

JAROSŁAW L. BOJARSKI (Warszawa)

REGULARITY OF SOLUTIONS IN PLASTICITY. II: PLATES

Abstract. The aim of this paper is to study the problem of regularity of displacement solutions in Hencky plasticity. We consider a plate made of a non-homogeneous material whose elastic-plastic properties change discontinuously. We prove that the displacement solutions belong to the space $W^{2,1}(\Omega)$ if the stress solution is continuous and belongs to the interior of the set of admissible stresses, at each point. The part of the functional which describes the work of boundary forces is relaxed.

1. Introduction. The principal aim of this contribution is to prove a theorem on regularity of displacement solutions in Hencky plasticity (see Theorem 24). We consider a plate made of a non-homogeneous Hencky material whose elastic-plastic properties change discontinuously.

In [6] and [15] the existence of solutions, for elastic-plastic plates, is proved in the space $SBH(\Omega)$ of *Special Bounded Hessian*. But the authors of those papers assume that the plate considered has elastic potential, i.e. the potential has nonlinear growth at infinity (cf. [6, (1) and (3)], [15, (2) and (6)]).

Demengel [7] solved the existence problem for an elastic-perfectly plastic plate made of a homogeneous and isotropic Hencky material. To prove that the functional of the total potential energy is weak* lower semicontinuous (l.s.c.) in the space $HB(\Omega)$, she used the method of relaxation of the kinematic boundary condition (see also [8], [14] and [3]).

In this contribution we show that the displacement solutions belong to the space $W^{2,1}(\Omega)$ if the stress solution is continuous and belongs to the

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interior of the set of admissible stresses, at each point. The part of the functional which describes the work of boundary forces is relaxed. We consider a non-homogeneous plate whose elastic-plastic properties change discontinuously.

2. Some basic definitions and theorems. In this paper Ω is a bounded, open, connected domain of class C^2 in \mathbb{R}^2 . We use the notation from [4]. We define the following Banach space of functions with bounded Hessian $HB(\Omega)$ (see [7], [14]):

$$(2.1) \quad HB(\Omega) \equiv \left\{ u \in W^{1,1}(\Omega) \mid \frac{\partial^2 u}{\partial x_i \partial x_j} \in \mathbb{M}_b(\Omega) \text{ for all } i, j \in \{1, 2\} \right\},$$

with the natural norm

$$(2.2) \quad \begin{aligned} \|u\|_{HB(\Omega)} &= \|u\|_{W^{1,1}} + \|\nabla \nabla u\|_{\mathbb{M}_b(\Omega, \mathbf{E}_s^2)} \\ &= \|u\|_{W^{1,1}} + \sum_{i,j=1}^2 \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{\mathbb{M}_b(\Omega, \mathbb{R})}. \end{aligned}$$

If $w \in D'(\Omega)$ and $\nabla \nabla w \in \mathbb{M}_b(\Omega, \mathbf{E}_s^2)$, then $w \in HB(\Omega)$ (see [14]). Moreover, we define $\Pi_2 \equiv \ker \nabla \nabla(\cdot) = \{u \in HB(\Omega) \mid \nabla \nabla u = 0\}$.

PROPOSITION 1 (see [14] and [3]). *Let $\mathbf{g} \in D'(\Omega, \mathbf{E}_s^2)$. Then the following conditions are equivalent:*

- (i) *there exists $u \in D'(\Omega)$ such that $\nabla \nabla u = \mathbf{g}$,*
- (ii) *for all $\boldsymbol{\varphi} \in C_c^\infty(\Omega, \mathbf{E}_s^2)$ with $\operatorname{div} \operatorname{div} \boldsymbol{\varphi} = 0$, $\sum_{i,j=1}^2 \langle g_{ij}; \varphi_{ij} \rangle_{D' \times D} = 0$.*

PROPOSITION 2 (see [14]). *Let $HB(\Omega)$ and $L^1(\operatorname{Fr} \Omega)$ be endowed with the norm topologies. There exists a continuous surjective linear operator*

$$(2.3) \quad (\gamma_0, \gamma_1) : HB(\Omega) \rightarrow \gamma_0(W^{2,1}(\Omega)) \times L^1(\operatorname{Fr} \Omega)$$

such that

$$(2.4) \quad \gamma_0(u) = u|_{\operatorname{Fr} \Omega}, \quad \gamma_1(u) = \frac{\partial u}{\partial \boldsymbol{\nu}}|_{\operatorname{Fr} \Omega}$$

for all $u \in C^2(\overline{\Omega})$, where $\boldsymbol{\nu}$ is the exterior unit vector normal to $\operatorname{Fr} \Omega$ and $\gamma_0(W^{2,1}(\Omega))$ is endowed with the norm topology of $L^1(\operatorname{Fr} \Omega)$, since $\gamma_0(W^{2,1}(\Omega)) \subset L^1(\operatorname{Fr} \Omega)$.

We define spaces

$$(2.5) \quad Z \equiv C_c(\Omega, \mathbb{R}) \times C_c(\Omega, \mathbf{E}_s^2), \quad Z_0 \equiv \{(a, \mathbf{b}) \in Z \mid \operatorname{div} \operatorname{div} \mathbf{b} = -a\},$$

endowed with the natural norm

$$(2.6) \quad \begin{aligned} \|(a, \mathbf{b})\|_Z &\equiv \|a\|_C + \|\mathbf{b}\|_C \\ &= \sup\{|a(x)| \mid x \in \Omega\} + \sup_{x \in \Omega} \sup_{i,j} \{|b_{ij}(x)| \mid x \in \Omega\} \end{aligned}$$

for $a \in C_c(\Omega, \mathbb{R})$ and $\mathbf{b} \in C_c(\Omega, \mathbf{E}_s^2)$. Then $HB(\Omega)$ is isomorphic to the dual of $[Z/Z_0, \|\cdot\|_Z]$ (see [3] and [14]). The topology $\sigma((Z/Z_0)^*; Z) \equiv \sigma(HB(\Omega); C_c(\Omega) \times C_c(\Omega, \mathbf{E}_s^2))$ is called the *weak* HB topology*. A net $\{u_\delta\}_{\delta \in D} \subset HB(\Omega)$ is convergent to $u_0 \in HB(\Omega)$ in this topology if and only if, for all $(a, \mathbf{b}) \in Z$,

$$(2.7) \quad \int_{\Omega} a(u_0 - u_\delta) dx + \int_{\Omega} \mathbf{b} : \nabla \nabla (u_0 - u_\delta) \rightarrow 0.$$

For every $(v, w) \in \gamma_0(W^{2,1}(\Omega)) \times L^1(\text{Fr } \Omega)$, the set $\{u \in HB(\Omega) \mid \gamma_0(u) = v \text{ and } \gamma_1(u) = w\}$ is dense in the space $HB(\Omega)$ endowed with the weak* topology (see [2, Proposition 2.5] and [3]). Then the trace operator $\gamma_0 : HB(\Omega) \rightarrow \gamma_0(W^{2,1}(\Omega))$ (resp. $\gamma_1 : HB(\Omega) \rightarrow L^1(\text{Fr } \Omega)$) is not continuous on $[HB(\Omega), \text{weak* topology}]$ if the space $\gamma_0(W^{2,1}(\Omega))$ (respectively $L^1(\text{Fr } \Omega)$) is endowed with a Hausdorff topology (or a T_1 -topology, see [11]).

DEFINITION 1 (see [14] and [11, Chap. 1, Sec. 6]). A net $\{u_\delta\}_{\delta \in D}$ converges to u_0 in the topology (2.8)–(2.9) if

$$(2.8) \quad u_\delta \rightarrow u_0 \quad \text{in } \|\cdot\|_{W^{1,p}(\Omega)}, \text{ where } 1 \leq p < 2,$$

$$(2.9) \quad \nabla \nabla u_\delta \rightharpoonup \nabla \nabla u_0 \quad \text{weak* in } \mathbb{M}_b(\Omega, \mathbf{E}_s^2).$$

PROPOSITION 3 (cf. [3]). *The weak* HB topology and the (2.8)–(2.9) topology are equivalent on bounded subsets of $HB(\Omega)$.*

Proof. Indeed, every bounded net $\{u_\delta\}_{\delta \in D}$ in HB contains a finer net, convergent in the topology (2.8)–(2.9) (see [14]). Thus $\text{cl}_{\|\cdot\|_{HB}} B(0, r)$ is a compact set in (2.8)–(2.9) and in the weak* HB topology. Note that the weak* HB topology is weaker than the (2.8)–(2.9) topology. Moreover, among all Hausdorff topologies, compact topologies are minimal (see [11, Corollary 3.1.14]). ■

The weak* $HB(\Omega)$ topology and the topology (2.8)–(2.9) are not equivalent, because the trace γ_0 , which is continuous in the topology (2.8)–(2.9), is not continuous in the weak* HB topology. Let $\gamma_0(W^{2,1}(\Omega))$ be endowed with the norm topology $L^1(\text{Fr } \Omega)$. Then (for all $r > 0$) $\gamma_0|_{\text{cl}_{\|\cdot\|_{HB}} B(0, r)}$ is a continuous function in the weak* HB topology.

PROPOSITION 4 (cf. [3, Theorem 5.13]). *The injection of the space $[HB(\Omega), \text{weak*}]$ into $[L^p(\Omega, \mathbb{R}), \|\cdot\|_{L^p}]$ is continuous on bounded subsets of $HB(\Omega)$, where $1 \leq p < 2$. ■*

We define the Banach space $W^2(\Omega, \text{div div})$ of measurable functions (cf. [14] and [3]):

$$(2.10) \quad W^2(\Omega, \text{div div}) \equiv \{\mathbf{M} \in L^\infty(\Omega, \mathbf{E}_s^2) \mid \text{div div } \mathbf{M} \in L^2(\Omega, \mathbb{R})\}$$

with the natural norm

$$(2.11) \quad \|\mathbf{M}\|_{W^2(\Omega, \text{div div})} = \|\mathbf{M}\|_{L^\infty(\Omega, \mathbf{E}_s^2)} + \|\text{div div } \mathbf{M}\|_{L^2(\Omega)}.$$

Moreover, $W_{\text{div}}^2(\Omega, \text{div div})$ denotes the following subspace of $W^2(\Omega, \text{div div})$:

$$(2.12) \quad W_{\text{div}}^2(\Omega, \text{div div}) \equiv \{\mathbf{M} \in W^2(\Omega, \text{div div}) \mid \text{div div } \mathbf{M} = 0 \text{ in } \Omega\}.$$

The distribution $\mathbf{M} : \nabla \nabla u$, where $\mathbf{M} \in W^2(\Omega, \text{div div})$, $u \in HB(\Omega)$, defined by

$$(2.13) \quad \langle \mathbf{M} : \nabla \nabla u; \varphi_1 \rangle_{D' \times D} = - \int_{\Omega} u \mathbf{M} : \nabla \nabla \varphi_1 \, dx \\ - 2 \int_{\Omega} \mathbf{M} : ((\nabla u) \otimes (\nabla \varphi_1)) \, dx + \int_{\Omega} u \varphi_1 (\text{div div } \mathbf{M}) \, dx$$

for $\varphi_1 \in C_c^\infty(\Omega)$, is a bounded measure on Ω , absolutely continuous with respect to $|\nabla \nabla u|$ (see [14]).

THEOREM 5 (cf. [14]). *Let $\gamma_0(W^{2,1}(\Omega))$ be endowed with the norm*

$$(2.14) \quad \gamma_0(W^{2,1}(\Omega)) \ni v \mapsto \|v\|_0 = \inf\{\|w\|_{W^{2,1}} \mid w \in W^{2,1}(\Omega), \text{ where } \gamma_0(w) = v\}.$$

Moreover, let $(\gamma_0(W^{2,1}(\Omega)))^*$ be endowed with the norm

$$(2.15) \quad (\gamma_0(W^{2,1}(\Omega)))^* \ni v^* \mapsto \|v^*\|_{0*} = \sup\{|v^*(v)| \mid \|v\|_0 \leq 1\}.$$

Then there exists a continuous, linear, surjective map (β_0, β_1) from $[W^2(\Omega, \text{div div}), \|\cdot\|_{W^2(\Omega, \text{div div})}]$ onto $[(\gamma_0(W^{2,1}(\Omega)))^*, \|\cdot\|_{0*}] \times [L^\infty(\text{Fr } \Omega), \|\cdot\|_{L^\infty}]$ such that

$$(2.16) \quad \beta_0(\mathbf{M}) = (\text{div } \mathbf{M})\nu + \frac{\partial}{\partial s}((\mathbf{M}\nu)\tau), \quad \beta_1(\mathbf{M}) = (\mathbf{M}\nu)\nu$$

for every $\mathbf{M} \in C^2(\overline{\Omega}, \mathbf{E}_s^2)$, where ν is the exterior unit vector normal to $\text{Fr } \Omega$, τ is a tangent unit vector and s the curvilinear abscissa on $\text{Fr } \Omega$ measured positively in the direction of τ . Furthermore, for all $u \in HB(\Omega)$ and $\mathbf{M} \in W^2(\Omega, \text{div div})$, the following Green formula holds:

$$(2.17) \quad \int_{\Omega} \mathbf{M} : \nabla \nabla u - \int_{\Omega} u (\text{div div } \mathbf{M}) \, dx \\ = \int_{\text{Fr } \Omega} \beta_1(\mathbf{M}) \gamma_1(u) \, ds - \langle \beta_0(\mathbf{M}); \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}(\Omega)))^* \times \gamma_0(W^{2,1}(\Omega))}.$$

LEMMA 6 (cf. [3] and [14]). *For every $\mathbf{M} \in W^2(\Omega, \text{div div})$, there exists a sequence $\{\mathbf{M}_n\}_{n \in \mathbb{N}} \subset W^2(\Omega, \text{div div}) \cap C^\infty(\Omega, \mathbf{E}_s^2)$ such that (for all $n \in \mathbb{N}$) $\beta_1(\mathbf{M}_n) = \beta_1(\mathbf{M})$, $\beta_0(\mathbf{M}_n) = \beta_0(\mathbf{M})$ and*

$$(2.18) \quad \mathbf{M}_n \rightarrow \mathbf{M} \quad \text{in the norm of } L^p(\Omega, \mathbf{E}_s^2), \text{ for all } p, 1 \leq p < \infty, \\ \text{and weak}^* L^\infty(\Omega, \mathbf{E}_s^2),$$

$$(2.19) \quad \text{div div } \mathbf{M}_n \rightarrow \text{div div } \mathbf{M} \quad \text{in the norm of } L^2(\Omega).$$

LEMMA 7 (see [3]). Let Ω_1 be an open bounded set of class C^2 . Let Ω be a bounded open connected set of class C^2 and suppose that $\Omega \subset\subset \Omega_1 \subset \mathbb{R}^2$. For every $\mathbf{M} \in W^2(\Omega, \text{div div})$, there exists $\mathbf{M}_1 \in W^2(\Omega_1, \text{div div})$ such that $\mathbf{M}_{1|\Omega} = \mathbf{M}$ and $\beta_0(\mathbf{M}_1) = 0$, $\beta_1(\mathbf{M}_1) = 0$ on $\text{Fr } \Omega_1$.

Proof. Let $\mathbf{M} \in W^2(\Omega, \text{div div})$. By Theorem 5, there exists $\mathbf{M}_2 \in W^2(\Omega_1 - \bar{\Omega}, \text{div div})$ such that $\beta_0(\mathbf{M}) = -\beta_0(\mathbf{M}_2)$ and $\beta_1(\mathbf{M}) = \beta_1(\mathbf{M}_2)$ on $\text{Fr } \Omega$. Indeed, by [1] we can extend any function $t \in W^{2,1}(\Omega)$ to a function $\hat{t} \in W^{2,1}(\Omega_1)$. Moreover, by Theorem 5, the trace (β_0, β_1) is a linear operator onto $[\gamma_0(W^{2,1}(\Omega))]^* \times L^\infty(\text{Fr } \Omega)$.

By Lemma 6 there exists $\mathbf{M}_3 \in W^2(\Omega_1 - \bar{\Omega}, \text{div div}) \cap C^\infty(\Omega_1 - \bar{\Omega}, \mathbf{E}_s^2)$ such that $\beta_0(\mathbf{M}_2) = \beta_0(\mathbf{M}_3)$ and $\beta_1(\mathbf{M}_2) = \beta_1(\mathbf{M}_3)$ on $\text{Fr}(\Omega_1 - \bar{\Omega})$. Due to the Green formula (2.17), we find $\mathbf{M}_4 \in W^2(\Omega_1, \text{div div})$ such that

$$(2.20) \quad \mathbf{M}_4 \equiv \begin{cases} \mathbf{M} & \text{on } \Omega, \\ \mathbf{M}_3 & \text{on } \Omega_1 - \bar{\Omega}, \\ \beta_1(\mathbf{M})\boldsymbol{\nu} \otimes \boldsymbol{\nu} & \text{on } \text{Fr } \Omega, \end{cases}$$

where $(\mathbf{p} \otimes \boldsymbol{\nu})_{ij} = p_i \nu_j$. If $h \in C_c^\infty(\Omega_1, \mathbb{R})$ and $h(x) = 1$ for all $x \in \Omega$ then $\mathbf{M}_1 \equiv h\mathbf{M}_4$. ■

3. Auxiliary theorems and spaces. In this paper the Lebesgue and Hausdorff measures on Ω and $\text{Fr } \Omega$ are denoted by dx and ds , respectively. Let Γ_0 and Γ_1 be Borel subsets of $\text{Fr } \Omega$ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \cup \Gamma_1 = \text{Fr } \Omega$ and $\Gamma_1 = \text{cl}(\text{int } \Gamma_1)$, where $\text{int } \Gamma_1$ denotes the interior of Γ_1 relative to the boundary. Let $\mathcal{K}_p : \bar{\Omega} \rightarrow 2^{\mathbf{E}_s^2}$ be a multifunction (defined for every $x \in \bar{\Omega}$) and suppose there exist $r_2 > r_1 > 0$ such that for every $x \in \bar{\Omega}$,

$$(3.1) \quad B_{\mathbf{E}_s^2}(0, r_1) \subset \mathcal{K}_p(x) \subset B_{\mathbf{E}_s^2}(0, r_2).$$

The set $\mathcal{K}_p(x)$ is the elasticity convex domain at the point x .

ASSUMPTION 1 (cf. [4]). Let Ω and Ω_1 ($\Omega \subset\subset \Omega_1$) be bounded open connected sets of class C^2 in \mathbb{R}^2 and suppose the inclusion (3.1) holds. For every $y \in \bar{\Omega}$,

$$(3.2) \quad \mathcal{K}_p(y) = \{\mathbf{z}(y) \mid \mathbf{z} \in C^1(\bar{\Omega}, \mathbf{E}_s^2), \mathbf{z}|_{\text{int } \Omega} \in W^2(\Omega, \text{div div}), \\ \mathbf{z}(x) \in \mathcal{K}_p(x) \text{ for } dx\text{-a.e. } x \in \Omega\}.$$

Moreover, for all $x \in \bar{\Omega}$, $\mathcal{K}_p(x)$ is a convex and closed subset in \mathbf{E}_s^2 .

ASSUMPTION 2. There exist $\bar{a} > 0$, $[q_{ij}] \in L^\infty(\Omega, \mathbf{E}_s^2)$ and $a_{ijkl} \in L^\infty(\Omega, \mathbb{R})$ for $i, j, k, l \in \{1, 2\}$ such that

$$(3.3) \quad \sum_{i,j,k,l=1}^2 a_{ijkl}(x) w_{ij}^* w_{kl}^* > \bar{a} \| [w_{ij}^*] \|_{\mathbf{E}_s^2}^2,$$

$$(3.4) \quad j^*(x, [w_{ij}^*]) = \sum_{i,j,k,l=1}^2 a_{ijkl}(x)(w_{ij}^* - q_{ij}(x))(w_{kl}^* - q_{kl}(x)) + I_{\mathcal{K}(x)}([w_{ij}^*])$$

for dx -a.e. $x \in \Omega$ and for every $[w_{ij}^*] \in \mathbf{E}_s^2$, where w_{ij}^* and w_{kl}^* are components of the matrix $[w_{ij}^*]$.

ASSUMPTION 3. Let $\Gamma_1 = \text{Fr } \Omega \cap \mathcal{C}$, where $\mathcal{C} = \text{cl int } \mathcal{C} \subset \Omega_1$ is a closed Caccioppoli set and $ds(\text{Fr } \Omega \cap \text{Fr } \mathcal{C}) = 0$ (cf. [12] and [4]). ■

We define

$$(3.5) \quad j(x, \mathbf{w}) \equiv j^{**}(x, \mathbf{w}) \equiv \sup\{\mathbf{w} : \mathbf{w}^* - j^*(x, \mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}_s^2\}$$

for dx -a.e. $x \in \Omega$ and all $\mathbf{w} \in \mathbf{E}_s^2$. The function j is a convex normal integrand (cf. [10, Chap. VIII, Proposition 1.2]). By (3.1) and (3.4) there exists $k > 0$ such that

$$(3.6) \quad cr_1 \|\mathbf{w}\|_{\mathbf{E}_s^2} - k \leq j(x, \mathbf{w}) \quad \text{for } dx\text{-a.e. } x \in \Omega$$

(cf. [4, (3.5)]). Define $j_\infty : \bar{\Omega} \times \mathbf{E}_s^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$(3.7) \quad j_\infty(x, \mathbf{w}) \equiv \sup\{\mathbf{w} : \mathbf{w}^* - I_{\mathcal{K}_p(x)}(\mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}_s^2\}$$

for $x \in \bar{\Omega}$ and $\mathbf{w} \in \mathbf{E}_s^2$. Because of (3.1) we have, for all $x \in \bar{\Omega}$ (see [4, (3.7)]),

$$(3.8) \quad c_1 r_1 \|\mathbf{w}\|_{\mathbf{E}_s^2} \leq j_\infty(x, \mathbf{w}) \leq c_2 r_2 \|\mathbf{w}\|_{\mathbf{E}_s^2}.$$

4. Regularity of displacement solutions. In this section we prove that the displacement solutions of the relaxed functional belong to the space $W^{2,1}(\Omega)$ if the stress solution is continuous and belongs to the interior of the set of admissible stresses, at each point (see Theorem 24).

Let $\mathcal{Z} : HB(\Omega) \rightarrow \mathbb{M}_b(\Omega, \mathbf{E}_s^2) \times L^1(\text{Fr } \Omega, \mathbb{R}) \times C(\text{Fr } \Omega)$ be given by

$$(4.1) \quad HB(\Omega) \ni u \mapsto \mathcal{Z}(u) = ((\nabla \nabla u), \gamma_1(u), \gamma_0(u)).$$

We define spaces

$$(4.2) \quad \mathcal{Z}(HB(\Omega)) \equiv \{(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in HB(\Omega)\}$$

and

$$(4.3) \quad W^2(\Omega, \text{div div}) \times C(\text{Fr } \Omega),$$

put in duality by the bilinear pairing $\langle \cdot; \cdot \rangle_{\mathcal{Z}}$, where

$$(4.4) \quad \begin{aligned} \langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); (\varphi^*, \psi^*) \rangle_{\mathcal{Z}} \\ = \int_{\Omega} \varphi^* : \nabla \nabla u - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) ds + \int_{\text{Fr } \Omega} \gamma_0(u) \psi^* ds, \end{aligned}$$

for every $\varphi^* \in W^2(\Omega, \text{div div})$ and

$$(4.5) \quad \psi^* \in C(\text{Fr } \Omega, \mathbb{R}), \quad (\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \in \mathcal{Z}(HB(\Omega)).$$

Moreover, we define a subspace $C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2)$ of $W^2(\Omega, \text{div div})$ by

$$(4.6) \quad C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2) \equiv \{\varphi^* \in C^1(\bar{\Omega}, \mathbf{E}_s^2) \mid \varphi_{|\Omega}^* \in W^2(\Omega, \text{div div})\}.$$

REMARK 1. The definition of spaces in duality requires that for every $(\varphi^*, \psi^*) \in W^2(\Omega, \text{div div}) \times C(\text{Fr } \Omega)$, $(\varphi^*, \psi^*) \neq (\mathbf{0}, 0)$, there exists $(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \in \mathcal{Z}(HB(\Omega))$ such that

$$(4.7) \quad \langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); (\varphi^*, \psi^*) \rangle_{\mathcal{Z}} \neq 0.$$

But for every $\varphi_s^* \in W^2(\Omega, \text{div div})$ such that $\text{div div } \varphi_s^* = 0$ in Ω , $\beta_0(\varphi_s^*) = 0$ on $\text{Fr } \Omega$, and for every $(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \in \mathcal{Z}(HB(\Omega))$,

$$(4.8) \quad \langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); (\varphi_s^*, 0) \rangle_{\mathcal{Z}} = \int_{\Omega} u(\text{div div } \varphi_s^*) dx = 0$$

(see (2.17) and [4]). We do not get a contradiction, since we do not use the Hausdorff property of the topology $\sigma(W^2(\Omega, \text{div div}) \times C(\text{Fr } \Omega); \mathcal{Z}(HB(\Omega)))$ (cf. [4, Remark 3]).

Let the bilinear form $\langle \cdot; \cdot \rangle_b$ between $\mathcal{Z}(HB(\Omega))$ and $W^2(\Omega, \text{div div})$ be given by

$$(4.9) \quad \begin{aligned} \langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); (\varphi^*, \beta_0(\varphi^*)) \rangle_b &= \int_{\Omega} \varphi^* : (\nabla \nabla u) \\ &\quad - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) ds + \langle \beta_0(\varphi^*); \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \end{aligned}$$

for $u \in HB(\Omega)$ and $\varphi^* \in W^2(\Omega, \text{div div})$. A net $\{u_{\delta}\}_{\delta \in D} \subset HB(\Omega)$ is said to converge to $\hat{u} \in HB(\Omega)$ in the topology $\sigma(\mathcal{Z}(HB(\Omega)); W^2(\Omega, \text{div div}))$ if

$$(4.10) \quad \langle (\nabla \nabla(u_{\delta} - \hat{u}), \gamma_1(u_{\delta} - \hat{u}), \gamma_0(u_{\delta} - \hat{u})); (\varphi^*, \beta_0(\varphi^*)) \rangle_b \rightarrow 0$$

for all $\varphi^* \in W^2(\Omega, \text{div div})$.

THEOREM 8. *Every closed ball $\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))$ (in $\mathcal{Z}(HB(\Omega))$) is compact in the topology $\sigma(\mathcal{Z}(HB(\Omega)); W^2(\Omega, \text{div div}))$.*

Proof. Step 1. Let a net $\{u_{\delta}\}_{\delta \in D} \subset HB(\Omega)$ be included in $B_{HB}(0, \hat{r})$. The injection $HB(\Omega) \subset H^1(\Omega)$ is continuous and injection $H^1(\Omega)$ into $L^2(\Omega)$ is compact (see [14, Chap. 3, (2.77)], [14, Chap. 1, (1.21)] and [1, Theorem 6.2]). Then there exists a finer net $\{u_{\delta_{\alpha}}\}_{\alpha \in A} \subset \{u_{\delta}\}_{\delta \in D}$ and a function $u_1 \in L^2(\Omega)$ such that

$$(4.11) \quad \|u_{\delta_{\alpha}} - u_1\|_{L^2(\Omega, \mathbb{R})} \rightarrow 0$$

and

$$(4.12) \quad \begin{aligned} \langle (\nabla \nabla u_{\delta_{\alpha}}, \gamma_1(u_{\delta_{\alpha}}), \gamma_0(u_{\delta_{\alpha}})); (\varphi^*, \beta_0(\varphi^*)) \rangle_b \\ = \int_{\Omega} u_{\delta_{\alpha}}(\text{div div } \varphi^*) dx \rightarrow \int_{\Omega} u_1(\text{div div } \varphi^*) dx \end{aligned}$$

for every $\varphi^* \in W^2(\Omega, \text{div div})$, because $\text{div div } \varphi^* \in L^2(\Omega)$ (cf. (2.17)). Since Ω is bounded subset of \mathbb{R}^2 and (4.11) holds, we have $\|u_{\delta_\alpha} - u_1\|_{L^1(\Omega, \mathbf{R})} \rightarrow 0$.

Step 2. By the compactness theorem for the space $BD(\Omega)$ (see [14, Chap. 2, Theorem 2.4]) the injection $\{\nabla u \in L^1(\Omega, \mathbb{R}^2) \mid u \in HB(\Omega)\}$ into $L^1(\Omega, \mathbb{R}^2)$ is compact. Then there exists a finer net $\{u_{\delta_{\alpha\beta}}\}_{\beta \in B}$ and a function $\widehat{w} \in L^1(\Omega, \mathbb{R}^2)$ such that $\|\nabla u_{\delta_{\alpha\beta}} - \widehat{w}\|_{L^1(\Omega, \mathbb{R}^2)} \rightarrow 0$. We obtain $-\int_\Omega u_{\delta_{\alpha\beta}} \text{div } \xi \, dx = \int_\Omega \nabla u_{\delta_{\alpha\beta}} \xi \, dx \rightarrow \int_\Omega \widehat{w} \xi \, dx$ for every $\xi \in C_c^1(\Omega, \mathbb{R}^2)$. Then $\nabla u_1 = \widehat{w}$ in the sense of distributions on Ω . Therefore, by Step 1, we obtain

$$(4.13) \quad \|u_{\delta_{\alpha\beta}} - u_1\|_{W^{1,1}(\Omega)} \rightarrow 0.$$

Step 3. The net $\{u_{\delta_{\alpha\beta}}\}_{\beta \in B}$ is bounded in $HB(\Omega)$, so there exists a finer net $\{u_{\delta_{\alpha\beta\gamma}}\}$ and a measure $\mu_1 \in \mathbb{M}_b(\Omega, \mathbf{E}_s^2)$ such that $\int_\Omega \varphi : (\nabla \nabla u_{\delta_{\alpha\beta\gamma}}) \rightarrow \int_\Omega \varphi : \mu_1$ for every $\varphi \in C_c^2(\Omega, \mathbf{E}_s^2)$. Since we have $\int_\Omega (\text{div div } \varphi) u_{\delta_{\alpha\beta\gamma}} \, dx \rightarrow \int_\Omega (\text{div div } \varphi) u_1 \, dx = \int_\Omega \varphi : \mu_1$ for such φ , we find that $\nabla \nabla u_1 = \mu_1$ in the sense of distributions on Ω . Therefore, by [14, Chap. 3, (2.79)], $u_1 \in HB(\Omega)$ and $\{u_{\delta_{\alpha\beta\gamma}}\}$ converges to u_1 in $\sigma(\mathcal{Z}(HB(\Omega)); W^2(\Omega, \text{div div}))$ (cf. (4.9) and (4.10)).

Step 4. We show that $\|u_1\|_{HB} \leq \widehat{r}$. Indeed, $\lim \|u_{\delta_{\alpha\beta\gamma}}\|_{W^{1,1}(\Omega)} = \|u_1\|_{W^{1,1}(\Omega)}$ by (4.13). Then for every $\varepsilon > 0$ there exists γ_ε such that for every $\gamma > \gamma_\varepsilon$ we obtain

$$(4.14) \quad \begin{aligned} \widehat{r} - \lim \|u_{\delta_{\alpha\beta\gamma}}\|_{W^{1,1}} + \varepsilon &\geq \sup_{\gamma > \gamma_\varepsilon} \|\nabla \nabla u_{\delta_{\alpha\beta\gamma}}\|_{\mathbb{M}_b(\Omega, \mathbf{E}_s^2)} \\ &= \sup_{\gamma > \gamma_\varepsilon} \sup_{\mathbf{w} \in C_0} \left\{ \int_\Omega \mathbf{w} : \nabla \nabla u_{\delta_{\alpha\beta\gamma}} \mid \mathbf{w} \in C_0(\Omega, \mathbf{E}_s^2), \sup_{i,j} \{\|w_{ij}\|_{C(\Omega)}\} \leq 1 \right\} \\ &= \sup_{\mathbf{w} \in C_0} \sup_{\gamma > \gamma_\varepsilon} \left\{ \int_\Omega \mathbf{w} : \nabla \nabla u_{\delta_{\alpha\beta\gamma}} \mid \mathbf{w} \in C_0(\Omega, \mathbf{E}_s^2), \sup_{i,j} \{\|w_{ij}\|_{C(\Omega)}\} \leq 1 \right\} \\ &\geq \sup_{\mathbf{w} \in C_0} \lim_{\gamma > \gamma_\varepsilon} \left\{ \int_\Omega \mathbf{w} : \nabla \nabla u_{\delta_{\alpha\beta\gamma}} \mid \mathbf{w} \in C_0(\Omega, \mathbf{E}_s^2), \sup_{i,j} \{\|w_{ij}\|_{C(\Omega)}\} \leq 1 \right\} \\ &= \sup_{\mathbf{w} \in C_0} \left\{ \int_\Omega \mathbf{w} : \nabla \nabla u_1 \mid \mathbf{w} \in C_0(\Omega, \mathbf{E}_s^2), \sup_{i,j} \{\|w_{ij}\|_{C(\Omega)}\} \leq 1 \right\} \\ &= \|\nabla \nabla u_1\|_{\mathbb{M}_b(\Omega, \mathbf{E}_s^2)}, \end{aligned}$$

where w_{ij} are the components of the matrix \mathbf{w} (cf. definition of $\|\cdot\|_{\mathbb{M}_b(\Omega, \mathbf{E}_s^2)}$). Therefore $[\text{cl}_{\|\cdot\|_{HB}}(B(0, \widehat{r})), \sigma(\mathcal{Z}(HB(\Omega)); W^2(\Omega, \text{div div}))]$ is compact. ■

THEOREM 9 (cf. [3]). *The space $[\text{cl}_{\|\cdot\|_{HB}} B(0, \widehat{r})]$, weak* HB topology] is homeomorphic to the space $\mathcal{Z}(\text{cl}_{\|\cdot\|_{HB}} B(0, r))$ endowed with the topology*

$$(4.15) \quad \sigma(\mathcal{Z}(HB(\Omega)); C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega, \mathbb{R})),$$

where the bilinear form $\langle \cdot; \cdot \rangle_{\mathcal{Z}}$ between $\mathcal{Z}(HB(\Omega))$ and $C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega, \mathbb{R})$ is given by (4.4) and (4.5) (see also (4.6)).

Proof. \mathcal{Z} is a one-to-one function. We will show that $\mathcal{Z}(\text{cl}_{\|\cdot\|_{HB}} B(0, \widehat{r}))$ is compact in the topology (4.15).

Step 1. Let a net $\{u_t\}_{t \in T}$ be a subset of $\text{cl}_{\|\cdot\|_{HB}} B(0, \widehat{r})$ (cf. [11, pp. 73–81]). Due to Steps 1 and 2 of the proof of Theorem 8, there exists a finer net $\{u_\delta\}_{\delta \in D}$ (where $D \subset T$) and $u_1 \in W^{1,1}(\Omega)$ such that $\|u_\delta - u_1\|_{W^{1,1}(\Omega)} \rightarrow 0$ and $\|u_\delta - u_1\|_{L^2(\Omega, \mathbb{R})} \rightarrow 0$.

Step 2. The net $\{\nabla \nabla u_\delta\}_{\delta \in D}$ is bounded in $\|\cdot\|_{\mathbb{M}_b}$. Thus there exists a finer net $\{\nabla \nabla u_\kappa\}_{\kappa \in K}$ (where $K \subset D$) such that $\nabla \nabla u_\kappa \rightharpoonup \mathbf{M}_0 \in \mathbb{M}_b(\Omega, \mathbf{E}_s^2)$ (in the weak* \mathbb{M}_b topology). By Proposition 1, there exists $u_g \in D'(\Omega)$ such that $\nabla \nabla u_g = \mathbf{M}_0$, and $\nabla \nabla u_\kappa \rightharpoonup \nabla \nabla u_g \in \mathbb{M}_b(\Omega, \mathbf{E}_s^2)$ in the weak* $\mathbb{M}_b(\Omega, \mathbf{E}_s^2)$ topology. The net $\{u_\kappa\}_{\kappa \in K}$ is finer than $\{u_\delta\}_{\delta \in D}$, so $u_1 = u_g + c$, where $c \in \ker(\nabla \nabla(\cdot))$, because

$$(4.16) \quad \int_{\Omega} \xi : \nabla \nabla u_\kappa = \int_{\Omega} (\text{div div } \xi) u_\kappa \, dx \rightarrow \int_{\Omega} (\text{div div } \xi) u_g \, dx \quad \forall \xi \in C_c^2(\Omega, \mathbf{E}_s^2).$$

Therefore, we can assume that $u_1 = u_g$ and we conclude that $u_g \in HB(\Omega)$.

Step 3. By Step 1 of the proof of Theorem 8 we get

$$(4.17) \quad \langle (\nabla \nabla(u_\kappa - u_g), \gamma_1(u_\kappa - u_g), \gamma_0(u_\kappa - u_g)); (\varphi^*, \beta_0(\varphi^*)) \rangle_b \rightarrow 0$$

for every $\varphi^* \in C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2)$, since $C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \subset W^2(\Omega, \text{div div})$. Moreover, $\|\gamma_0(u_\kappa - u_g)\|_{L^1(\text{Fr } \Omega)} \rightarrow 0$, because $\|u_\kappa - u_g\|_{W^{1,1}(\Omega)} \rightarrow 0$ (cf. Step 1). Then

$$(4.18) \quad \int_{\text{Fr } \Omega} \gamma_0(u_\kappa - u_g) \psi^* \, ds \rightarrow 0$$

for every $\psi^* \in C(\text{Fr } \Omega, \mathbb{R})$, since the closed and bounded set $\text{Fr } \Omega$ is compact and $C(\text{Fr } \Omega, \mathbb{R}) \subset L^\infty(\text{Fr } \Omega, \mathbb{R})$.

Step 4. Because of Step 1, we can show (similarly to Step 4 of the proof of Theorem 8) that $\|u_g\|_{HB} \leq \widehat{r}$.

Step 5. By (2.16), $\beta_0(\varphi^*) \in C(\text{Fr } \Omega, \mathbb{R})$ if $\varphi^* \in C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2)$. Then

$$(4.19) \quad \int_{\Omega} \varphi^* : \nabla \nabla(u_\kappa - u_g) - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u_\kappa - u_g) \, ds + \int_{\text{Fr } \Omega} \gamma_0(u_\kappa - u_g) \psi^* \, ds \rightarrow 0$$

for every $\varphi^* \in C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2)$ and $\psi^* \in C(\text{Fr } \Omega, \mathbb{R})$ (see (4.4), (4.9), (4.17) and (4.18)). By Steps 1 and 2 we obtain

$$(4.20) \quad \int_{\Omega} \mathbf{M} : \nabla \nabla (u_{\kappa} - u_g) + \int_{\Omega} h(u_{\kappa} - u_g) dx \rightarrow 0$$

for every $\mathbf{M} \in C_c(\Omega, \mathbf{E}_s^2)$ and $h \in C_c(\Omega, \mathbb{R})$, because $\nabla \nabla u_{\kappa} \rightharpoonup \nabla \nabla u_g$ (in the weak* \mathbb{M}_b topology) and $\|u_{\kappa} - u_g\|_{L^2(\text{Fr } \Omega)} \rightarrow 0$. Therefore the set $\text{cl}_{\|\cdot\|_{HB}} B(0, \hat{r})$ (endowed with the topology where convergence of nets is defined by (4.19) and (4.20)) is compact. The Hausdorff topologies (4.15) and weak* HB are weaker than the topology given by (4.19)–(4.20). By [11, Corollary 3.1.14] the weak* HB topology and (4.15) are homeomorphic on the set $\text{cl}_{\|\cdot\|_{HB}} B(0, \hat{r})$. ■

THEOREM 10. *Every closed ball $\text{cl}_{\|\cdot\|_{HB}} B(0, \hat{r})$ endowed with the topology $\sigma(\mathcal{Z}(HB(\Omega)); W^2(\Omega, \text{div div}))$ is homeomorphic to $[\text{cl}_{\|\cdot\|} B_{HB}(0, \hat{r})]$; weak* HB topology].*

Proof. Similarly to Step 5 of the proof of Theorem 9, we define a compact topology on $\text{cl}_{\|\cdot\|_{HB}} B(0, \hat{r})$, which is stronger than the weak* HB topology and $\sigma(\mathcal{Z}(HB(\Omega)); W^2(\Omega, \text{div div}))$. Next we use Corollary 3.1.14 of [11] to deduce the equivalence of the three topologies on $\text{cl}_{\|\cdot\|_{HB}} B(0, \hat{r})$. ■

We study a thin plate occupying a given surface Ω . This plate is subjected to a force, of surface density $f \in L^2(\Omega, \mathbb{R})$, acting in a direction at right angles to the surface. We shall suppose the plate to be subject to force $g_0 \in L^\infty(\text{Fr } \Omega) \subset (\gamma_0(W^{2,1}(\Omega)))^*$ on Γ_1 and bending moment of linear density $g_1 \in L^\infty(\Gamma_1, \mathbb{R})$. Let λ (where $\infty > \lambda \geq 0$) stand for the load multiplier.

Define

$$(4.21) \quad W^2(\Omega, \text{div div}) \ni \mathbf{M} \mapsto [P_{\lambda, r}^*](\mathbf{M}) = - \int_{\Gamma_1} I_{B_{\mathbb{R}}(0, r)}(\lambda g_1 - \beta_1(\mathbf{M})) ds \\ - I_{B_{(\gamma_0(W^{2,1}))^*(0, r)}}(\beta_0(\mathbf{M})|_{\text{int } \Gamma_1} - \lambda g_0) - \int_{\Omega} j^*(x, \mathbf{M}) dx$$

if $\text{div div } \mathbf{M} = +\lambda f$ in Ω and $[P_{\lambda, r}^*](\mathbf{M}) = -\infty$ otherwise. Here $\beta_0(\mathbf{M})|_{\text{int } \Gamma_1}$ denotes the functional $\tilde{\psi}^* \in (\gamma_0(W^{2,1}(\Omega)))^*$ with $\tilde{\psi}^* = \beta_0(\mathbf{M})$ on $\text{int } \Gamma_1$ and $\tilde{\psi}^* = 0$ on $\text{cl } \Gamma_0$ (see [3, Proposition 5.16]). Moreover, $I_A(\cdot)$ denotes the indicator function of a set A . The dual relaxed problem, for $r \geq 0$, is given by the formula

$$(4.22) \quad (P_{\lambda, r}^*) \text{ find } \sup\{[P_{\lambda, r}^*](\mathbf{M}) \mid \mathbf{M} \in W^2(\Omega, \text{div div}), \\ \text{div div } \mathbf{M} = \lambda f \text{ in } \Omega\}.$$

ASSUMPTION 4. There exist $\mathbf{M}_L \in C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2)$ such that $[P_{\lambda,r}^*](\mathbf{M}_L) = \sup(P_{\lambda,r}^*) > -\infty$, $\beta_1(\mathbf{M}_L) = \lambda g_1$ on Γ_1 , $\beta_0(\mathbf{M}_L) = \lambda g_0$ on Γ_1 and $\mathbf{M}_L(x) \in \mathcal{K}_p(x)$ for all $x \in \Omega$. Moreover, there exist $\delta_0 > 0$ such that $\text{dist}(\mathbf{M}_L(x), \text{Fr } \mathcal{K}_p(x)) = \inf\{\|\mathbf{M}_L(x) - \mathbf{z}\|_{\mathbf{E}_s^2} \mid \mathbf{z} \in \text{Fr } \mathcal{K}_p(x)\} > \delta_0$ for every $x \in \Omega$. ■

Define $H_r : \mathcal{Z}(HB) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$(4.23) \quad \begin{aligned} H_r(\nabla\nabla u, \gamma_1(u), \gamma_0(u)) & \\ \equiv & - \int_{\Omega} \mathbf{M}_L : \nabla\nabla u \, dx + r \int_{\Gamma_1} |\gamma_1(u)| \, ds + r \|\gamma_0(u)\|_{\gamma_0(W^{2,1})} \\ & + \int_{\Gamma_0} I_{\{\gamma_1(u)=0\}}(\gamma_1(u)) \, ds + \int_{\Gamma_0} I_{\{\gamma_0(u)=0\}}(\gamma_0(u)) \, ds + \int_{\Omega} j(x, \nabla\nabla u) \, dx \end{aligned}$$

if $u \in W^{2,1}(\Omega)$, and $H_r(\nabla\nabla u, \gamma_1(u), \gamma_0(u)) = +\infty$ otherwise (where $\|\cdot\|_{\gamma_0(W^{2,1})} = \|\cdot\|_0$, see (2.14)). Because of the equalities

$$(4.24) \quad \begin{aligned} \lambda \int_{\Omega} f u \, dx &= \int_{\Omega} \mathbf{M}_L : \nabla\nabla u \, dx - \int_{\text{Fr } \Omega} \beta_1(\mathbf{M}_L) \gamma_1(u) \, ds \\ &+ \langle \beta_0(\mathbf{M}_L); \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})}, \end{aligned}$$

$$(4.25) \quad \lambda \int_{\Gamma_1} g_1 \gamma_1(u) \, ds = \int_{\Gamma_1} \beta_1(\mathbf{M}_L) \gamma_1(u) \, ds,$$

$$(4.26) \quad \lambda \int_{\Gamma_1} g_0 \gamma_0(u) \, ds = \int_{\Gamma_1} \beta_0(\mathbf{M}_L) \gamma_0(u) \, ds$$

we obtain

$$(4.27) \quad \begin{aligned} H_r(\nabla\nabla u, \gamma_1(u), \gamma_0(u)) &= -\lambda \int_{\Omega} f u \, dx - \lambda \int_{\Gamma_1} g_1 \gamma_1(u) \, ds \\ &+ r \int_{\Gamma_1} |\gamma_1(u)| \, ds + \lambda \int_{\Gamma_1} g_0 \gamma_0(u) \, ds + r \|\gamma_0(u)\|_{\gamma_0(W^{2,1})} + \int_{\Omega} j(x, \nabla\nabla u) \, dx \end{aligned}$$

if $u \in W^{2,1}(\Omega)$, $\gamma_1(u) = 0$ on Γ_0 , $\gamma_0(u) = 0$ on Γ_0 , and $H_r(\nabla\nabla u, \gamma_1(u), \gamma_0(u)) = +\infty$ otherwise (cf. (2.17), (4.21), Assumption 4 and [14, Chap. 3, (2.33)]). The functional H_r describe the primal displacement problem for plates, with relaxed boundary conditions on Γ_1 .

We define a linear space by

$$(4.28) \quad \begin{aligned} A(\Omega) &\equiv \{(\varphi^*, \psi^*) \in W^2(\Omega, \text{div div}) \times (\gamma_0(W^{2,1}(\Omega)))^* \mid \\ &\exists \psi_1^* \in C(\text{Fr } \Omega), \exists \varphi_1^* \in C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2), \exists \varphi_s^* \in W^2_{\text{div}}(\Omega, \text{div div}), \\ &\varphi^* = \varphi_1^* + \varphi_s^* \text{ and } \psi^* = \psi_1^* + \beta_0(\varphi_s^*)\} \end{aligned}$$

(see (2.12)). Moreover, a net $\{u_\delta\}_{\delta \in D} \subset HB(\Omega)$ converges to $\hat{u} \in HB(\Omega)$

in the topology $\sigma(\mathcal{Z}(HB(\Omega)); A(\Omega))$ if and only if

$$(4.29) \quad \int_{\Omega} \varphi^* : \nabla \nabla (u_{\delta} - \widehat{u}) - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u_{\delta} - \widehat{u}) ds \\ + \langle \psi^*; \gamma_0(u_{\delta} - \widehat{u}) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \rightarrow 0$$

for every $(\varphi^*, \psi^*) \in A(\Omega)$. Because of the duality between $\mathcal{Z}(HB(\Omega))$ and $A(\Omega)$ we obtain the dual functional $H_r^* : A(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ by setting

$$(4.30) \quad H_r^*(\varphi^*, \psi^*) = \sup \left\{ \int_{\Omega} \varphi^* : \nabla \nabla u dx - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) ds \right. \\ \left. + \langle \psi^*; \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} - H_r(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in W^{2,1}(\Omega) \right\}$$

for $(\varphi^*, \psi^*) \in A(\Omega)$. The bidual functional $H_r^{**} : \mathcal{Z}(HB(\Omega)) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(4.31) \quad H_r^{**}(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) = \sup \left\{ \int_{\Omega} \varphi^* : \nabla \nabla u - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) ds \right. \\ \left. + \langle \psi^*; \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} - H_r^*(\varphi^*, \psi^*) \mid (\varphi^*, \psi^*) \in A(\Omega) \right\}$$

for $u \in HB(\Omega)$. The extension $\widetilde{\mathcal{Z}}(HB(\Omega))$ of $\mathcal{Z}(HB(\Omega))$ is given by

$$(4.32) \quad \widetilde{\mathcal{Z}}(HB(\Omega)) = \{(\mathbf{z}, \gamma_1(u), w) \in \text{span}[\nabla \nabla(HB(\Omega)), L^1(\Omega, \mathbf{E}_s^2)] \\ \times L^1(\text{Fr } \Omega) \times \gamma_0(W^{2,1}(\Omega)) \mid \\ \exists \widehat{\mathbf{z}} \in L^1(\Omega, \mathbf{E}_s^2), \exists \widehat{w} \in \gamma_0(W^{2,1}(\Omega)), \exists \widetilde{u} \in HB(\Omega), \\ \mathbf{z} = \widehat{\mathbf{z}} dx + \nabla \nabla \widetilde{u}, \gamma_1(u) = \gamma_1(\widetilde{u}) \text{ and } w = \widehat{w} + \gamma_0(\widetilde{u})\}.$$

The bilinear pairing between $\widetilde{\mathcal{Z}}(HB(\Omega))$ and $A(\Omega)$ is given by

$$(4.33) \quad \langle (\mathbf{z}, \gamma_1(u), w); (\varphi^*, \psi^*) \rangle_1 \equiv \int_{\Omega} \varphi^* : \mathbf{z} - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) ds \\ + \langle \psi^*; w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})}$$

for $(\mathbf{z}, \gamma_1(u), w) \in \widetilde{\mathcal{Z}}(HB(\Omega))$ and $(\varphi^*, \psi^*) \in A(\Omega)$. The extension \widetilde{H}_r of H_r onto the space $\widetilde{\mathcal{Z}}(HB(\Omega))$ is given by

$$(4.34) \quad \widetilde{H}_r(\mathbf{z}, \gamma_1(u), w) \\ = - \int_{\Omega} \mathbf{M}_L : \mathbf{z} dx + \int_{\text{Fr } \Omega} \beta_1(\mathbf{M}_L) \gamma_1(u) ds \\ - \langle \beta_0(\mathbf{M}_L); w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} - \int_{\Gamma_1} \beta_1(\mathbf{M}_L) \gamma_1(u) ds$$

$$\begin{aligned}
& + \int_{\Gamma_1} \beta_0(\mathbf{M}_L) w \, ds + r \int_{\Gamma_1} |\gamma_1(u)| \, ds + r \|w\|_{\gamma_0(W^{2,1})} \\
& + \int_{\Gamma_0} I_{\{\gamma_1(u)=0\}}(\gamma_1(u)) \, ds + \int_{\Gamma_0} I_{\{w=0\}}(w) \, ds + \int_{\Omega} j(x, \mathbf{z}) \, dx
\end{aligned}$$

if $\mathbf{z} = \widehat{\mathbf{z}} \, dx + \nabla \nabla \tilde{u}$, $\gamma_1(u) = \gamma_1(\tilde{u})$ and $w = \widehat{w} + \gamma_0(\tilde{u})$, with $\widehat{\mathbf{z}} \in L^1(\Omega, \mathbf{E}_s^2)$, $\widehat{w} \in \gamma_0(W^{2,1}(\Omega))$, $\tilde{u} \in W^{2,1}(\Omega)$, and $\widetilde{H}_r(\mathbf{z}, \gamma_1(u), w) = +\infty$ otherwise. By the duality between $\widetilde{\mathcal{Z}}(HB(\Omega))$ and $A(\Omega)$ we define a functional \widetilde{H}_r^* on the linear space $A(\Omega)$ by

$$(4.35) \quad \widetilde{H}_r^*(\varphi^*, \psi^*) = \sup\{\langle (\mathbf{z}, \gamma_1(u), w); (\varphi^*, \psi^*) \rangle_1 - \widetilde{H}_r(\mathbf{z}, \gamma_1(u), w) \mid (\mathbf{z}, \gamma_1(u), w) \in \widetilde{\mathcal{Z}}(HB(\Omega))\}$$

for $(\varphi^*, \psi^*) \in A(\Omega)$. The bidual functional $\widetilde{H}_r^{**} : \widetilde{\mathcal{Z}}(HB(\Omega)) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(4.36) \quad \widetilde{H}_r^{**}(\mathbf{z}, \gamma_1(u), w) = \sup\{\langle (\mathbf{z}, \gamma_1(u), w); (\varphi^*, \psi^*) \rangle_1 - \widetilde{H}_r^*(\varphi^*, \psi^*) \mid (\varphi^*, \psi^*) \in A(\Omega)\}$$

for $(\mathbf{z}, \gamma_1(u), w) \in \widetilde{\mathcal{Z}}(HB(\Omega))$.

PROPOSITION 11. *The explicit form of \widetilde{H}_r^* is*

$$(4.37) \quad \begin{aligned} \widetilde{H}_r^*(\varphi^*, \psi^*) &= \int_{\Gamma_1} I_{B_{\mathbb{R}}(0,r)}(-\beta_1(\varphi^*)) \, ds + I_{B_{(\gamma_0(W^{2,1}))^*}(0,r)}(\psi^*_{\text{int } \Gamma_1}) \\ &+ \int_{\Omega} j^*(x, (\varphi^* + \mathbf{M}_L)) \, dx \end{aligned}$$

for $(\varphi^*, \psi^*) \in A(\Omega)$. Here $\psi^*_{\text{int } \Gamma_1}$ denotes the functional $\widetilde{\psi}^* \in (\gamma_0(W^{2,1}(\Omega)))^*$ such that $\widetilde{\psi}^* = \psi^*$ on $\text{int } \Gamma_1$ and $\widetilde{\psi}^* = 0$ on $\text{cl } \Gamma_0$ (see (4.21)). If we extend \widetilde{H}_r^* onto the space $W^2(\Omega, \text{div div}) \times (\gamma_0(W^{2,1}))^*$ by (4.35), then \widetilde{H}_r^* is given by (4.37) for every $(\varphi^*, \psi^*) \in W^2(\Omega, \text{div div}) \times (\gamma_0(W^{2,1}))^*$.

Proof. By Theorem 3A of [13] we obtain

$$(4.38) \quad \begin{aligned} \widetilde{H}_r^*(\varphi^*, \psi^*) &= \sup \left\{ \int_{\Omega} \varphi^* : \mathbf{z} \, dx - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) \, ds \right. \\ &+ \langle \psi^*; w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} + \int_{\Omega} \mathbf{M}_L : \mathbf{z} \, dx - r \int_{\Gamma_1} |\gamma_1(u)| \, ds \\ &- r \|w\|_{\gamma_0(W^{2,1})} - \int_{\Gamma_0} I_{\{\gamma_1(u)=0\}}(\gamma_1(u)) \, ds - \int_{\Gamma_0} I_{\{w=0\}}(w) \, ds - \int_{\Omega} j(x, \mathbf{z}) \, dx \left. \right\} \\ &\quad \mathbf{z} \in L^1(\Omega, \mathbf{E}_s^2), u \in W^{2,1}(\Omega), w \in \gamma_0(W^{2,1}(\Omega)) \end{aligned}$$

$$\begin{aligned}
&= \sup \left\{ \int_{\Omega} (\varphi^* + \mathbf{M}_L) : \mathbf{z} \, dx - \int_{\Omega} j(x, \mathbf{z}) \, dx \mid \mathbf{z} \in L^1(\Omega, \mathbf{E}_s^2) \right\} \\
&\quad + \sup \left\{ - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) \, ds - r \int_{\Gamma_1} |\gamma_1(u)| \, ds \right. \\
&\quad \quad \left. - \int_{\Gamma_0} I_{\{\gamma_1(u)=0\}}(\gamma_1(u)) \, ds \mid u \in W^{2,1}(\Omega) \right\} \\
&\quad + \sup \{ \langle \psi^*; w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} - r \|w\|_{\gamma_0(W^{2,1})} \mid \\
&\quad \quad w \in \gamma_0(W^{2,1}(\Omega)) \text{ and } w = 0 \text{ on } \text{cl } \Gamma_0 \}
\end{aligned}$$

for every $(\varphi^*, \psi^*) \in A(\Omega)$, since γ_1 is a surjection on $L^1(\text{Fr } \Omega)$ (cf. [8]). Then we obtain (4.37). ■

Define $\tilde{H}_r^{*\#} : \tilde{\mathcal{Z}}(HB) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned}
(4.39) \quad \tilde{H}_r^{*\#}(\mathbf{z}, \gamma_1(u), w) &= \sup \{ \langle (\mathbf{z}, \gamma_1(u), w); (\varphi^*, \psi^*) \rangle_1 - \tilde{H}_r^*(\varphi^*, \psi^*) \mid \\
&\quad (\varphi^*, \psi^*) \in C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega) \}
\end{aligned}$$

for $(\mathbf{z}, \gamma_1(u), w) \in \tilde{\mathcal{Z}}(HB(\Omega))$.

PROPOSITION 12. *The explicit form of $\tilde{H}_r^{*\#}$ is*

$$\begin{aligned}
(4.40) \quad \tilde{H}_r^{*\#}(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) &= -\lambda \int_{\Omega} f u \, dx - \lambda \int_{\Gamma_1} g_1 \gamma_1(u) \, ds \\
&\quad + r \int_{\Gamma_1} |\gamma_1(u)| \, ds + \lambda \int_{\Gamma_1} g_0 \gamma_0(u) \, ds + r \|\gamma_0(u)\|_{\gamma_0(W^{2,1})} \\
&\quad + \int_{\Gamma_0} j_{\infty}(x, -\boldsymbol{\nu} \otimes \boldsymbol{\nu} \gamma_1(u)) \, ds + \int_{\Gamma_0} I_{\{\gamma_0(u)=0\}}(\gamma_0(u)) \, ds \\
&\quad + \int_{\Omega} j(x, (\nabla \nabla u)_a) \, dx + \int_{\Omega} j_{\infty} \left(x, \frac{d(\nabla \nabla u)_s}{d|(\nabla \nabla u)_s|} \right) d|(\nabla \nabla u)_s|
\end{aligned}$$

for every $u \in HB(\Omega)$.

Proof. The field \mathbf{M}_L is a solution of $(P_{\lambda,r}^*)$, i.e. $[P_{\lambda,r}^*](\mathbf{M}_L) = \sup(P_{\lambda,r}^*)$ (cf. (4.21)). Thus $\tilde{H}_r^*(0, 0) < \infty$. Moreover, $C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2)$ and $C(\text{Fr } \Omega)$ are PCU-stable, so by [5] and by Theorem 3A of [13] we get

$$(4.41) \quad \tilde{H}_r^{*\#}(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) = \sup \left\{ \int_{\Omega} \varphi^* : \nabla \nabla u - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(u) \, ds \right.$$

$$\begin{aligned}
& + \langle \psi^*; \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \\
& - \int_{\Gamma_1} I_{B_{\mathbb{R}}(0,r)}(-\beta_1(\varphi^*)) ds - I_{B_{(\gamma_0(W^{2,1}))^*}(0,r)}(\psi^*_{|\text{int } \Gamma_1}) \\
& - \int_{\Omega} j^*(x, (\varphi^* + \mathbf{M}_L)) dx \left| (\varphi^*, \psi^*) \in C^1_{\text{div div}}(\bar{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega) \right\} \\
= & \sup \left\{ \int_{\Omega} [(\varphi^* + \mathbf{M}_L) : (\nabla \nabla u)_a - j^*(x, (\varphi^* + \mathbf{M}_L))] dx \right. \\
& + \int_{\Omega} \left[(\varphi^* + \mathbf{M}_L) : \frac{d(\nabla \nabla u)_s}{d|(\nabla \nabla u)_s|} - j^*_{\infty}(x, (\varphi^* + \mathbf{M}_L)) \right] d|(\nabla \nabla u)_s| \\
& + \int_{\Gamma_1} [(-\beta_1(\varphi^*))\gamma_1(u) - I_{B_{\mathbb{R}}(0,r)}(-\beta_1(\varphi^*))] ds \\
& + \int_{\Gamma_0} [(\varphi^* + \mathbf{M}_L) : (-\nu \otimes \nu \gamma_1(u)) - j^*_{\infty}(x, (\varphi^* + \mathbf{M}_L))] ds \left| \right. \\
& \left. \varphi^* \in C^1_{\text{div div}}(\bar{\Omega}, \mathbf{E}_s^2) \right\} \\
& - \int_{\Omega} \mathbf{M}_L : (\nabla \nabla u) + \int_{\Gamma_0} \beta_1(\mathbf{M}_L)\gamma_1(u) ds \\
& + \sup \{ \langle \psi^*; \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} - I_{B_{(\gamma_0(W^{2,1}))^*}(0,r)}(\psi^*_{|\text{int } \Gamma_1}) \mid \\
& \left. \psi^* \in C(\text{Fr } \Omega) \right\}
\end{aligned}$$

for every $u \in HB(\Omega)$, which is (4.40) (cf. (4.24), (4.25) and (4.26)). Here $\nabla \nabla u = (\nabla \nabla u)_a + (\nabla \nabla u)_s$ is the Lebesgue decomposition of $\nabla \nabla u$ into absolutely continuous and singular parts with respect to dx . In the above calculations we have used the equality $j^*_{\infty}(x, \varphi^*) = I_{\mathcal{K}_p(x)}(\varphi^*)$, which holds for every $\varphi^* \in \mathbf{E}_s^2$ and $x \in \bar{\Omega}$. ■

DEFINITION 2. We say that H_r is *coercive* if

$$(4.42) \quad H_r(\nabla \nabla u_m, \gamma_1(u_m), \gamma_0(u_m)) \rightarrow +\infty$$

for every sequence $\{u_m\}_{m \in \mathbb{N}} \subset HB(\Omega)$ such that $\|u_m\|_{HB(\Omega)} \rightarrow \infty$.

ASSUMPTION 5. Let H_r be a coercive function. Moreover, let $\tilde{H}_r^{\#}$ be the largest minorant that is less than H_r and l.s.c. in the weak* HB topology.

DEFINITION 3. We say that a net $\{u_{\delta}\}_{\delta \in D} \subset HB(\Omega)$ converges to $u_0 \in HB(\Omega)$ in the topology (4.43)–(4.44) if

$$(4.43) \quad \langle (\nabla \nabla(u_{\delta} - u_0), \gamma_1(u_{\delta} - u_0), \gamma_0(u_{\delta} - u_0)); (\varphi^*, \beta_0(\varphi^*)) \rangle_1 \rightarrow 0$$

for every $\varphi^* \in W^2(\Omega, \text{div div})$ and

$$(4.44) \quad \langle (\nabla \nabla(u_\delta - u_0), \gamma_1(u_\delta - u_0), \gamma_0(u_\delta - u_0)); (\tilde{\varphi}^*, \psi^*) \rangle_1 \rightarrow 0$$

for every $(\tilde{\varphi}^*, \psi^*) \in C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$.

LEMMA 13. *Every closed ball $\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))$ (in $\mathcal{Z}(HB(\Omega))$) is compact in the topology (4.43)–(4.44).*

Proof. Let $\{u_\delta\}_{\delta \in D} \subset HB(\Omega)$ be a net in $B_{HB}(0, \hat{r})$. Then, by Theorem 8, there exists a finer net $\{u_{\delta_\beta}\}_{\beta \in B} \subset HB(\Omega)$ and $\tilde{u} \in \text{cl}_{\|\cdot\|} B_{HB}(0, \hat{r})$ such that $\{u_{\delta_\beta}\}$ converges to \tilde{u} in the sense of (4.43). Moreover, by Theorem 9, there exists a net $\{u_{\delta_{\beta_\gamma}}\}$ (finer than $\{u_{\delta_\beta}\}_{\beta \in B}$) and $\hat{u} \in \text{cl}_{\|\cdot\|} B_{HB}(0, \hat{r})$ such that $\{u_{\delta_{\beta_\gamma}}\}$ converges to \hat{u} in the sense of (4.44). Therefore $\{u_{\delta_{\beta_\gamma}}\}$ converges to $\hat{u} \in \text{cl}_{\|\cdot\|} B_{HB}(0, \hat{r})$ in the topology (4.43)–(4.44). ■

COROLLARY 14. *Every closed ball $\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))$ (in $\mathcal{Z}(HB(\Omega))$) is compact in the topology $\sigma(\mathcal{Z}(HB(\Omega)); A(\Omega))$ (cf. (4.29)). Moreover, the topologies (4.43)–(4.44) and $\sigma(\mathcal{Z}(HB(\Omega)); A(\Omega))$ are equivalent on each such ball.*

Proof. The Hausdorff topology $\sigma(\mathcal{Z}(HB(\Omega)); A(\Omega))$ is weaker than the topology (4.43)–(4.44). Then, by [11, Corollary 3.1.14] and Lemma 13, we obtain the equivalence of the two topologies on $\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))$. ■

PROPOSITION 15. *The space $[\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))]$, weak* topology] is homeomorphic to the space $[\mathcal{Z}(\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))), \sigma(\mathcal{Z}(HB(\Omega)); A(\Omega))]$.*

Proof. The Hausdorff topology $\sigma(\mathcal{Z}(HB); W^2(\Omega, \text{div div}))$ is weaker than (4.43)–(4.44). Then (by [11, Corollary 3.1.14], Corollary 14 and Theorem 10) we obtain the equivalence of these topologies on $\text{cl}_{\|\cdot\|_{HB}}(B(0, \hat{r}))$. ■

LEMMA 16. *For every $(\varphi^*, \psi^*) \in A(\Omega)$ we have $\tilde{H}_r^*(\varphi^*, \psi^*) \geq H_r^*(\varphi^*, \psi^*)$. Moreover, $\tilde{H}_r^{**}(\hat{\mathbf{z}}) \leq H_r^{**}(\hat{\mathbf{z}})$ for every $\hat{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))$.*

PROPOSITION 17. *Under Assumption 5, $\tilde{H}_r^{\#\#}(\hat{\mathbf{z}}) = \tilde{H}_r^{**}(\hat{\mathbf{z}}) = H_r^{**}(\hat{\mathbf{z}})$ for every $\hat{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))$.*

Proof. In the definition of \tilde{H}_r^{**} we take the supremum over a larger domain, so $\tilde{H}_r^{\#\#} \leq \tilde{H}_r^{**}$. By Lemma 16, $\tilde{H}_r^{\#\#} \leq \tilde{H}_r^{**} \leq H_r^{**}$. Because of (4.30) and (4.31), H_r^{**} is the l.s.c. regularization of H_r in $\sigma(\mathcal{Z}(HB(\Omega)); A(\Omega))$. By Assumption 5 and Proposition 15 we get $\tilde{H}_r^{\#\#} = H_r^{**}$. ■

LEMMA 18. *Let Assumption 5 hold. For every $u \in W^{2,1}(\Omega)$ such that $\gamma_1(u)|_{\Gamma_0} = 0$, we have $\tilde{H}_r^{**}(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) = H_r(\nabla \nabla u, \gamma_1(u), \gamma_0(u))$.*

Proof. By Lemma 16, $\tilde{H}_r^{**} \leq H_r^{**} \leq H_r$. Then, by (4.40) and Proposition 17, we obtain the conclusion of the lemma. ■

LEMMA 19. Let $\varphi_s^* \in W_{\text{div}}^2(\Omega, \text{div div})$. Then

$$(4.45) \quad H_r^*(\varphi^*, \psi^*) = H_r^*(\varphi^* + \varphi_s^*, \psi^* + \beta_0(\varphi_s^*)),$$

for every $\varphi^* \in C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2)$ and $\psi^* \in C(\text{Fr } \Omega, \mathbb{R})$ (cf. (2.12)).

Proof. By (4.30) and (2.17) we obtain

$$(4.46) \quad \begin{aligned} & H_r^*(\varphi^*, \psi^*) \\ &= \sup \left\{ \int_{\Omega} u(\text{div div } \varphi^*) dx \right. \\ &\quad - \langle (\beta_0(\varphi^*) - \psi^*); \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \\ &\quad \left. - H_r(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in W^{2,1}(\Omega) \right\} \\ &= \sup \left\{ \int_{\Omega} u(\text{div div}(\varphi^* + \varphi_s^*)) dx \right. \\ &\quad + \langle (\psi^* + \beta_0(\varphi_s^*) - \beta_0(\varphi^* + \varphi_s^*)); \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \\ &\quad \left. - H_r(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in W^{2,1}(\Omega) \right\} \\ &= H_r^*(\varphi^* + \varphi_s^*, \psi^* + \beta_0(\varphi_s^*)) \end{aligned}$$

for every $(\varphi^*, \psi^*) \in C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$. ■

Let $\ker \nabla \nabla(\cdot) \equiv \{u \in HB \mid \nabla \nabla u = 0\}$. Then

$$(4.47) \quad \ker \nabla \nabla(\cdot) = \{u \in W^{2,1} \mid u(z, y) = az + by + c \text{ and } a, b, c \in \mathbb{R}\}.$$

There exist continuous projections of the space $W^{2,1}(\Omega)$ on

$$(4.48) \quad \tilde{P}_I = \{u \in W^{2,1}(\Omega) \mid u(z, y) = c \text{ and } c \in \mathbb{R}\},$$

$$(4.49) \quad \tilde{P}_{II} = \{u \in W^{2,1}(\Omega) \mid u(z, y) = az \text{ and } a \in \mathbb{R}\},$$

$$(4.50) \quad \tilde{P}_{III} = \{u \in W^{2,1}(\Omega) \mid u(z, y) = by \text{ and } b \in \mathbb{R}\},$$

given by

$$(4.51) \quad W^{2,1}(\Omega) \ni u \mapsto P_I(u) = \frac{1}{\text{meas } \Omega} \int_{\Omega} u(z, y) dz dy,$$

$$(4.52) \quad W^{2,1}(\Omega) \ni u \mapsto P_{II}(u) = \frac{z}{\text{meas } \Omega} \left(\int_{\Omega} \frac{\partial u(z, y)}{\partial z} dz dy \right).$$

The projection P_{III} of $W^{2,1}(\Omega)$ on \tilde{P}_{III} is defined similarly to (4.52). Hence there exists a decomposition $W^{2,1}(\Omega) = \ker \nabla \nabla(\cdot) \oplus [W^{2,1}(\Omega)/\ker \nabla \nabla(\cdot)]$, given by

$$(4.53) \quad u = (P_I(u) + P_{II}(u) + P_{III}(u)) + \{u - (P_I(u) + P_{II}(u) + P_{III}(u))\}.$$

LEMMA 20. *The trace $(\beta_0, \beta_1) : W_{\text{div}}^2(\Omega, \text{div div}) \rightarrow \{(\gamma_0(u), \gamma_1(u)) \in \gamma_0(W^{2,1}(\Omega)) \times L^1(\text{Fr } \Omega) \mid u \in [W^{2,1}(\Omega)/\ker \nabla \nabla(\cdot)]^*\}$ is a surjection.*

Proof. The proof is similar to Step 2 of the proof of [14, Chap. 3, Proposition 2.11]. Let $(L_0, L_1) \in (\gamma_0(W^{2,1}(\Omega)))^* \times L^\infty(\text{Fr } \Omega)$, $L_0|_{\gamma_0(\ker \nabla \nabla(\cdot))} = 0$ and $L_1|_{\gamma_1(\ker \nabla \nabla(\cdot))} = 0$. Then

$$(4.54) \quad \begin{aligned} W^{2,1}(\Omega) \ni u &\mapsto l(u) \\ &= \int_{\text{Fr } \Omega} L_1 \gamma_1(u) \, ds - \langle L_0; \gamma_0(u) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \end{aligned}$$

is a continuous, linear functional on $W^{2,1}(\Omega)$ such that $l(u) = 0$ if $u \in \ker \nabla \nabla(\cdot)$. Because of [14, Chap. 3, (2.82)] the operator $u \mapsto \nabla \nabla u$ is an isomorphism of $W^{2,1}(\Omega)/\ker \nabla \nabla(\cdot)$ into $L^1(\Omega, \mathbf{E}_s^2)$. By the Hahn–Banach theorem, there exists an extension of the form l , with the same norm, on the space $L^1(\Omega, \mathbf{E}_s^2)$. Then there exists $\mathbf{M} \in L^\infty(\Omega, \mathbf{E}_s^2)$ such that $l(u) = \int_{\Omega} \mathbf{M} : \nabla \nabla u \, dx$ for all $u \in W^{2,1}(\Omega)/\ker \nabla \nabla(\cdot)$. If $\tilde{u} \in C_c^\infty(\Omega)$ then $l(\tilde{u}) = 0$. Since $l(u) = \int_{\Omega} \mathbf{M} : \nabla \nabla u \, dx - \int_{\Omega} 0u \, dx$ for all $u \in W^{2,1}(\Omega)/\ker \nabla \nabla(\cdot)$, we have $\text{div}(\text{div } \mathbf{M}) = 0$ in Ω . Due to Step 1 of the proof of [14, Chap. 3, Proposition 2.11] we see that $L_0 = \beta_0(\mathbf{M})$ and $L_1 = \beta_1(\mathbf{M})$. Finally since l is extended to a linear mapping with the same norm,

$$(4.55) \quad \begin{aligned} \|l\| &= \{\|L_0\|_{(\gamma_0(W^{2,1}))^*} + \|L_1\|_{L^\infty}\} = \|\mathbf{M}\|_{L^\infty(\Omega, \mathbf{E}_s^2)} \\ &= \|\mathbf{M}\|_{W^2(\Omega, \text{div div})}, \end{aligned}$$

and this completes the proof of surjectivity. ■

We say that a net $\{(\varphi_k^*, \psi_k^*)\}_{k \in K} \subset A(\Omega)$ converges to $(\widehat{\varphi}^*, \widehat{\psi}^*) \in A(\Omega)$ in the topology

$$(4.56) \quad \sigma(A(\Omega); L^\infty(\Omega, \mathbf{E}_s^2) \times L^\infty(\Gamma_1, \mathbb{R}) \times \gamma_0(W^{2,1}(\Omega))|_{\Gamma_1})$$

if

$$(4.57) \quad \begin{aligned} \int_{\Omega} (\varphi_k^* - \widehat{\varphi}_k^*) : \mathbf{z} \, dx - \int_{\Gamma_1} \beta_1(\varphi_k^* - \widehat{\varphi}_k^*) h \, ds \\ + \langle (\psi_k^* - \widehat{\psi}_k^*); w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \rightarrow 0 \end{aligned}$$

for every $\mathbf{z} \in L^\infty(\Omega, \mathbf{E}_s^2)$, $h \in L^\infty(\Gamma_1)$ and $w \in \gamma_0(W^{2,1})$ with $w|_{\Gamma_0} = 0$.

LEMMA 21. *Let $\widehat{f} : A(\Omega) \rightarrow \mathbb{R}$ be a linear functional, continuous in the topology (4.56), such that $\widehat{f}(\varphi_s^*, \beta_0(\varphi_s^*)) = 0$ for every $\varphi_s^* \in W_{\text{div}}^2(\Omega, \text{div div})$ (cf. (2.12)). Then there exists $\tilde{u} \in W^{2,1}(\Omega)$ such that $\gamma_1(\tilde{u}) = 0 = \gamma_0(\tilde{u})$ on Γ_0 , $\nabla \nabla \tilde{u} \in L^\infty(\Omega, \mathbf{E}_s^2)$, $\gamma_1(\tilde{u}) \in L^\infty(\text{Fr } \Omega)$ and*

$$(4.58) \quad \begin{aligned} \widehat{f}(\varphi^*, \psi^*) &= \int_{\Omega} \varphi^* : (\nabla \nabla \tilde{u}) \, dx - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(\tilde{u}) \, ds \\ &\quad + \langle \psi^*; \gamma_0(\tilde{u}) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \end{aligned}$$

for $(\varphi^*, \psi^*) \in A(\Omega)$.

Proof. Step 1. The functional \widehat{f} is continuous in the topology (4.56). So, by Theorem V.3.9 of [9], there exist $\widetilde{\boldsymbol{\varkappa}} \in L^\infty(\Omega, \mathbf{E}_s^2)$ and $\widehat{u} \in W^{2,1}(\Omega)$ such that $\gamma_1(\widehat{u}) = 0 = \gamma_0(\widehat{u})$ on Γ_0 , $\gamma_1(\widehat{u}) \in L^\infty(\text{Fr } \Omega)$ and

$$(4.59) \quad \widehat{f}(\boldsymbol{\varphi}^*, \psi^*) = \int_{\Omega} \boldsymbol{\varphi}^* : \widetilde{\boldsymbol{\varkappa}} \, dx - \int_{\text{Fr } \Omega} \beta_1(\boldsymbol{\varphi}^*) \gamma_1(\widehat{u}) \, ds \\ + \langle \psi^*; \gamma_0(\widehat{u}) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})}$$

for all $(\boldsymbol{\varphi}^*, \psi^*) \in A(\Omega)$. Indeed, the trace $(\gamma_1, \gamma_0) : W^{2,1}(\Omega) \rightarrow L^1(\text{Fr } \Omega) \times \gamma_0(W^{2,1}(\Omega))$ is a surjection and $L^\infty(\text{Fr } \Omega) \subset L^1(\text{Fr } \Omega)$, because Ω is a bounded domain of class C^2 (see [8] and [14, Chap. 3, Section 2, proof of Proposition 2.11]). Let $\bar{f} : C_c^\infty(\Omega_1, \mathbf{E}_s^2) \rightarrow \mathbb{R}$ be given by $\bar{f}(\boldsymbol{\varphi}_2^*) \equiv \widehat{f}(\boldsymbol{\varphi}_{2|\bar{\Omega}}^*, \beta_0(\boldsymbol{\varphi}_{2|\bar{\Omega}}^*))$ for $\boldsymbol{\varphi}_2^* \in C_c^\infty(\Omega_1, \mathbf{E}_s^2)$, where $\Omega \subset\subset \Omega_1$. Then, by (4.59), for every $\boldsymbol{\varphi}_1^* \in C_c^\infty(\Omega_1, \mathbf{E}_s^2)$ such that $\text{div div } \boldsymbol{\varphi}_1^* = 0$ on Ω_1 , we get

$$(4.60) \quad \bar{f}(\boldsymbol{\varphi}_1^*) = \int_{\Omega} \boldsymbol{\varphi}_1^* : \widetilde{\boldsymbol{\varkappa}} \, dx - \int_{\text{Fr } \Omega} \boldsymbol{\varphi}_1^* : (\boldsymbol{\nu} \otimes \boldsymbol{\nu} \gamma_1(\widehat{u})) \, ds \\ + \langle \beta_0(\boldsymbol{\varphi}_{1|\bar{\Omega}}^*); \gamma_0(\widehat{u}) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \\ = \langle (\widetilde{\boldsymbol{\varkappa}}, (\boldsymbol{\nu} \otimes \boldsymbol{\nu} \gamma_1(\widehat{u})), \gamma_0(\widehat{u})); \boldsymbol{\varphi}_1^* \rangle_{D'(\Omega_1) \times D(\Omega_1)} = 0.$$

Therefore, by Proposition 1, there exists $\widetilde{u}_1 \in D'(\Omega_1)$ such that $\nabla \nabla \widetilde{u}_1 = (\widetilde{\boldsymbol{\varkappa}}, (\boldsymbol{\nu} \otimes \boldsymbol{\nu} \gamma_1(\widehat{u})), \gamma_0(\widehat{u})) \in D'(\Omega_1)$.

Step 2. By (4.60) we can assume that $\text{supp } \widetilde{u}_1 \subset \bar{\Omega}$. For every $\boldsymbol{\varphi}_3^* \in C_c^\infty(\Omega, \mathbf{E}_s^2)$ we have $\langle \nabla \nabla \widetilde{u}_1; \boldsymbol{\varphi}_3^* \rangle_{D'(\Omega_1) \times D(\Omega_1)} = \int_{\Omega} \boldsymbol{\varphi}_3^* : \widetilde{\boldsymbol{\varkappa}} \, dx$, since $C_c^\infty(\Omega, \mathbf{E}_s^2) \subset C_c^\infty(\Omega_1, \mathbf{E}_s^2)$. Then $\nabla \nabla \widetilde{u}_1(x) = \widetilde{\boldsymbol{\varkappa}}(x)$ for dx -a.e. $x \in \Omega$. By [14, Chap. 3, (2.79)], $\widetilde{u}_{1|\Omega} \in W^{2,1}(\Omega)$, because $\widetilde{\boldsymbol{\varkappa}} \in L^\infty(\Omega, \mathbf{E}_s^2)$.

Step 3. Let \widetilde{u}_2 be the projection of $\widetilde{u}_{1|\Omega}$ onto $W^{2,1}(\Omega)/\ker \nabla \nabla(\cdot)$, given by $\widetilde{u}_2 = \widetilde{u}_{1|\Omega} - (P_I(\widetilde{u}_{1|\Omega}) + P_{II}(\widetilde{u}_{1|\Omega}) + P_{III}(\widetilde{u}_{1|\Omega}))$ (cf. (4.51)–(4.53)). By Lemma 20 and Green's formula (2.17) we obtain

$$(4.61) \quad 0 = \int_{\Omega} \mathbf{M} : \nabla \nabla \widetilde{u}_2 \, dx \\ - \int_{\text{Fr } \Omega} \beta_1(\mathbf{M}) \gamma_1(\widehat{u}) \, ds + \langle \beta_0(\mathbf{M}); \gamma_0(\widehat{u}) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \\ = - \int_{\text{Fr } \Omega} \beta_1(\mathbf{M}) (\gamma_1(\widehat{u}) - \gamma_1(\widetilde{u}_2)) \, ds \\ + \langle \beta_0(\mathbf{M}); (\gamma_0(\widehat{u}) - \gamma_0(\widetilde{u}_2)) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})}$$

for every $\mathbf{M} \in W_{\text{div}}^2(\Omega, \text{div div})$. Then there exists $\widetilde{u}_3 \in \ker \nabla \nabla(\cdot)$ such that $\gamma_1(\widetilde{u}_2 + \widetilde{u}_3) = \gamma_1(\widehat{u})$ and $\gamma_0(\widetilde{u}_2 + \widetilde{u}_3) = \gamma_0(\widehat{u})$. Therefore $\widetilde{u} = \widetilde{u}_2 + \widetilde{u}_3$ satisfies (4.58). ■

PROPOSITION 22. *Let $r > 0$ (in the definition of H_r , cf. (4.23)) and let the space $W_{\text{div}}^2(\Omega, \text{div div})$ be defined by (2.12). Then*

$$(4.62) \quad H_r^*(\mathbf{0}, 0) = \inf\{\tilde{H}_r^*(\varphi_s^*, \beta_0(\varphi_s^*)) \mid \varphi_s^* \in W_{\text{div}}^2(\Omega, \text{div div})\}.$$

Proof. Step 1. Suppose there exists $\delta_1 > 0$ such that

$$(4.63) \quad H_r^*(\mathbf{0}, 0) + \delta_1 < \inf\{\tilde{H}_r^*(\varphi_s^*, \beta_0(\varphi_s^*)) \mid \varphi_s^* \in W_{\text{div}}^2(\Omega, \text{div div})\}.$$

In view of Lemmas 16 and 19, it suffices to show that this assumption leads to a contradiction.

Let $\tilde{H}_{r|L^\infty} : L^\infty(\Omega, \mathbf{E}_s^2) \times L^\infty(\Gamma_1) \times \gamma_0(W^{2,1}(\Omega))|_{\Gamma_1} \rightarrow \mathbb{R} \cup \{+\infty\}$ be the restriction of \tilde{H}_r , given by $\tilde{H}_{r|L^\infty}(\mathbf{z}, h, w) = \tilde{H}_r(\mathbf{z}, h, w)$ for $\mathbf{z} \in L^\infty(\Omega, \mathbf{E}_s^2)$, $h \in L^\infty(\Gamma_1, \mathbb{R})$ and $w \in \gamma_0(W^{2,1}(\Omega))$ such that $w|_{\Gamma_0} = 0$ (cf. (4.45)). Define the dual functional to $\tilde{H}_{r|L^\infty}$ by

$$(4.64) \quad \begin{aligned} \tilde{H}_{r|L^\infty}^*(\varphi^*, \psi^*) = \sup \left\{ \int_{\Omega} \varphi^* : \mathbf{z} \, dx - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) h \, ds \right. \\ \left. + \langle \psi^*; w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} - \tilde{H}_{r|L^\infty}(\mathbf{z}, h, w) \right\} \\ \mathbf{z} \in L^\infty(\Omega, \mathbf{E}_s^2), h \in L^\infty(\Gamma_1), w \in \gamma_0(W^{2,1}) \text{ and } w|_{\Gamma_0} = 0 \end{aligned}$$

for $(\varphi^*, \psi^*) \in A(\Omega)$. By (4.34) and (4.38) we obtain $\tilde{H}_{r|L^\infty}^*(\varphi^*, \psi^*) = \tilde{H}_r^*(\varphi^*, \psi^*)$ for every $(\varphi^*, \psi^*) \in A(\Omega)$. Therefore H_r^* is l.s.c. on $A(\Omega)$ in the topology (4.56).

Step 2. The linear space $W_{\text{div}}^2(\Omega, \text{div div})$ is closed in $L^\infty(\Omega, \mathbf{E}_s^2)$ endowed with the topology $\sigma(L^\infty(\Omega, \mathbf{E}_s^2); L^\infty(\Omega, \mathbf{E}_s^2))$. Indeed, let $\{\mathbf{M}_k\}_{k \in K} \subset W_{\text{div}}^2(\Omega, \text{div div})$ be a net convergent to \mathbf{M}_0 in $\sigma(L^\infty(\Omega, \mathbf{E}_s^2); L^\infty(\Omega, \mathbf{E}_s^2))$, i.e. $\int_{\Omega} (\mathbf{M}_k - \mathbf{M}_0) : \mathbf{p} \, dx \rightarrow 0$ for every $\mathbf{p} \in L^\infty(\Omega, \mathbf{E}_s^2)$. Then for every $u \in C_c^2(\Omega, \mathbb{R})$,

$$(4.65) \quad 0 = \int_{\Omega} (\text{div div } \mathbf{M}_k) u \, dx = \int_{\Omega} \mathbf{M}_k : (\nabla \nabla u) \, dx \rightarrow \int_{\Omega} \mathbf{M}_0 : (\nabla \nabla u) \, dx.$$

Therefore $\text{div div } \mathbf{M}_0 = 0$ in the sense of distributions on Ω .

Step 3. The set $W_{\text{div}}^2(\Omega, \text{div div}) \cap B_{L^\infty(\Omega, \mathbf{E}_s^2)}(0, \hat{r})$ endowed with the topology (4.56) is compact.

Indeed, let $\{\mathbf{M}_k\}_{k \in K} \subset W_{\text{div}}^2(\Omega, \text{div div}) \cap B_{L^\infty(\Omega, \mathbf{E}_s^2)}(0, \hat{r})$ be a net. Then there exists a finer net $\{\mathbf{M}_{k_t}\}_{t \in T}$ and $\mathbf{M}_0 \in W_{\text{div}}^2(\Omega, \text{div div}) \cap B_{L^\infty(\Omega, \mathbf{E}_s^2)}(0, \hat{r})$ such that $\{\mathbf{M}_{k_t}\}_{t \in T}$ converges to \mathbf{M}_0 in $\sigma(L^\infty(\Omega, \mathbf{E}_s^2); L^\infty(\Omega, \mathbf{E}_s^2))$, because the Hausdorff topology $\sigma(L^\infty; L^\infty)$ is weaker than $\sigma(L^\infty; L^1)$ (cf. [11, Corollary 3.1.14]). The trace (β_0, β_1) is a continuous linear map from $[W^2(\Omega, \text{div div}), \|\cdot\|_{W^2(\Omega, \text{div div})}]$ into $[(\gamma_0(W^{2,1}(\Omega)))^*, \|\cdot\|_{0^*}] \times [L^\infty(\text{Fr } \Omega), \|\cdot\|_{L^\infty}]$. Hence the net $\{(\beta_0(\mathbf{M}_{k_t}), \beta_1(\mathbf{M}_{k_t}))\}_{t \in T}$ is bounded in $(\gamma_0(W^{2,1}(\Omega)))^*$

$\times L^\infty(\text{Fr } \Omega)$. Therefore there exists a finer net $\{\mathbf{M}_{k_{t_p}}\}_{p \in P}$ and $(\tilde{\varrho}_0, \tilde{\varrho}_1) \in (\gamma_0(W^{2,1}(\Omega)))^* \times L^\infty(\text{Fr } \Omega)$ such that

$$(4.66) \quad \int_{\text{Fr } \Omega} (\beta_1(\mathbf{M}_{k_{t_p}}) - \tilde{\varrho}_1) h \, ds - \langle (\beta_0(\mathbf{M}_{k_{t_p}}) - \tilde{\varrho}_0); w \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})} \rightarrow 0$$

for every $(h, w) \in L^1(\text{Fr } \Omega) \times \gamma_0(W^{2,1}(\Omega))$. Moreover, by (2.17), similarly to Step 4 of the proof of [4, Proposition 19] we obtain $\beta_0(\mathbf{M}_0) = \tilde{\varrho}_0$ and $\beta_1(\mathbf{M}_0) = \tilde{\varrho}_1$.

Step 4. By the Hahn–Banach theorem, for every $m \in \mathbb{N}$, there exists an affine functional $A(\Omega) \ni (\varphi^*, \psi^*) \mapsto \tilde{f}_m(\varphi^*, \psi^*) + \tilde{a}_m \in \mathbb{R}$ such that

$$(4.67) \quad \tilde{f}_m(\varphi^*, \psi^*) + \tilde{a}_m < \tilde{H}_r^*(\varphi^*, \psi^*)$$

for $(\varphi^*, \psi^*) \in A(\Omega)$,

$$(4.68) \quad \tilde{f}_m(\tilde{\mathbf{M}}_s, \beta_0(\tilde{\mathbf{M}}_s)) + \tilde{a}_m > \inf_{\varphi_s^*} \{ \tilde{H}_r^*(\varphi_s^*, \beta_0(\varphi_s^*)) \mid$$

$$\varphi_s^* \in W_{\text{div}}^2(\Omega, \text{div div}) \} - \delta_1/2^m$$

for $\tilde{\mathbf{M}}_s \in W_{\text{div}}^2(\Omega, \text{div div}) \cap B_{L^\infty(\Omega, \mathbf{E}_s^2)}(0, \hat{r} 2^m)$ and \tilde{f}_m is continuous in the topology (4.56). Indeed, \tilde{H}_r^* is l.s.c. and $W_{\text{div}}^2(\Omega, \text{div div}) \cap B_{L^\infty(\Omega, \mathbf{E}_s^2)}(0, \hat{r} 2^m)$ is compact in this topology. By (4.37) and Assumption 4, $\tilde{H}_r^*(\mathbf{0}, 0) < \infty$. Because of (4.21), (4.22) and Assumption 4,

$$\tilde{H}_r^*(\mathbf{0}, 0) = \inf \{ \tilde{H}_r^*(\varphi_s^*, \beta_0(\varphi_s^*)) \mid \varphi_s^* \in W_{\text{div}}^2(\Omega, \text{div div}) \}.$$

Step 5. Due to Steps 5, 6 and 7 of the proof of [4, Proposition 19] we obtain the existence of $\tilde{u} \in W^{2,1}(\Omega)$ such that $\gamma_1(\tilde{u}) = 0 = \gamma_0(\tilde{u})$ on Γ_0 , $\nabla \nabla \tilde{u} \in L^\infty(\Omega, \mathbf{E}_s^2)$, $\gamma_1(\tilde{u}) \in L^\infty(\text{Fr } \Omega)$ and

$$(4.69) \quad A(\Omega) \ni (\varphi^*, \psi^*) \mapsto \tilde{f}_0(\varphi^*, \psi^*) = \int_{\Omega} \varphi^* : (\nabla \nabla \tilde{u}) \, dx \\ - \int_{\text{Fr } \Omega} \beta_1(\varphi^*) \gamma_1(\tilde{u}) \, ds + \langle \psi^*; \gamma_0(\tilde{u}) \rangle_{(\gamma_0(W^{2,1}))^* \times \gamma_0(W^{2,1})}$$

satisfies the inequalities

$$(4.70) \quad \tilde{f}_0(\varphi^*, \psi^*) + \tilde{H}_r^*(\mathbf{0}, 0) - \delta_1/2 < \tilde{H}_r^*(\varphi^*, \psi^*), \quad \tilde{f}_0(\tilde{\mathbf{M}}_s, \beta_0(\tilde{\mathbf{M}}_s)) \geq 0$$

for every $(\varphi^*, \psi^*) \in A(\Omega)$ and every $\tilde{\mathbf{M}}_s \in W_{\text{div}}^2(\Omega, \text{div div})$ (cf. Lemma 21).

Step 6. We say that a net $\{(\varphi_k^*, \psi_k^*)\}_{k \in K} \subset C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$ converges to $(\hat{\varphi}^*, \hat{\psi}^*)$ in the topology $\sigma(C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega); W^{2,1}(\Omega))$ if

$$(4.71) \quad \langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); ((\varphi_k^* - \hat{\varphi}^*), (\psi_k^* - \hat{\psi}^*)) \rangle_1 \rightarrow 0$$

for every $u \in W^{2,1}(\Omega)$ such that $\gamma_1(u)|_{\Gamma_0} = 0$ on Γ_0 (see (4.33)). The l.s.c. regularization of \tilde{H}_r^* in the topology $\sigma(C_{\text{div div}}^1(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega); W^{2,1}(\Omega))$

(denoted by $\text{cl}_{(4.71)} \tilde{H}_r^*$) is given by

$$\begin{aligned}
(4.72) \quad \text{cl}_{(4.71)} \tilde{H}_r^*(\varphi^*, \psi^*) &= \sup\{\langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); (\varphi^*, \psi^*) \rangle_1 \\
&\quad - \tilde{H}_r^{\#\#}(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in W^{2,1}(\Omega), \gamma_1(u)|_{\Gamma_0} = 0\} \\
&= \sup\{\langle (\nabla \nabla u, \gamma_1(u), \gamma_0(u)); (\varphi^*, \psi^*) \rangle_1 \\
&\quad - H_r(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in W^{2,1}(\Omega), \gamma_1(u)|_{\Gamma_0} = 0\} \\
&= H_r^*(\varphi^*, \psi^*),
\end{aligned}$$

for every $(\varphi^*, \psi^*) \in C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$ (cf. Proposition 17 and Lemma 18). From (4.63), (4.69), (4.70) and (4.72) we obtain a contradiction. ■

PROPOSITION 23. *For every $r > 0$,*

$$(4.73) \quad \inf\{H_r(\bar{\mathbf{z}}) \mid \bar{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))\} = \inf\{\tilde{H}_r(\bar{\mathbf{z}}) \mid \bar{\mathbf{z}} \in \tilde{\mathcal{Z}}(HB(\Omega))\}.$$

Proof. By (4.35), (4.37), (4.21), (4.22), Assumption 4, (4.37), Proposition 22, and (4.30), we get

$$\begin{aligned}
(4.74) \quad \sup\{-\tilde{H}_r(\bar{\mathbf{z}}) \mid \bar{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))\} &= \tilde{H}_r^*(\mathbf{0}, 0) \\
&= -[P_{\lambda,r}^*](\mathbf{M}_L) = \inf\{-[P_{\lambda,r}^*](\mathbf{M}) \mid \mathbf{M} \in W^2(\Omega, \text{div div})\} \\
&= \inf\{\tilde{H}_r^*(\mathbf{M}_s, \beta_0(\mathbf{M}_s)) \mid \mathbf{M}_s \in W_{\text{div}}^2(\Omega, \text{div div})\} \\
&= H_r^*(\mathbf{0}, 0) = \sup\{-H_r(\nabla \nabla u, \gamma_1(u), \gamma_0(u)) \mid u \in HB(\Omega)\}. \quad \blacksquare
\end{aligned}$$

Let

$$(4.75) \quad \text{cl}_{\sigma(\mathcal{Z}(HB); [C \cap W^2] \times C)} H_r \quad (\text{resp. } \text{cl}_{\sigma(\tilde{\mathcal{Z}}(HB); [C \cap W^2] \times C)} \tilde{H}_r)$$

denote the largest l.s.c. minorant of H_r in the topology $\sigma(\mathcal{Z}(HB(\Omega)); C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega))$ (respectively, the largest l.s.c. minorant of \tilde{H}_r in $\sigma(\tilde{\mathcal{Z}}(HB(\Omega)); C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega))$), i.e. (4.75) stands for the l.s.c. regularizations of H_r and \tilde{H}_r in the above mentioned topologies.

Because $(\mathbf{0}, 0) \in C_{\text{div div}}^1(\bar{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$ and from Proposition 23 we get

$$\begin{aligned}
(4.76) \quad \inf\{\text{cl}_{\sigma(\mathcal{Z}(HB); [C \cap W^2] \times C)} H_r(\bar{\mathbf{z}}) \mid \bar{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))\} \\
= \inf\{\text{cl}_{\sigma(\tilde{\mathcal{Z}}(HB); [C \cap W^2] \times C)} \tilde{H}_r(\bar{\mathbf{z}}) \mid \bar{\mathbf{z}} \in \tilde{\mathcal{Z}}(HB(\Omega))\}.
\end{aligned}$$

The main result of this paper is the following.

THEOREM 24. *Let $r > 0$ and suppose Assumptions 2, 4 and 5 hold. If H_r is a coercive function, then by Proposition 17 the functional H_r^{**} is given by (4.39), since $H_r^{**} = \tilde{H}_r^{\#\#}$. Moreover, every minimum point $\hat{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))$ of H_r^{**} is given by a function $\hat{u} \in W^{2,1}(\Omega)$ such that $\gamma_1(\hat{u}) = 0$ on Γ_0 and $\hat{\mathbf{z}} = (\nabla \nabla \hat{u}, \gamma_1(\hat{u}), \gamma_0(\hat{u}))$.*

Proof. Step 1. Let $\widehat{\mathbf{z}} = (\nabla\nabla\widehat{u}, \gamma_1(\widehat{u}), \gamma_0(\widehat{u})) \in \mathcal{Z}(HB(\Omega))$ be a minimum point of H_r^{**} . By (4.30), (4.31), Theorem 9 and Proposition 15 the functional H_r^{**} is the l.s.c. regularization of H_r in $\sigma(\widetilde{\mathcal{Z}}(HB(\Omega)); C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega))$. Then, by (4.76), we obtain

$$(4.77) \quad \begin{aligned} H_r^{**}(\widehat{\mathbf{z}}) &= \text{cl}_{\sigma(\mathcal{Z}(HB); [C\cap W^2] \times C)} H_r(\widehat{\mathbf{z}}) \\ &= \inf\{\text{cl}_{\sigma(\widetilde{\mathcal{Z}}(HB); [C\cap W^2] \times C)} \widetilde{H}_r(\overline{\mathbf{z}}) \mid \overline{\mathbf{z}} \in \widetilde{\mathcal{Z}}(HB(\Omega))\}. \end{aligned}$$

For every $\overline{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))$ we have $H_r(\overline{\mathbf{z}}) = \widetilde{H}_r(\overline{\mathbf{z}})$, hence

$$\text{cl}_{\sigma(\mathcal{Z}(HB); [C\cap W^2] \times C)} H_r(\overline{\mathbf{z}}) \geq \text{cl}_{\sigma(\mathcal{Z}(HB); [C\cap W^2] \times C)} \widetilde{H}_r(\overline{\mathbf{z}}).$$

Because of (4.77) and (4.76), $\widehat{\mathbf{z}} \in \mathcal{Z}(HB(\Omega))$ is a minimum point of $\text{cl}_{\sigma(\mathcal{Z}(HB); [C\cap W^2] \times C)} \widetilde{H}_r$ on $\widetilde{\mathcal{Z}}(HB(\Omega))$. By [10, Chap. 1, (5.2)], $(\mathbf{0}, 0) \in \partial(\text{cl}_{\sigma(\widetilde{\mathcal{Z}}(HB); [C\cap W^2] \times C)} \widetilde{H}_r)(\widehat{\mathbf{z}})$, where ∂ is a subgradient and $(\mathbf{0}, 0) \in C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$. Then $\widehat{\mathbf{z}} \in \partial(\text{cl}_{\sigma(\widetilde{\mathcal{Z}}(HB); [C\cap W^2] \times C)} \widetilde{H}_r)^*(\mathbf{0}, 0)$ (see [10, Chap. 1, Corollary 5.2]). By [10, Chap. 1, Corollary 4.1] we have $\widehat{\mathbf{z}} \in \partial(\widetilde{H}_r^*)(\mathbf{0}, 0)$. Then by [10, Chap. 1, (5.2)] we get

$$(4.78) \quad \langle (\nabla\nabla\widehat{u}, \gamma_1(\widehat{u}), \gamma_0(\widehat{u})); ((\varphi^*, \psi^*) - (\mathbf{0}, 0)) \rangle_1 + \widetilde{H}_r^*(\mathbf{0}, 0) \leq \widetilde{H}_r^*(\varphi^*, \psi^*)$$

for every $(\varphi^*, \psi^*) \in C^1_{\text{div div}}(\overline{\Omega}, \mathbf{E}_s^2) \times C(\text{Fr } \Omega)$, since $\widehat{\mathbf{z}} = (\nabla\nabla\widehat{u}, \gamma_1(\widehat{u}), \gamma_0(\widehat{u}))$.

Step 2. Because of Assumption 3, $\Gamma_1 = \text{Fr } \Omega \cap \mathcal{C}$, where $\mathcal{C} = \text{cl int } \mathcal{C} \subset \Omega_1$ is a closed Caccioppoli set, $ds(\text{Fr } \Omega \cap \text{Fr } \mathcal{C}) = 0$ and $\Omega \subset\subset \Omega_1$. Let $\mathcal{O}_{\Gamma_0} \equiv \Omega_1 - \mathcal{C}$. Then $\text{Fr } \Omega \cap \mathcal{O}_{\Gamma_0} = \Gamma_0$. Therefore for every $k \in \mathbb{N}$ there exists an open set Ω'_k such that $\Omega'_k \subset \mathcal{O}_{\Gamma_0}$, $\Omega'_k \subset\subset \Omega_1$, $dx(\Omega'_k) < 1/(2k)$ and $\{x \in \Gamma_0 \mid \gamma_1(\widehat{u})(x) \neq 0\} \subset \Omega'_k$.

Step 3. Suppose the singular part $(\nabla\nabla\widehat{u})_s$ is not 0 or $ds(\{x \in \Gamma_0 \mid \gamma_1(\widehat{u})(x) \neq 0\}) > 0$. Then there exists $\zeta > 0$ such that $\|(\nabla\nabla\widehat{u})_s\|_{\mathbb{M}_b} + \int_{\Gamma_0} |\gamma_1(\widehat{u})(x)| ds > \zeta$. Therefore, for every $k \in \mathbb{N}$, there exist open sets $\Omega''_k \subset\subset \Omega$ with $\Omega''_k \cup \Omega'_k \subset\subset \Omega_1$ such that $dx(\Omega''_k) < 1/k$ and $\|(\nabla\nabla\widehat{u})_s\|_{\mathbb{M}_b(\Omega''_k, \mathbf{E}_s^2)} + \int_{\Omega''_k} |\gamma_1(\widehat{u})(x)| ds > \frac{1}{2}\zeta$. The existence of the sequence $\{\Omega''_k\}_{k \in \mathbb{N}}$ follows from the regularity of the measure $\nabla\nabla\widehat{u}$. By Assumption 4, $B_{\mathbf{E}_s^2}(\mathbf{M}_L(x), \delta_0) \subset \mathcal{K}_p(x)$ for every $x \in \Omega$. Then for every $k \in \mathbb{N}$ there exists $\varphi_k^* \in C^2_c(\Omega_1, \mathbf{E}_s^2)$ such that $\varphi_k^*|_{\Omega_1 - \Omega''_k} = 0$,

$$(4.79) \quad \|\varphi_k^*(x)\|_{\mathbf{E}_s^2} < \frac{1}{2}\delta_0 \quad \forall x \in \Omega''_k,$$

$$(4.80) \quad \langle (\nabla\nabla\widehat{u}, \gamma_1(\widehat{u}), \gamma_0(\widehat{u})); (\varphi_k^*, 0) \rangle_1 > \frac{1}{8}\zeta\delta_0$$

(cf. Step 3 of the proof of [4, Theorem 21]). Due to Step 4 of the proof of [4, Theorem 21] we obtain the conclusion of Theorem 24. ■

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Institute of Fundamental Technological Research
 Polish Academy of Sciences
 Świętokrzyska 21
 00-049 Warszawa, Poland
 E-mail: jbojar@ippt.gov.pl

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