+1,j)} = \omega^{(i,j)} \left(1 + \frac{h_0}{h_1} c^{(i,j)}[\overline{z}]\right) &- \frac{h_0}{h_1} c^{(i,j)}[\overline{z}] \omega^{(i,j+1)} \\ &+ \frac{h_0}{h_1} (u^{(i,j+1)} - u^{(i,j)})(c^{(i,j)}[z] - c^{(i,j)}[\overline{z}]) \\ &+ h_0(\lambda^{(i,j)}[\overline{u}, \overline{z}] - \lambda^{(i,j)}[u, z]) + h_0 \xi^{(i,j)}. \end{aligned} \tag{15}$$

Using Assumption $[\Lambda]$ and the stability condition (iii) we get

$$\begin{aligned} |\omega^{(i+1,j)}| \le |\omega^{(i,j)}| \left(1 + \frac{h_0}{h_1} c^{(i,j)}[\overline{z}]\right) &- \frac{h_0}{h_1} c^{(i,j)}[\overline{z}] |\omega^{(i,j+1)}| \\ &+ \frac{h_0}{h_1} |u^{(i,j+1)} - u^{(i,j)}| \, |c^{(i,j)}[z] - c^{(i,j)}[\overline{z}]| \\ &+ h_0 L_\lambda |\omega^{(i,j)}| + h_0 L_z(x^{(j)}) |\overline{z}^{(i)} - z^{(i)}| + h_0 |\xi^{(i,j)}|. \end{aligned} \tag{16}$$

Recall that

$$z^{(i)} = \frac{h_1}{2}\sum_{j=0}^{\infty}(u^{(i,j+1)} + u^{(i,j)}) = \frac{h_1}{2}\,u^{(i,0)} + h_1\sum_{j=1}^{\infty}u^{(i,j)},$$

$$\overline{z}^{(i)} = \frac{h_1}{2}\,\sum_{j=0}^{\infty}(\overline{u}^{(i,j+1)} + \overline{u}^{(i,j)}) = \frac{h_1}{2}\,\overline{u}^{(i,0)} + h_1\sum_{j=1}^{\infty}\overline{u}^{(i,j)}.$$

Hence

$$|\overline{z}^{(i)} - z^{(i)}| \le h_1\sum_{j=0}^{\infty}|\overline{u}^{(i,j)} - u^{(i,j)}| = \|\omega^{(i)}\|_1. \tag{17}$$

By (17), Assumptions $[C]$, $[\varLambda]$ and (i) we write (16) in the form

$$|\omega^{(i+1,j)}| \le |\omega^{(i,j)}|\left(1 + \frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]\right) - \frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]|\omega^{(i,j+1)}|$$
$$+ h_0L_uL_c\|\omega^{(i)}\|_1 + h_0L_\lambda|\omega^{(i,j)}| + h_0\|L_z\|_\infty\|\omega^{(i)}\|_1 + h_0|\xi^{(i,j)}|.$$

Hence

$$\|\omega^{(i+1)}\|_\infty \le (1 + h_0L_\lambda)\|\omega^{(i)}\|_\infty + h_0(L_uL_c + \|L_z\|_\infty)\|\omega^{(i)}\|_1 + h_0\|\xi^{(i)}\|_\infty. \tag{18}$$

From Assumption $[C]$ we obtain

$$\sum_{j=0}^{\infty}c^{(i,j)}[\overline{z}](|\omega^{(i,j)}| - |\omega^{(i,j+1)}|) \le L_c\|\omega^{(i)}\|_1. \tag{19}$$

Multiplying (16) by h_1, summing over $j = 0, 1, \ldots$ and taking into consideration (i), (17), (19), Assumptions $[C]$ and $[\varLambda]$ we obtain

$$\|\omega^{(i+1)}\|_1 \le \|\omega^{(i)}\|_1(1 + h_0L_\lambda + h_0L_c + 2h_0L_cU + h_0\|L_z\|_1) + h_0\|\xi^{(i)}\|_1, \tag{20}$$

where

$$U = \max_{i=0,\ldots,N_0} h_1\sum_{j=0}^{\infty}u^{(i,j)}.$$

Consider the comparison recurrence equations with respect to (18) and (20):

$$\begin{aligned}\eta^{(i+1)} &= \eta^{(i)}(1 + h_0L_\lambda) + h_0(L_uL_c + \|L_z\|_\infty)\widetilde{\eta}^{(i)} + h_0\|\xi^{(i)}\|_\infty\\ \widetilde{\eta}^{(i+1)} &= \widetilde{\eta}^{(i)}(1 + h_0L_\lambda + h_0L_c + 2h_0L_cU + h_0\|L_z\|_1) + h_0\|\xi^{(i)}\|_1.\end{aligned} \tag{21}$$

Taking into consideration the initial conditions

$$\|\omega^{(0)}\|_1 \le \widetilde{\eta}^{(0)} = \overline{C}_{0,h} \to 0, \qquad \|\omega^{(0)}\|_\infty \le \eta^{(0)} = C_{0,h} \to 0,$$

we obtain the estimates $\|\omega^{(i)}\|_\infty \le \eta^{(i)}$ and $\|\omega^{(i)}\|_1 \le \widetilde{\eta}^{(i)}$, hence the solutions of (18), (20) satisfy

$$\|\omega^{(i)}\|_1 \le \widetilde{\eta}^{(i)} \le e^{La}\biggl(\overline{C}_{0,h} + \frac{\overline{C}_h}{L}\biggr) =: \widehat{C}_h,$$

$$\|\omega^{(i)}\|_\infty \le \eta^{(i)} \le e^{L_\lambda a}\biggl(C_{0,h} + \frac{(L_u L_c + \|L_z\|_\infty)\widehat{C}_h + C_h}{L_\lambda}\biggr)$$

for $i = 0, 1, \ldots, N_0$, where $L = L_\lambda + L_c + 2L_u U + \|L_z\|_1$. The right-hand sides of these estimates are derived from (21), because $\eta^{(i)} \le \eta^{(N_0)}$, $\widetilde{\eta}^{(i)} \le \widetilde{\eta}^{(N_0)}$ and $(1 + h_0 L)^i \le e^{h_0 iL} \le e^{aL}$. ■

We now prove the stability lemma for the backward scheme (7)–(9). The discretization error for problem (1)–(3) on the mesh E_h^- is defined in the same way as for the forward scheme.

LEMMA 4. *Suppose that* $u, \overline{u}\colon E_h \to \mathbb{R}_+$ *and*

(i) u *is a bounded and* l^1*-bounded solution of problem* (7)–(9), *satisfying the discrete Lipschitz condition with some* $L_u > 0$,

(ii) $\overline{u}$ *is a bounded and* l^1*-bounded solution of problem* (11)–(14) *with*

$$\|\xi^{(i)}\|_\infty \le C_h, \quad \|\widehat{\xi}^{(0)}\|_\infty \le C_{0,h}, \quad \|\xi^{(i)}\|_1 \le \overline{C}_h, \quad \|\widehat{\xi}^{(0)}\|_1 \le \overline{C}_{0,h},$$

where $C_{0,h}, C_h, \overline{C}_{0,h}, \overline{C}_h \to 0$ *as* $\|h\| \to 0$,

(iii) c *satisfies the stability conditions*

$$1 - \frac{h_0}{h_1}\, c(t,x,q) \ge 0, \qquad c(t,x,q) \ge 0 \quad \textit{for } (t,x,q) \in E \times \mathbb{R}_+,$$

(iv) *Assumptions* $[C]$ *and* $[\Lambda]$ *hold.*

Then $|\overline{u}^{(i,j)} - u^{(i,j)}|$ *converges uniformly to* 0 *as* $\|h\| \to 0$ *in both the supremum norm and* l^1 *norm.*

REMARK 5. As in the forward scheme, if c does not depend on the last variable, then the discrete Lipschitz condition in (i) can be omitted.

REMARK 6. From Assumption $[C]$ and (iii) it follows that $c(t, 0, q) = 0$ for $t \in [0, a]$, $q \in \mathbb{R}_+$, that is, the characteristic which meets the lateral boundary at the point $(t_0, 0)$ is tangent to it there. By the Lipschitz condition, $\eta(t) \equiv 0$ is the only characteristic which meets the lateral boundary.

Proof of Lemma 4. Set $\omega^{(i,j)} = \overline{u}^{(i,j)} - u^{(i,j)}$. As in Lemma 2, we obtain

$$\begin{aligned} \omega^{(i+1,j)} &= \omega^{(i,j)}\biggl(1 - \frac{h_0}{h_1}\, c^{(i,j)}[\overline{z}]\biggr) + \frac{h_0}{h_1}\, c^{(i,j)}[\overline{z}]\omega^{(i,j-1)} \\ &\quad + \frac{h_0}{h_1}\,(u^{(i,j)} - u^{(i,j-1)})(c^{(i,j)}[z] - c^{(i,j)}[\overline{z}]) \\ &\quad + h_0(\lambda^{(i,j)}[\overline{u}, \overline{z}] - \lambda^{(i,j)}[u, z]) + h_0 \xi^{(i,j)} \end{aligned} \tag{22}$$

for $j = 1, 2, \ldots$. By Assumption $[\Lambda]$ and (iii) we obtain

$$(23)\quad |\omega^{(i+1,j)}| \le |\omega^{(i,j)}|\left(1-\frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]\right)+\frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]|\omega^{(i,j-1)}| + \frac{h_0}{h_1}\,|u^{(i,j)}-u^{(i,j-1)}|\,|c^{(i,j)}[z]-c^{(i,j)}[\overline{z}]| + h_0L_\lambda|\omega^{(i,j)}|+h_0L_z(x^{(j)})|\overline{z}^{(i)}-z^{(i)}|+h_0|\xi^{(i,j)}|$$

for $j=1,2,\ldots$. Using Assumptions $[C]$, $[\Lambda]$, (i) and (17) we get

$$(24)\quad |\omega^{(i+1,j)}| \le |\omega^{(i,j)}|\left(1-\frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]\right)+\frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]|\omega^{(i,j-1)}| + h_0L_uL_c\|\omega^{(i)}\|_1+h_0L_\lambda|\omega^{(i,j)}| + h_0\|L_z\|_\infty\|\omega^{(i)}\|_1+h_0|\xi^{(i,j)}|$$

for $j=1,2,\ldots$. We also write an analogous conclusion for (8):

$$(25)\quad |\omega^{(i+1,0)}| \le |\omega^{(i,0)}|+h_0L_\lambda|\omega^{(i,0)}|+h_0L_z(0)\|\omega^{(i)}\|_1+h_0|\xi^{(i,0)}|.$$

Using the stability condition (iii) and adding (24) and (25) we obtain the recurrence inequality

$$(26)\quad \|\omega^{(i+1)}\|_\infty \le (1+h_0L_\lambda)\|\omega^{(i)}\|_\infty+h_0(L_uL_c+\|L_z\|_\infty)\|\omega^{(i)}\|_1 + h_0\|\xi^{(i)}\|_\infty.$$

Multiplying (23) by h_1, summing over $j=1,2,\ldots$, using Assumption $[C]$ and (17), we obtain

$$(27)\quad h_1\sum_{j=1}^{\infty}|\omega^{(i+1,j)}| \le h_1\sum_{j=1}^{\infty}\left(1-\frac{h_0}{h_1}\,c^{(i,j)}[\overline{z}]\right)|\omega^{(i,j)}|+h_0\sum_{j=1}^{\infty}c^{(i,j)}[\overline{z}]|\omega^{(i,j-1)}| + h_0L_c\|\omega^{(i)}\|_1\sum_{j=1}^{\infty}|u^{(i,j)}-u^{(i,j-1)}|+h_0h_1L_\lambda\sum_{j=1}^{\infty}|\omega^{(i,j)}| + h_0h_1\,\|\omega^{(i)}\|_1\sum_{j=1}^{\infty}L_z(x_j)+h_0h_1\sum_{j=1}^{\infty}|\xi^{(i,j)}|.$$

By Assumption $[C]$ we get

$$(28)\quad \sum_{j=1}^{\infty}c^{(i,j)}[\overline{z}](|\omega^{(i,j-1)}|-|\omega^{(i,j)}|) \le L_c\|\omega^{(i)}\|.$$

Taking into consideration (28), multiplying (25) by h_1 and adding to (27), we obtain

$$(29)\quad \|\omega^{(i+1)}\|_1 \le (1+h_0L_c+h_0L_\lambda+2h_0L_cU+h_0\|L_z\|_1)\|\omega^{(i)}\|_1 + h_0\|\xi^{(i)}\|_1,$$

where

$$U = \max_{i=0,\dots,N_0} h_1 \sum_{j=0}^{\infty} u^{(i,j)}.$$

The initial conditions for inequalities (26) and (29) have the estimates

$$\|\omega_0\|_1 \le \overline{C}_{0,h} \to 0, \qquad \|\omega_0\|_\infty \le C_{0,h} \to 0.$$

Hence, as in Lemma 2, we have estimates for any solution of (26), (29):

$$\|\omega^{(i)}\|_1 \le e^{La}\left(\overline{C}_{0,h} + \frac{\overline{C}_h}{L}\right) =: \widehat{C}_h,$$

$$\|\omega^{(i)}\|_\infty \le e^{L_\lambda a}\left(C_{0,h} + \frac{l(L_u L_c + \|L_z\|_\infty)\widehat{C}_h + C_h}{L_\lambda}\right)$$

for $i = 0, 1, \dots, N_0$, where $L = L_\lambda + L_c + 2L_c U + \|L_z\|_1$. ■

3.1. *Stability—the case of finite quadrature.* Since only a finite number of terms can be involved in practical computations, we shall prove a lemma on stability with respect to cut-offs of the quadrature for the forward scheme. Denoting by u_h the solution of this scheme with the finite quadrature $Q_h^{N_h}$, we write it as follows:

$$\delta_0 u_h^{(i,j)} + c^{(i,j)}[z_h]\delta_+ u_h^{(i,j)} = \lambda^{(i,j)}[u_h, z_h] \quad \text{on } E_h^+ \tag{30}$$

with

$$z_h^{(i)} = (Q_h^{N_h} u_h)_i, \tag{31}$$

and the initial condition

$$u_h^{(0,j)} = v^{(j)} \quad \text{for } j \in \mathbb{N}. \tag{32}$$

Lemma 7. *Suppose that*

(i) $h_1 N_h \to \infty$ *as* $\|h\| \to 0$,

(ii) c *satisfies the stability conditions*

$$1 + \frac{h_0}{h_1} c(t,x,q) \ge 0, \qquad c(t,x,q) \le 0 \quad \textit{for } (t,x,q) \in E \times \mathbb{R}_+,$$

(iii) *Assumptions* [C] *and* [Λ] *hold.*

Then the scheme (4)–(6) *is stable with respect to cut-offs of the quadrature.*

Proof. Suppose that a discrete function u: $E_h^+ \to \mathbb{R}_+$ is a bounded and l^1-bounded solution of problem (4)–(6), satisfying the discrete Lipschitz condition $|u^{(i,j+1)} - u^{(i,j)}| \le L_u h_1$ with some $L_u > 0$. Denote by u_h the unique solution of (30)–(32), which clearly exists. Observe that u_h is also bounded and l^1-bounded. Set $\varepsilon^{(i,j)} = u^{(i,j)} - u_h^{(i,j)}$. Subtracting (4) and (30), we obtain the explicit recurrence error equation

$$\varepsilon^{(i+1,j)} = \varepsilon^{(i,j)}\left(1 + \frac{h_0}{h_1}\, c^{(i,j)}[z_h]\right) - \frac{h_0}{h_1}\, c^{(i,j)}[z_h]\varepsilon^{(i,j+1)} + \frac{h_0}{h_1}\,(u^{(i,j+1)} - u^{(i,j)})(c^{(i,j)}[z] - c^{(i,j)}[z_h]) + h_0(\lambda^{(i,j)}[u,z] - \lambda^{(i,j)}[u_h, z_h]) \tag{33}$$

with the initial condition $\varepsilon^{(0,j)} = 0$ for $j = 0, 1, \ldots$. By Assumptions $[C]$, $[\Lambda]$ and (ii) we get

$$|\varepsilon^{(i+1,j)}| \le |\varepsilon^{(i,j)}|\left(1 + \frac{h_0}{h_1}\, c^{(i,j)}[z_h]\right) - \frac{h_0}{h_1}\, c^{(i,j)}[z_h]|\varepsilon^{(i,j+1)}| + h_0(L_z(x^{(j)}) + L_u L_c)|z_h^{(i)} - z^{(i)}| + h_0 L_\lambda |\varepsilon^{(i,j)}|. \tag{34}$$

Notice that

$$|z^{(i)} - \widetilde{z}^{(i)}| \le \|\varepsilon^{(i)}\|_1 + \widetilde{U}_h^{(i)}, \tag{35}$$

where

$$\widetilde{U}_h^{(i)} = h_1 \sum_{j=N_h+1}^{\infty} u^{(i,j)},$$

and the remainder $\widetilde{U}_h^{(i)}$ tends to 0 as $h_1 \to 0$. Using Assumption $[\Lambda]$ and (35), we obtain

$$\|\varepsilon^{(i+1)}\|_\infty \le (1 + h_0 L_\lambda)\|\varepsilon^{(i)}\|_\infty + h_0(\|L_z\|_\infty + L_c L_u)(\|\varepsilon^{(i)}\|_1 + \widetilde{U}_h), \tag{36}$$

where $\widetilde{U}_h = \max_{i=0,\ldots,N_0} \widetilde{U}_h^{(i)}$.

Multiplying (33) by h_1, summing over $j = 0, 1, \ldots$, taking into consideration (19), Assumptions $[\Lambda]$, $[C]$ and (35), we obtain

$$\|\varepsilon^{(i+1)}\|_1 \le (1 + h_0 L_\lambda + h_0 L_c + 2h_0 L_c U + h_0\|L_z\|_1)\|\varepsilon^{(i)}\|_1 + h_0(\|L_z\|_1 + 2L_c U)\widetilde{U}_h, \tag{37}$$

where $U = \max_{i=0,\ldots,N_0} \sum_{j=1}^{\infty} u^{(i,j)}$. Writing, as in Lemma 2, the comparison recurrence equations with respect to (36) and (37) and taking into consideration the initial conditions $\|\varepsilon^{(0)}\|_\infty = 0$, $\|\varepsilon^{(0)}\|_1 = 0$, we obtain the estimates

$$\|\varepsilon^{(i)}\|_1 \le e^{aL}\, \frac{(2L_c U + \|L_z\|_1)\widetilde{U}_h}{L} =: \widehat{C}_h,$$

$$\|\varepsilon^{(i)}\|_\infty \le e^{aL_\lambda}\, \frac{(\|L_z\|_\infty + L_c L_u)(\widehat{C}_h + \widetilde{U}_h)}{L_\lambda},$$

where $L = L_\lambda + L_c + 2L_c U + \|L_z\|_1$. Since $h_1 N_h \to \infty$, it follows that $\widetilde{U}_h \to 0$ as $\|h\| \to 0$, and we have the desired assertion $\|\varepsilon^{(i)}\|_\infty \to 0$, $\|\varepsilon^{(i)}\|_1 \to 0$ as $\|h\| \to 0$. ∎

Now we state the main result of our paper.

THEOREM 8. *If the assumptions of Lemmas* 2 *and* 7 *are satisfied*, *then the forward schemes are stable with respect to the right-hand side*, *the initial condition and the cuts-off of the quadrature.*

Proof. This follows immediately from the proofs of Lemmas 2 and 7. ■

REMARK 9. Results similar to Lemma 7 and Theorem 8 can also be formulated for the backward scheme. We skip the details.

4. Consistency of backward schemes. Denote by $\xi^{(i,j)}$ the defect of the difference scheme (7), (8). The scheme is *consistent* with the differential equation (1) if

$$\max_i \|\xi^{(i)}\|_\infty \to 0, \quad \max_i \|\xi^{(i)}\|_1 \to 0 \quad \text{as } \|h\| \to 0$$

for each bounded solution $u \in C^1(E, \mathbb{R}_+)$ such that $u(t,\cdot) \in L^\infty(\mathbb{R}_+)$ and

(a) $\partial_t u(\cdot, x), \partial_t u(x, \cdot)$ are uniformly continuous on $\mathbb{R}_+$,
(b) $[0,a] \ni t \mapsto \partial_t u(t,\cdot) \in L^1(\mathbb{R}_+)$ is bounded and continuous,
(c) there is an integrable modulus of continuity for $\partial_x u(t,\cdot)$.

We now prove a consistency theorem for backward schemes. The proof of consistency for forward schemes is similar.

THEOREM 10. *Suppose that* $u \in C^1(E)$ *and*

(i) $\partial_t u(\cdot, x), \partial_x u(t,\cdot)$ *are uniformly continuous*,
(ii) *Assumption* $[\Lambda]$ *holds*,
(iii) $u(t,\cdot), \partial_x u(t,\cdot) \in L^1(\mathbb{R}_+)$ *for* $t \in [0,a]$,
(iv) *there is a modulus of continuity* ω_t: $[0,a) \times E \to \mathbb{R}_+$ *of the function* $\partial_t u(\cdot, x)$ *which satisfies*

$$\|\omega_t(h_0;\cdot,\cdot)\|_\infty + \sup_{s\in[0,a]} \int_0^\infty \omega_t(h_0; s, x)\, dx \to 0 \quad \text{as } h_0 \to 0,$$

(v) *there is a modulus of continuity* ω_x: $[0,a) \times E \to \mathbb{R}_+$ *of the function* $\partial_x u(s,\cdot)$ *which satisfies*

$$\|\omega_x(h_1;\cdot,\cdot)\|_\infty + \sup_{s\in[0,a]} \int_0^\infty \omega_x(h_1; s, x)\, dx \to 0 \quad \text{as } h_1 \to 0,$$

(vi) c *is bounded*,
(vii) $[0,a] \ni t \mapsto \|\partial_x u(t,\cdot)\|_{L^1}$ *is bounded.*

Then the difference scheme (7)–(9) *is consistent with the differential problem*

(1)–(3) *on the solution* u. *Moreover*, *we have the estimate*

$$|(Q_h u)_i - (Q_h^{N_h} u)_i| \le h_1 \sum_{j=N_h+1}^{\infty} |u(t_i, x_j)|,$$

which tends to 0 *as* $\|h\| \to 0$ *and* $h_1 N_h \to \infty$.

Proof. Notice that (5) is an extended trapezoidal rule for (2) at the point $t = t_i$. The error of the trapezoidal rule on the interval $[x_j, x_{j+1}]$ is given by

$$\varrho_{ij} = \int_{x_j}^{x_{j+1}} u(t_i, x)\, dx - \frac{u(t_i, x_j) + u(t_i, x_{j+1})}{2} h_1 \tag{38}$$

for $i = 0, \ldots, N_0$ and $j = 0, 1, \ldots$. Denote by

$$\varrho_i = \sum_{j=0}^{\infty} \varrho_{ij}$$

the error of discretization for (2) at $t = t_i$. From the Taylor formula we obtain

$$\begin{aligned} h_1[u(t_i, x_j) + u(t_i, x_{j+1})] &= \int_{x_j}^{x_{j+1}} [u(t_i, x_j) + u(t_i, x_{j+1})]\, dx \\ &= \int_{x_j}^{x_{j+1}} \Big[2u(t_i, x) + \int_{x}^{x_j} \partial_x u(t_i, \eta)\, d\eta + \int_{x}^{x_{j+1}} \partial_x u(t_i, \eta)\, d\eta\Big]\, dx. \end{aligned}$$

By (38) we have the estimate

$$|\varrho_{ij}| \le \frac{h_1}{2} \int_{x_j}^{x_{j+1}} |\partial_x u(t_i, \eta)|\, d\eta,$$

and consequently

$$|\varrho_i| \le \frac{h_1}{2} \sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} |\partial_x u(t_i, \eta)|\, d\eta = \frac{h_1}{2} \|\partial_x u(t_i, \cdot)\|_{L^1}$$

for $t_i \in [0, a]$, $i = 0, \ldots, N_0$.

The defect $\xi^{(i,j)}$ of the discretization of scheme (13)–(14) is defined by the equation

$$\begin{aligned} \frac{u(t_{i+1}, x_j) - u(t_i, x_j)}{h_0} &+ c(t_i, x_j, \widetilde{z}(t_i))\, \frac{u(t_i, x_j) - u(t_i, x_{j-1})}{h_1} \\ &= \lambda(t_i, x_j, u(t_i, x_j), \widetilde{z}(t_i)) + \xi^{(i,j)}, \end{aligned} \tag{39}$$

where

$$\widetilde{z}(t_i) = \frac{h_1}{2} \sum_{j=0}^{\infty} (u(t_i, x_j) + u(t_i, x_{j+1})).$$

We show that $\sup_{i,j} |\xi^{(i,j)}| \to 0$ as $\|h\| \to 0$. By the Taylor formula we have

$$u(t_{i+1}, x_j) = u(t_i, x_j) + h_0 \partial_t u(t_i + \theta_i h_0, x_j),$$

where $\theta_i \in [0,1]$ for $i = 0, 1, \ldots, N_0 - 1$, $j = 0, 1, \ldots,$ and

$$u(t_i, x_{j-1}) = u(t_i, x_j) - h_1 \partial_x u(t_i, x_j - \theta_j h_1),$$

where $\theta_j \in [0,1]$ for $i = 0, 1, \ldots, N_0,\ j = 1, 2, \ldots$. If we substitute $u(t_{i+1}, x_j)$ and $u(t_i, x_{j-1})$ in (39) we get

$$\begin{aligned}
\xi^{(i,j)} &= \partial_t u(t_i + h_0\theta_i, x_j) - \partial_t u(t_i, x_j) \\
&\quad + c(t_i, x_j, \widetilde{z}(t_i))(\partial_x u(t_i, x_j - \theta_j h_1) - \partial_x u(t_i, x_j)) \\
&\quad + \lambda(t_i, x_j, u(t_i, x_j), z(t_i)) - \lambda(t_i, x_j, u(t_i, x_j), \widetilde{z}(t_i)).
\end{aligned} \tag{40}$$

From Assumption $[\Lambda]$ it follows that $\|L_z\|_\infty < \infty$ and

$$\begin{aligned}
&|\lambda(t_i, x_j, u(t_i, x_j), z(t_i)) - \lambda(t_i, x_j, u(t_i, x_j), \widetilde{z}(t_i))| \\
&\qquad \le L_z(x_j)|z(t_i) - \widetilde{z}(t_i)| = L_z(x_j)|\varrho_i| \le \frac{h_1}{2}\, \|L_z\|_\infty \|\partial_x u(t_i, \cdot)\|_{L^1}.
\end{aligned}$$

From uniform continuity of the functions $\partial_t u(\cdot, x)$ and $\partial_x u(t, \cdot)$, by (vii) and the above estimate we have

$$|\xi^{(i,j)}| \le \|\omega_t(h_0; \cdot, \cdot)\|_\infty + \|c\|_\infty \|\omega_x(h_1; \cdot, \cdot)\|_\infty + \frac{h_1 B}{2}\, \|L_z\|_\infty$$

for $i = 0, 1, \ldots, N_0$ and $j = 1, 2, \ldots,$ where

$$B = \max_i \|\partial_x u(t_i, \cdot)\|_{L^1}.$$

The Euler difference scheme (8) for the differential problem

$$u'(t, 0) = \lambda(t, 0, u(t, 0), z(t)), \qquad u(0, 0) = v(0),$$

has the local error estimate

$$\begin{aligned}
|\xi^{(i,0)}| &\le |\partial_t u(t_i + \theta_i h_0, 0) - \partial_t u(t_i, 0)| \\
&\qquad + |\lambda(t_i, 0, u(t_i, 0), z(t_i)) - \lambda(t_i, 0, u(t_i, 0), \widetilde{z}(t_i))|,
\end{aligned}$$

and consequently

$$|\xi^{(i,0)}| \le \omega_t(h_0; \cdot, 0) + \frac{h_1 B}{2}\, L_z(0) \tag{41}$$

for $i = 0, 1, \ldots, N_0$. Hence

$$\|\xi^{(i)}\|_\infty \to 0 \qquad \text{as } \|h\| \to 0 \text{ for } i = 0, 1, \ldots, N_0.$$

Now we prove that

$$\max_{i=0,1,\ldots,N_0} \|\xi^{(i)}\|_1 \to 0 \quad \text{ as } \|h\| \to 0.$$

If we apply Assumption [Λ] and (vi) to (40), we obtain

$$|\xi^{(i,j)}| \le |\partial_t u(t_i + h_0\theta_i, x_j) - \partial_t u(t_i, x_j)| + \|c\|_\infty |\partial_x u(t_i, x_j - \theta_j h_1) - \partial_x u(t_i, x_j)| + \frac{h_1}{2} L_z(x_j) \|\partial_x u(t_i, \cdot)\|_{L^1}. \tag{42}$$

Multiplying (41) and (42) by h_1 and summing over $j = 0, 1, \ldots,$ we obtain

$$h_1 \sum_{j=0}^{\infty} |\xi^{(i,j)}| \le h_1 \omega_t(h_0; \cdot, 0) + h_1 \sum_{j=1}^{\infty} |\partial_t u(t_i + h_0\theta_i, x_j) - \partial_t u(t_i, x_j)| + h_1 \|c\|_\infty \sum_{j=1}^{\infty} |\partial_x u(t_i, x_j - \theta_j h_1) - \partial_x u(t_i, x_j)| + \frac{h_1 B}{2} \|L_z\|_1. \tag{43}$$

Notice that

$$h_1 \omega_t(h_0; \cdot, 0) + h_1 \sum_{j=1}^{\infty} |\partial_t u(t_i + h_0\theta_i, x_j) - \partial_t u(t_i, x_j)| \le h_1 \sum_{j=0}^{\infty} \omega_t(h_0; t_i, x_j)$$

and

$$h_1 \sum_{j=1}^{\infty} |\partial_x u(t_i, x_j - \theta_j h_1) - \partial_x u(t_i, x_j)| \le h_1 \sum_{j=1}^{\infty} \omega_x(h_1; t_i, x_j).$$

Moreover

$$h_1 \sum_{j=0}^{\infty} \omega_t(h_0; t_i, x_j) \to \int_0^\infty \omega_t(h_0; t_i, x_j)\, dx,$$
$$h_1 \sum_{j=0}^{\infty} \omega_x(h_1; t_i, x_j) \to \int_0^\infty \omega_x(h_1; t_i, x_j)\, dx$$

as $\|h\| \to 0$. By (iv) and (v) we get

$$\|\xi^{(i)}\|_1 \to 0 \quad \text{as } \|h\| \to 0 \text{ for } i = 0, 1, \ldots, N_0.$$

The last assertion is obvious:

$$|(Q_h u)_i - (Q_h^{N_h} u)_i| \le h_1 \sum_{j=N_h+1}^{\infty} \frac{u(t_i, x_j) + u(t_i, x_{j+1})}{2} \le \int_{x_j}^{\infty} u(t_i, x)\, dx + h_1 \int_{x_j}^{\infty} |\partial_x u(t_i, x)|\, dx,$$

which tends to 0 as $\|h\| \to 0$ and $h_1 N_h \to \infty$. ∎

Remark 11. Consistency for schemes with cut-off trapezoidal rule contains one additional term: the remainder of this quadrature, which tends to zero. This follows from the last assertion of Theorem 10. In fact, assumptions (iv)–(vi) can be weakened, but the calculations become more involved.

Remark 12. Similar consistency and stability results can be obtained for Lax–Friedrichs type schemes.

5. Numerical experiments. We present numerical tests which illustrate our theoretical results. The computations were performed for forward and backward schemes. With prescribed functions $u : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$, $v(x) = u(0,x)$, c: $[0,1] \times \mathbb{R}_+^2 \to \mathbb{R}$ we determine the respective right-hand sides of the difference equations. For the sake of computer technical constraints the unbounded domain is restricted to a bounded rectangle $[0,1] \times [0,1000]$.

For the backward scheme we take $c(t,x,z) = t\sin^2(xz)$, and for the forward scheme we put $c(t,x,z) = -t\sin^2(xz)$. The right-hand side of (1) has the form

$$\lambda(t,x,p,q) = \kappa\cos(t(p+q)) + g_i(t,x), \qquad \kappa \in \mathbb{R},\ i = 1,2.$$

The first solution

$$u_1(t,x) = \frac{1+\cos(tx)}{1+x^2}, \qquad (t,x) \in [0,1] \times \mathbb{R}_+,$$

generates the initial condition

$$u_1(0,x) = v_1(x) = \frac{2}{1+x^2} \qquad \text{for } x \in \mathbb{R}_+,$$

and g_1 is given by

$$\begin{aligned} g_1(t,x) &= \partial_t u_1(t,x) + c(t,x,z_1(t))\partial_x u_1(t,x) \\ &\quad - \kappa\cos(t(u_1(t,x) + z_1(t))), \\ z_1(t) &= \frac{\pi}{2}(1+e^{-t}) \end{aligned}$$

for $(t,x) \in [0,1] \times \mathbb{R}_+$.

The second solution u_2 generates the corresponding initial condition and the integral z_2:

$$\begin{aligned} u_2(t,x) &= \frac{x\cos t}{(1+t+x^2)^2}, \qquad (t,x) \in [0,1] \times \mathbb{R}_+, \\ u_2(0,x) &= v_2(x) = \frac{x}{(1+x^2)^2}, \qquad x \in \mathbb{R}_+, \\ z_2(t) &= \frac{1}{2}\,\frac{\cos t}{1+t}, \qquad t \in [0,a]. \end{aligned}$$

In these examples we take $\kappa = 0.2$, $h_0 = h_1 = 0.001$.

Table 1. Forward scheme, exact solution u_1

x	Solution	Error
10.000	0.000106314	−0.000007230
50.000	0.000103045	−0.000006959
100.000	0.000110857	0.000004556
200.000	0.000000905	0.000000285
500.000	0.000006422	0.000000375
750.000	0.000002179	0.000000252
999.699	0.000000801	0.000000088

$z_{300} = 2.734449, \;$ quadrature error $= -0.000021$

Table 2. Backward scheme, exact solution u_1

x	Solution	Error
10.000	0.000106496	−0.000007412
50.000	0.000103038	−0.000006952
100.000	0.000110863	0.000004551
200.000	0.000000906	0.000000284
500.000	0.000006423	0.000000374
750.000	0.000002179	0.000000251
999.999	0.000000880	0.000000098

$z_{300} = 2.734449, \;$ quadrature error $= -0.000022$

Table 3. Forward scheme, exact solution u_2

x	Solution	Error
10.000	0.000896038	−0.000000235
50.000	0.000007403	−0.000000043
100.000	0.000000963	−0.000000042
200.000	0.000000157	−0.000000042
500.000	0.000000049	−0.000000042
750.000	0.000000044	−0.000000042
999.599	0.000000043	−0.000000042

$z_{400} = 0.328931, \;$ quadrature error $= -0.000019$

Table 1 shows the numerical results and errors when applying the forward scheme for the exact solution u_1. A similar error propagation is observed when performing the computations for the backward scheme (see Table 2). In Tables 3 and 4 we list numerical results for the exact solution u_2 (forward and backward schemes).

Table 4. Backward scheme, exact solution u_2

x	Solution	Error
10.000	0.000896032	−0.000000229
50.000	0.000007404	−0.000000044
100.000	0.000000964	−0.000000043
200.000	0.000000158	−0.000000043
500.000	0.000000050	−0.000000043
750.000	0.000000045	−0.000000043
999.999	0.000000044	−0.000000043

$z_{400} = 0.328931$, quadrature error $= -0.000020$

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