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FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF MAGNETOHYDRODYNAMIC INCOMPRESSIBLE VISCOUS FLUID

Abstract. The existence of a global motion of magnetohydrodynamic fluid in a domain bounded by a free surface and under the external electrodynamic field is proved. The motion is such that the velocity and magnetic field are small in the H^3-space.

1. Introduction. In this paper we prove the existence of global solutions to the equations describing the motion of a magnetohydrodynamic incompressible viscous fluid in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface S_t. In the domain $D_t \subset \mathbb{R}^3$ which is exterior to Ω_t we have a gas under constant pressure p_0. Moreover in D_t we have an electromagnetic field which is generated by some currents which are located on a fixed boundary B of D_t.

In the domain Ω_t the motion is described by the following problem:

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla\overset{1}{H} + \mu_1\nabla\frac{\overset{1}{H}{}^2}{2} = f && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div} v = 0, && \text{in } \widetilde{\Omega}^T,\\
&\mu_1\overset{1}{H}_t = -\operatorname{rot}\overset{1}{E} && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}) && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div}(\mu_1\overset{1}{H}) = 0 && \text{in } \widetilde{\Omega}^T,
\end{aligned}
\tag{1.1}
$$

where $\widetilde{\Omega}^T = \bigcup_{0\le t\le T}\Omega_t\times\{t\}$, $v=v(x,t)$ is the velocity of fluid, $p=p(x,t)$ is the pressure, $\overset{1}{H}=\overset{1}{H}(x,t)$ is the magnetic field, $f=f(x,t)$ is the external

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force field per unit mass, μ_1 is the constant magnetic permeability, σ_1 is the constant electric conductivity, $\overset{1}{E} = \overset{1}{E}(x,t)$ is the electric field, and

$$\mathbb{T}(v,p) = \{\nu(\partial_{x_i} v_j + \partial_{x_j} v_i) - p\delta_{ij}\} \tag{1.2}$$

is the stress tensor, where ν is the viscosity of the fluid. Moreover,

$$\mathbb{D}(v) = \{\nu(\partial_{x_i} v_j + \partial_{x_j} v_i)\} \tag{1.3}$$

is the dilatation tensor.

In the domain D_t in which there is a dielectric (gas) we assume that there is no fluid motion inside ($v = 0$). Therefore we have the electromagnetic field only, described by the following system:

$$\begin{aligned} &\mu_2 \overset{2}{H}_t = -\operatorname{rot} \overset{2}{E} && \text{in } \widetilde{D}^T, \\ &\operatorname{rot} \overset{2}{H} = \sigma_2 \overset{2}{E} && \text{in } \widetilde{D}^T, \\ &\operatorname{div}(\mu_2 \overset{2}{H}) = 0 && \text{in } \widetilde{D}^T, \end{aligned} \tag{1.4}$$

where $\widetilde{D}^T = \bigcup_{0\le t\le T} D_t \times \{t\}$.

On $S_t = \partial\Omega_t \cap \partial D_t$ we assume the following transmission and boundary conditions:

$$\begin{aligned} &\mathbb{T}(v,p)n = \Big(-p_0 I - \mu_1 \overset{1}{H} \otimes \overset{1}{H} + \mu_1 \frac{\overset{1}{H}{}^2}{2} I\Big) n && \text{on } \widetilde{S}^T, \\ &\frac{1}{\sigma_1} \overset{1}{H} = \frac{1}{\sigma_2} \overset{2}{H} && \text{on } \widetilde{S}^T, \\ &\overset{1}{E} \cdot \tau_\alpha = \overset{2}{E} \cdot \tau_\alpha, \quad \alpha = 1,2, && \text{on } \widetilde{S}^T, \\ &v \cdot n = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \widetilde{S}^T, \end{aligned} \tag{1.5}$$

where $\widetilde{S}^T = \bigcup_{0\le t\le T} S_t \times \{t\}$, n is the unit outward vector to Ω_t and normal to S_t, τ_α, $\alpha = 1,2$, is the tangent vector to S_t, and $\phi(x,t) = 0$ describes S_t at least locally.

Next we assume the boundary conditions on B:

$$\begin{aligned} &\overset{2}{H} = H_* && \text{on } B, \\ &\overset{2}{E} = E_* && \text{on } B. \end{aligned} \tag{1.6}$$

Finally, we assume the initial conditions

$$\begin{aligned} &\Omega_t|_{t=0} = \Omega, \quad S_t|_{t=0} = S, \quad D_t|_{t=0} = D, \\ &v|_{t=0} = v_0, \quad \overset{1}{H}|_{t=0} = \overset{1}{H}_0, \quad \text{in } \Omega, \\ &\overset{2}{H}|_{t=0} = \overset{2}{H}_0, \quad \text{in } D. \end{aligned} \tag{1.7}$$

Now we make some comments on the literature concerning free boundary problems for the nonstationary incompressible Navier–Stokes system. Local existence of solutions in the case without surface tension was proved in Hölder and Sobolev anisotropic spaces by V. A. Solonnikov in [4, 5]. To prove the existence of solutions of corresponding linear problems in Hölder and in Sobolev spaces the potential theory techniques were used (see [6, 7], respectively). In [4] V. A. Solonnikov showed the existence of global motions of a viscous incompressible fluid bounded by a free surface. The proof was based on the Korn inequality. To prove the existence of solutions in the case of surface tension V. A. Solonnikov used the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}$ with noninteger positive l. In all papers by Solonnikov, Lagrangian coordinates are used.

To prove existence of solutions to the above problem we introduce the Lagrangian coordinates $\xi \in \Omega$. They are the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \qquad x|_{t=0} = \xi \in \Omega. \tag{1.8}$$

Therefore $x_v(\xi,t) = \xi + \int_0^t \overline{v}(\xi,\tau)\,d\tau$, where

$$\overline{v}(\xi,t) = v(x_v(\xi,t),t).$$

To introduce the Lagrangian coordinates in D_t we extend v onto D_t. Let us denote the extended function by v'. Then we define $\xi \in D$ to be the Cauchy data to the problem

$$\frac{dx}{dt} = v'(x,t), \qquad x|_{t=0} = \xi \in D. \tag{1.9}$$

Therefore $x_{v'}(\xi,t) = \xi + \int_0^t \overline{v}'(\xi,\tau)\,d\tau$, where $\overline{v}'(\xi,t) = v'(x_{v'}(\xi,t),t)$. Then by $(1.1)_5$,

$$\Omega_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in \Omega\},$$
$$S_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in S\}.$$

Since S_t is determined at least locally by the equation $\phi(x,t) = 0$, S is described by $\phi(x_v(\xi,t),t)|_{t=0} = 0$. Moreover, we have

$$\overline{n}_v = n(x_v(\xi,t),t) = \frac{\nabla_x \phi(x,t)}{|\nabla_x \phi(x,t)|}\bigg|_{x=x_v(\xi,t)}.$$

To simplify considerations we introduce the following notation:

$$\|u\|_{l,Q} = \|u\|_{H^l(Q)}, \qquad Q \in \{\Omega, S, D, \Pi, B\}, \qquad 0 \le l \in \mathbb{Z},$$

$$\|u\|_{k,p,q,Q^T} = \|u\|_{L_q(0,T,W_p^k(Q))}, \qquad Q \in \{\Omega, S, D, \Pi, B\},$$
$$p,q \in [1,\infty], \quad 0 \le k \in \mathbb{Z},$$

where $Q^t = Q \times (0,t)$,

$$|u|_{p,Q} = \|u\|_{L_p(Q)}, \quad Q \in \{\Omega, S, D, \Pi, B\}, \quad p \in [1,\infty].$$

2. Weak solutions. Weak solutions to problem (1.1)–(1.7) are formulated in Lagrangian coordinates.

DEFINITION 2.1. A *weak solution* of problem (1.1)–(1.7) is a pair of functions $\overline{v}, \overline{H}$ which satisfy the integral identities

$$\begin{aligned}(2.1)\quad & \int_0^T\int_\Omega (-\overline{v}\cdot\overline{\varphi}_t + \mathbb{D}_v(\overline{v})\cdot\mathbb{D}_v(\overline{\varphi}))\,d\xi\,dt \\ &\quad - \int_0^T\int_\Omega \Big(\mu_1 \overset{1}{\overline{H}}\cdot\nabla_v\overset{1}{\overline{H}}\cdot\overline{\varphi} - \mu_1\nabla_v\frac{\overset{1}{\overline{H}}{}^2}{2}\cdot\overline{\varphi}\Big)d\xi\,dt \\ &= \int_0^T\int_\Omega \overline{f}\cdot\overline{\varphi}\,d\xi\,dt + \int_0^T\int_S\Big(-p_0 I - \mu_1\overset{1}{\overline{H}}\otimes\overset{1}{\overline{H}} + \mu_1\frac{\overset{1}{\overline{H}}{}^2}{2}I\Big)\overline{n}\cdot\overline{\varphi}\,d\xi_S\,dt - \int_\Omega \overline{v}_0\cdot\overline{\varphi}(0)\,d\xi,\end{aligned}$$

$$\begin{aligned}(2.2)\quad & \int_0^T\int_\Pi\Big(-\mu\overline{H}\cdot\overline{\psi}_t - \mu\overline{v}\cdot\nabla_v\overline{H}\cdot\overline{\psi} + \frac{1}{\sigma}\operatorname{rot}_v\overline{H}\cdot\operatorname{rot}_v\overline{\psi}\Big)\,d\xi\,dt \\ &\quad - \int_0^T\int_\Omega \mu_1(\overline{v}\times\overset{1}{\overline{H}})\cdot\operatorname{rot}_v\overline{\psi}\,d\xi\,dt \\ &= \frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_v\times\overline{E}_*)\cdot\overline{\psi}\,d\xi_B\,dt - \mu\int_\Pi \overline{H}_0\cdot\overline{\psi}(0)\,d\xi,\end{aligned}$$

where φ, ψ are sufficiently regular and $\varphi(x,T) = \psi(x,T) = 0$, $\overline{n}_v$ is the unit outward vector normal to S or B.

In (2.1), (2.2) we use the notation $\overline{A}(\xi,t) = A(x_v(\xi,t),t)$, $\overline{H}_{|\Omega} = \overset{1}{\overline{H}}$, $\overline{H}_{|D} = \overset{2}{\overline{H}}$, $\sigma_{|\Omega} = \sigma_1$, $\sigma_{|D} = \sigma_2$, $\Pi = \Omega\cup D$, $\mu_{|\Omega} = \mu_1$, $\mu_{|D} = \mu_2$, v in (2.2) is an extension onto Π,

$$\mathbb{D}_v(\overline{v}) = \{\nu(\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_j + \partial_{x_j}\xi_k\nabla_{\xi_k}\overline{v}_i)\}, \quad \operatorname{rot}_v\overline{v} = \nabla_v\times\overline{v},$$

$$\nabla_v = \partial_x\xi_i\nabla_{\xi_i}, \quad \operatorname{div}_v\overline{v} = \nabla_v\cdot\overline{v} = \partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_i, \quad \partial_{\xi_i} = \nabla_{\xi_i}.$$

Let A be the Jacobi matrix of the transformation $x = x_v(\xi,t)$. Then $\det A = \exp(\int_0^t \operatorname{div}_v\overline{v}\,d\tau) = 1$.

Moreover

$$x^i_{\xi_j} = \delta_{ij} + \int_0^t \partial_{\xi_j}\overline{v}_i(\xi,\tau)\,d\tau \quad\text{and}\quad \xi_x = x_\xi^{-1}.$$

Hence we get

$$\sup_{\xi\in\Omega}|x_\xi| \le 1+\sup_{\xi\in\Omega}\int_0^t |\overline{v}_\xi(\xi,\tau)|\,d\tau \le 1+c\int_0^t \|\overline{v}\|_{3,\Omega}\,d\tau$$
$$\le 1+c\sqrt{t}\sqrt{\int_0^t \|\overline{v}\|^2_{3,\Omega}\,d\tau} \le 1+c\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}.$$

Therefore $\sup_{x\in\Omega_t}|\xi_x| \le \alpha(a)$, where $a=\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}$ and α is an increasing positive function.

To prove the existence of a solution to the above problem we linearize (2.1), (2.2) to the form

$$\int_0^T\int_\Omega (-\overline{v}\,\overline{\varphi}_t + \mathbb{D}_u(\overline{v})\cdot\mathbb{D}_u(\overline{\varphi}))\,d\xi\,dt \tag{2.3}$$
$$-\int_0^T\int_\Omega \left(\mu_1\overset{1}{\overline{H}}{}'\cdot\nabla_u\overset{1}{\overline{H}}{}'\cdot\overline{\varphi}-\mu_1\nabla_u\frac{\overset{1}{\overline{H}}{}'^2}{2}\cdot\overline{\varphi}\right)d\xi\,dt = \int_0^T\int_\Omega \overline{f}\cdot\overline{\varphi}\,d\xi\,dt$$
$$+\int_0^T\int_S\left(-p_0I-\mu_1\overset{1}{\overline{H}}\otimes\overset{1}{\overline{H}}+\mu_1\frac{\overset{1}{\overline{H}}{}^2}{2}I\right)\overline{n}_u\cdot\overline{\varphi}\,d\xi_S\,dt-\int_\Omega \overline{v}_0\cdot\overline{\varphi}(0)\,d\xi,$$

$$\int_0^T\int_\Pi\left(-\mu\overline{H}\cdot\overline{\psi}_t-\mu\overline{u}\cdot\nabla_u\overline{H}\cdot\overline{\psi}+\frac{1}{\sigma}\operatorname{rot}_u\overline{H}\cdot\operatorname{rot}_u\overline{\psi}\right)d\xi\,dt \tag{2.4}$$
$$-\int_0^T\int_\Omega \mu_1(\overline{u}\times\overset{1}{\overline{H}})\cdot\operatorname{rot}_u\overline{\psi}\,d\xi\,dt$$
$$=\frac{1}{\sigma_2}\int_0^T\int_B(\overline{n}_u\times\overline{E}_*)\cdot\overline{\psi}\,d\xi_B\,dt-\mu\int_\Pi \overline{H}_0\cdot\overline{\psi}(0)\,d\xi,$$

where $\overset{1}{\overline{H}}{}'$ and u with $\operatorname{div}u=0$ are given functions.

Similarly to [1], [2] we prove

THEOREM 2.1. *Assume that* $\overline{v}_0\in H^2(\Omega)$; $\overline{v}_t(0),\overline{v}_{tt}(0)\in L_2(\Omega)$; $\overline{f}_t,\overline{f}_{tt}\in L_2(0,T,L_2(\Omega))$; $\overline{f}\in L_2(0,T,H^2(\Omega))$; $\overline{H}_0\in H^2(\Pi)$; $\overline{H}_t(0)\in H^1(\Pi)$; $\overline{E}_*\in L_\infty(0,T,H^1(B))$; $\overline{E}_{*t},\overline{H}_{*tt}\in L_2(0,T,L_2(B))$; $\overline{H}_{*t}\in L_2(0,T,H^2(B))$; $\overline{H}_*\in L_2(0,T,H^3(B))$, $S,B\in H^{5/2}$. *Then there exists* $T^*>0$ *such that for* $T\le T^*$ *there exists a solution to problem* (1.1)–(1.7) *such that*

$$\overline{v}\in L_2(0,T,H^3(\Omega))\cap L_\infty(0,T,H^1(\Omega));$$
$$\overline{v}_t\in L_\infty(0,T,H^1(\Omega))\cap L_2(0,T,H^2(\Omega));$$
$$\overline{v}_{tt}\in L_\infty(0,T,L_2(\Omega))\cap L_2(0,T,H^1(\Omega));$$

$$\overline{p} \in L_2(0,T,H^2(\Omega));\ \overline{p}_t \in L_2(0,T,H^1(\Omega));$$
$$\overline{H} \in L_2(0,T,H^3(\Pi)) \cap L_\infty(0,T,H^1(\Pi));$$
$$\overline{H}_t \in L_\infty(0,T,H^1(\Pi)) \cap L_2(0,T,H^2(\Pi));$$
$$\overline{H}_{tt} \in L_\infty(0,T,L_2(\Pi)) \cap L_2(0,T,H^1(\Pi)),$$

where $(T^*)^\gamma(\varphi(0)+\beta) \le b$, $b>0$ *is a sufficiently small constant*, $\gamma>0$ *is a constant and*

$$\beta = \|\overline{E}_*\|^2_{0,2,2,B^t} + \|\overline{E}_{*t}\|^2_{0,2,2,B^t} + \|\overline{H}_*\|^2_{3,2,2,B^t} + \|\overline{H}_{*t}\|^2_{2,2,2,B^t} + \|\overline{H}_{*tt}\|^2_{0,2,2,B^t} + \|\bar{f}_t\|^2_{0,2,2,B^t} + \|\bar{f}\|^2_{1,2,2,B^t}, \tag{2.5}$$

$$\varphi(0) = \sum_{i+k\le 2} (\|\partial_t^i \overline{v}(0)\|^2_{k,\Omega} + \|\partial_t^i \overline{H}(0)\|^2_{k,\Pi}). \tag{2.6}$$

Moreover, if $\varphi(0)$, β *are sufficiently small then*

$$\|\overline{v}_t\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{2,2,\infty,\Omega^T} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_t\|^2_{2,2,2,\Omega^T} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^T} + \|\overline{p}'\|^2_{2,2,2,\Omega^T} + \|\overline{p}_t\|^2_{1,2,2,\Omega^T} + \|\overline{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{3,2,2,\Pi^T} + \|\overline{H}_t\|^2_{2,2,2,\Pi^T} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^T} \le c(\varphi(0)+\beta), \tag{2.7}$$

where $\overline{p}' = \overline{p} - p_0$.

First, in Section 3, we derive a differential inequality (3.23) which enables a step by step extension of the local solution of (1.1)–(1.7) from $[0,T]$ to $[0,\infty)$. In Section 4 we establish Korn type inequalities which are necessary to prove inequality (3.23). In Section 5 we prove the following

Main Theorem. *Assume that* $f = \int_\Omega v_0\,dx = \int_\Omega v_0\cdot\varphi_i\,dx = 0$, $i = 1,2,3$, *where* φ_i, $i=1,2,3$, *are defined in Lemma* 4.1, $H_* \in H^3(B)$, $H_{**} \in H^2(B)$, $H_{*tt} \in H^1(B)$, $S_t, B \in H^{5/2}$, $(v(0),p'(0),H(0)) \in \mathcal{N}(0)$, $\varphi(0) \le \varepsilon_1$ *where* ε_1 *is sufficiently small. Assume also that* $\alpha(t) \le e^{-\mu t}$, *where* $\mu > 1/2$ *is sufficiently large and* $\alpha(t)$ *is defined in Lemma* 5.2. *Then there exists a global solution of* (1.1)–(1.7) *such that* $(v(t),p'(t),H(t)) \in \mathcal{M}(t)$, $t\in\mathbb{R}_+$, *where* $\mathcal{N}(0)$ *and* $\mathcal{M}(t)$ *are defined in Section* 5.

In Lemmas 3.1–3.12 we need

Lemma 2.1. *For a solution of problem* (1.1)–(1.7) *we have*

$$\|H_t\|^2_{1,\Omega} \le \alpha(a)(\|H_t\|^2_{1,\Omega_t} + \|H\|^2_{2,\Omega_t}\|v\|^2_{2,\Omega_t}),$$
$$\|\overline{H}\|^2_{i,\Omega} \le \alpha(a)\|H\|^2_{i,\Omega_t}, \qquad i=1,2,3,$$
$$\|\overline{v}\|^2_{i,\Omega} \le \alpha(a)\|v\|^2_{i,\Omega_t}, \qquad i=1,2,3,$$
$$\|\overline{H}_t\|^2_{2,\Omega} \le \alpha(a)[\|H_t\|^2_{2,\Omega_t} + \|H\|^2_{3,\Omega_t}\|v\|^2_{2,\Omega_t} + \|H\|^2_{2,\Omega_t}\|v\|^2_{3,\Omega_t}],$$
$$\|\overline{v}_t\|^2_{2,\Omega_t} \le \alpha(a)[\|v_t\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t}\|v\|^2_{2,\Omega_t}],$$

$$\begin{aligned}
\|\overline{H}_{tt}\|_{0,\Omega}^2 &\le \alpha(a)[\|H_{tt}\|_{0,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2(\|H_t\|_{1,\Omega_t}^2 + \|H\|_{2,\Omega_t}^2) \\
&\quad + \|v_t\|_{1,\Omega_t}^2\|H\|_{2,\Omega_t}^2 + \|H\|_{2,\Omega_t}^2\|v\|_{2,\Omega_t}^4], \\
\|\overline{v}_{tt}\|_{0,\Omega}^2 &\le \alpha(a)[\|v_{tt}\|_{0,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2(\|v_t\|_{1,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2) + \|v\|_{2,\Omega_t}^6], \\
\|\overline{H}_{tt}\|_{1,\Omega}^2 &\le \alpha(a)[\|H_{tt}\|_{1,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2(\|H_t\|_{2,\Omega_t}^2 + \|H\|_{3,\Omega_t}^2) \\
&\quad + \|H\|_{3,\Omega_t}^2\|v_t\|_{2,\Omega_t}^2 + \|H\|_{2,\Omega_t}^2\|v\|_{2,\Omega_t}^4], \\
\|\overline{v}_{tt}\|_{1,\Omega}^2 &\le \alpha(a)[\|v_{tt}\|_{1,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2(\|v_t\|_{2,\Omega_t}^2 + \|v\|_{3,\Omega_t}^2) + \|v\|_{2,\Omega_t}^6],
\end{aligned}$$

where $a = \sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}$ and α is an increasing positive function.

Proof. Differentiating $\overline{H}(\xi,t) = H(x(\xi,t),t)$ with respect to t and ξ we get $\overline{H}_t = H_x v + H_t$ and $\overline{H}_\xi = H_x x_\xi$. Then

$$\begin{aligned}
\|\overline{H}_t\|_{1,\Omega}^2 &\le c(\|H_x(x(\xi,t),t)v(x(\xi,t),t)\|_{1,\Omega}^2 + \alpha(a)\|H_t\|_{1,\Omega_t}^2) \\
&\le c(\|H_x(x(\xi,t),t)\|_{1,\Omega}^2\|v(x(\xi,t),t)\|_{2,\Omega}^2 + \alpha(a)\|H_t\|_{1,\Omega_t}^2) \\
&\le c\gamma\Big(t\int_0^t \|v\|_{3,\Omega_\tau}^2\,d\tau\Big)(\|H\|_{2,\Omega_t}^2\|v\|_{2,\Omega_t}^2 + \|H_t\|_{1,\Omega_t}^2).
\end{aligned}$$

Hence the first inequality is proved. Similarly we can show the other inequalities. ∎

In Lemmas 3.7, 3.8, 3.10 and 3.11 we need

Lemma 2.2. *For a solution of problem* (1.1)–(1.7) *we have*

$$\|\overline{v}_t\|_{1,\Omega}^2 \ge c\gamma\Big(t\int_0^t \|v\|_{3,\Omega_\tau}^2\,d\tau\Big)(\|v_t\|_{1,\Omega_t}^2 - \|v\|_{2,\Omega_t}^2\|v\|_{3,\Omega_t}^2), \tag{2.8}$$

where γ is a positive function.

Proof. From $\overline{v}_t = v_x v + v_t$ we get $\int_\Omega \overline{v}_t^2\,d\xi = \int_\Omega v_t^2\,d\xi + 2\int_\Omega v_t\cdot v_x v\,d\xi + \int_\Omega (v_x v)^2\,d\xi$ and from $\overline{v}_{t\xi} = v_{tx}x_\xi + v_{xx}vx_\xi + v_x^2 x_\xi$ we get

$$\int_\Omega \overline{v}_{t\xi}^2\,d\xi = \int_\Omega (v_{tx}x_\xi)^2\,d\xi + 2\int_\Omega (v_{tx}x_\xi)b\,d\xi + \int_\Omega b^2\,d\xi,$$

where $b = v_{xx}vx_\xi + v_x^2 x_\xi$. Hence we obtain (2.8). ∎

Similarly we obtain an inequality for

$$\|\overline{v}\|_{i,\Omega}^2;\ \|\overline{H}\|_{i,\Omega}^2,\quad i=1,2,3;\quad \|\overline{p}\|_{2,\Omega}^2;\ \|\overline{v}_t\|_{2,\Omega}^2;\ \|\overline{p}_t\|_{1,\Omega}^2;\ \|\overline{H}_t\|_{2,\Omega}^2.$$

In Lemmas 3.10, 3.11 we use inequalities (3.16), (3.19), (3.20) in local coordinates z, connected with $\{\xi\}$ (see [1]).

3. Differential inequality. Assume that the existence of a sufficiently smooth local solution of problem (1.1)–(1.7) has been proved and

$$(*)\quad \overset{2}{H}_{|B} = 0 \quad \text{on } B; \qquad \mathbb{T}(v,p')n = \Big(-\mu_1 \overset{1}{H} \otimes \overset{1}{H} + \mu_1 \frac{\overset{1}{H}{}^2}{2} I\Big) n \quad \text{on } \widetilde{S}^T,$$

where $p' = p - p_0$, $f = 0$, $\int_\Omega v_0\,dx = 0$, $\int_\Omega v_0 \cdot \varphi_i\,dx = 0$, and φ_i, $i = 1,2,3$, are defined in Lemma 4.1.

In this section we obtain a special differential inequality which enables us to prove the existence of a global solution.

Remark 3.1. Integrating $(1.1)_1$ over Ω_t we get

$$\frac{d}{dt}\int_{\Omega_t} v\,dx - \int_{\Omega_t} \operatorname{div}\mathbb{T}(v,p')\,dx + \mu_1 \int_{\Omega_t} \Big(-\operatorname{div}(\overset{1}{H} \otimes \overset{1}{H}) + \nabla \frac{\overset{1}{H}{}^2}{2} I\Big)\,dx = \int_{\Omega_t} f\,dx.$$

Then from $(*)$ we get

$$\frac{d}{dt}\int_{\Omega_t} v\,dx + \int_{S_t} \Big(\mu_1 \overset{1}{H} \otimes \overset{1}{H} - \mu_1 \frac{\overset{1}{H}{}^2}{2} I\Big) n\,dx_{S_t} + \mu_1 \int_{\Omega_t} \Big(-\operatorname{div}(\overset{1}{H} \otimes \overset{1}{H}) + \nabla \frac{\overset{1}{H}{}^2}{2}\Big)\,dx = 0.$$

Integrating the last equality by parts we get

$$\int_{\Omega_t} v\,dx = \int_\Omega v_0\,dx = 0.$$

Remark 3.2. Let φ_i, $i = 1,2,3$, be defined in Lemma 4.1. Multiplying $(1.1)_1$ by φ_i, $i = 1,2,3$, and integrating over Ω_t we get

$$\frac{d}{dt}\int_{\Omega_t} v \cdot \varphi_i\,dx - \int_{\Omega_t} \operatorname{div}(\mathbb{T}(v,p')\varphi_i)\,dx + \mu_1 \int_{\Omega_t} \Big(-\operatorname{div}(\overset{1}{H} \otimes \overset{1}{H}\varphi_i) + \nabla \frac{\overset{1}{H}{}^2}{2} \varphi_i\Big)\,dx = \int_{\Omega_t} f \cdot \varphi_i\,dx.$$

Then from $(*)$ we get

$$\frac{d}{dt}\int_{\Omega_t} v\varphi_i\,dx + \int_{S_t} \Big(\mu_1 \overset{1}{H} \otimes \overset{1}{H} - \mu_1 \frac{\overset{1}{H}{}^2}{2} I\Big)\varphi_i \cdot n\,dx_{S_t} + \mu_1 \int_{\Omega_t} \Big(-\operatorname{div}(\overset{1}{H} \otimes \overset{1}{H}) \cdot \varphi_i + \nabla \frac{\overset{1}{H}{}^2}{2} \cdot \varphi_i\Big)\,dx_{S_t} = 0.$$

Integrating the last equality by parts we get

$$\int_{\Omega_t} v\cdot\varphi_i\,dx=\int_{\Omega} v_0\cdot\varphi_i\,dx=0,\qquad i=1,2,3.$$

LEMMA 3.1. *For a sufficiently smooth solution* (v,p',H) *of* (1.1)–(1.7), *we have*

$$\frac{d}{dt}\|v\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t}\le c\|\overset{1}{H}\|^4_{1,\Omega_t}. \tag{3.1}$$

Proof. Multiplying $(1.1)_1$ by v and integrating over Ω_t we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t}\partial_t v^2\,dx+\int_{\Omega_t} v\cdot\nabla v\cdot v\,dx+\int_{\Omega_t}\mathbb{D}^2(v)\,dx-\mu_1\int_{\Omega_t}\overset{1}{H}\cdot\nabla\overset{1}{H}v\,dx\\ +\mu_1\int_{\Omega_t}\nabla\frac{\overset{1}{H}{}^2}{2}v\,dx+\mu_1\int_{S_t}\bigg(\overset{1}{H}\otimes\overset{1}{H}-\frac{\overset{1}{H}{}^2}{2}I\bigg)vn\,dx_{S_t}=0.\end{aligned} \tag{3.2}$$

Using

$$\int_{\Omega_t} v\cdot v_t\,dx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} v^2\,dx-\int_{\Omega_t} v\cdot\nabla vv\,dx$$

and Lemma 4.1 we get (3.1).

LEMMA 3.2. *For a sufficiently smooth solution* (v,p',H) *of* (1.1)–(1.7), *we have*

$$\begin{aligned}\frac{d}{dt}\|v_t\|^2_{0,\Omega_t}+\|v_t\|^2_{1,\Omega_t}\le c[\|v_t\|^2_{1,\Omega_t}\|v\|^2_{1,\Omega_t}+\|\overset{1}{H}_t\|^2_{1,\Omega_t}\|\overset{1}{H}\|^2_{1,\Omega_t}\\ +(\|v\|^2_{1,\Omega_t}+\|H\|^2_{1,\Omega_t})^2]\equiv X_1.\end{aligned} \tag{3.3}$$

Proof. Differentiating $(1.1)_1$ with respect to t, multiplying by v_t and integrating over Ω_t we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t}(v_t)^2_t\,dx+\int_{\Omega_t} v_t\cdot\nabla v\cdot v_t\,dx+\int_{\Omega_t} v\cdot\nabla v_t\cdot v_t\,dx\\ +\int_{\Omega_t}\mathbb{D}^2(v_t)\,dx-\mu_1\int_{\Omega_t}(\overset{1}{H}\cdot\nabla\overset{1}{H})_t v_t\,dx+\mu_1\int_{\Omega_t}(\nabla\overset{1}{H}{}^2)_t v_t\,dx\\ +\mu_1\int_{S_t}\bigg[\bigg(\overset{1}{H}\otimes\overset{1}{H}-\frac{\overset{1}{H}{}^2}{2}I\bigg)n\bigg]_t v_t\,dx_{S_t}=0.\end{aligned} \tag{3.4}$$

Using

$$\int_{\Omega_t} v_t\cdot v_{tt}\,dx=\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} v_t^2\,dx-\int_{\Omega_t} v_t\cdot\nabla v_t v\,dx$$

and Lemma 4.2 we get (3.3).

Lemma 3.3. *For a sufficiently smooth solution* (v, p', H) *of* (1.1)–(1.7), *we have*

$$\frac{d}{dt}\|v_{tt}\|_{0,\Omega_t}^2 + \|v_{tt}\|_{1,\Omega_t}^2 \le c[\|v_{tt}\|_{1,\Omega_t}^2\|v\|_{2,\Omega_t}^2 + \|v_t\|_{1,\Omega_t}^4 + \|\overset{1}{H}_{tt}\|_{0,\Omega_t}^2\|\overset{1}{H}\|_{1,\Omega_t}^2 + \|\overset{1}{H}_t\|_{1,\Omega_t}^4 + (\|v_t\|_{1,\Omega_t}^2 + \|v\|_{2,\Omega_t}^2(\|v\|_{1,\Omega_t}^2 + 1) + \|\overset{1}{H}_t\|_{1,\Omega_t}^2 + \|\overset{1}{H}\|_{1,\Omega_t}^2)^2] \equiv X_2. \tag{3.5}$$

Proof. Differentiating $(1.1)_1$ twice with respect to t, multiplying by v_{tt}, integrating over Ω_t and using the equality

$$\int_{\Omega_t} v_{tt}\cdot v_{ttt}\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(v_{tt})^2\,dx - \int_{\Omega_t} v_{tt}\cdot\nabla v_{tt} v\,dx$$

and Lemma 4.3, we get (3.5).

Lemma 3.4. *For a sufficiently smooth solution* (v, p', H) *of* (1.1)–(1.7), *we have*

$$\frac{d}{dt}\|H\|_{0,\Pi_t}^2 + \|H\|_{1,\Pi_t}^2 \le c\|H\|_{1,\Pi_t}^2\|v\|_{1,\Pi_t}^2. \tag{3.6}$$

Proof. Multiplying $(1.1)_{3,4}$ by H and integrating over Π_t, and using the equality

$$\int_{\Pi_t} H\cdot H_t\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Pi_t} H^2\,dx - \int_{\Pi_t} H\cdot\nabla H v\,dx,$$

we get (3.6).

Lemma 3.5. *For a sufficiently smooth solution* (v, p', H) *of* (1.1)–(1.7), *we have*

$$\frac{d}{dt}\|H_t\|_{0,\Pi_t}^2 + \|H_t\|_{1,\Pi_t}^2 \le c(\|H_t\|_{1,\Pi_t}^2\|v\|_{1,\Pi_t}^2 + \|v_t\|_{1,\Pi_t}^2\|H\|_{1,\Pi_t}^2). \tag{3.7}$$

Proof. Differentiating $(1.1)_{3,4}$ with respect to t, multiplying by H_t, integrating over Π_t and using the equality

$$\int_{\Pi_t} H_t\cdot H_{tt}\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Pi_t}(H_t)^2\,dx - \int_{\Pi_t} H_t\cdot\nabla H_t v\,dx,$$

we get (3.7).

Lemma 3.6. *For a sufficiently smooth solution* (v, p', H) *of* (1.1)–(1.7), *we have*

$$\frac{d}{dt}\|H_{tt}\|_{0,\Pi_t}^2 + \|H_{tt}\|_{1,\Pi_t}^2 \le c(\|H_{tt}\|_{1,\Pi_t}^2\|v\|_{1,\Pi_t}^2 + \|v_{tt}\|_{1,\Pi_t}^2\|H\|_{1,\Pi_t}^2) + \|v_t\|_{1,\Pi_t}^2\|H_t\|_{1,\Pi_t}^2 \equiv X_3. \tag{3.8}$$

Proof. Differentiating $(1.1)_{3,4}$ twice with respect to t, multiplying by H_{tt}, integrating over Π_t and using the equality

$$\int_{\Pi_t} H_{tt}\cdot H_{ttt}\,dx = \frac{1}{2}\frac{d}{dt}\int_{\Pi_t}(H_{tt})^2\,dx - \int_{\Pi_t} H_{tt}\cdot\nabla H_{tt}v\,dx,$$

we get (3.8).

LEMMA 3.7. *For a sufficiently smooth solution* (v,p',H) *of* (1.1)–(1.7), *we have*

$$\begin{aligned}(3.9)\qquad &\frac{d}{dt}\|v_t\|^2_{0,\Omega_t} + \frac{d}{dt}\|v\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t} + \|p'\|^2_{2,\Omega_t}\\ &\le c\Big[\gamma\Big(t\int_0^t\|v\|^2_{3,\Omega_\tau}\,d\tau\Big)\|\overset{1}{H}\|^2_{1,\Omega_t}\|\overset{1}{H}\|^2_{3,\Omega_t}\\ &\quad + \varepsilon(\|v_t\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t}) + \|v\|^2_{3,\Omega_t}\|v\|^2_{1,\Omega_t} + X_1\Big] \equiv X_4.\end{aligned}$$

Proof. Similarly to [1] we prove the inequality

$$\begin{aligned}(3.10)\qquad \|\overline{v}\|^2_{3,\Omega} + \|\overline{p}'\|^2_{2,\Omega} \le \alpha(a)\bigg(&\|\overset{1}{\overline{H}}\|^2_{2,\Omega}\|\overset{1}{\overline{H}}\|^2_{3,\Omega} + \|\overline{v}_t\|^2_{1,\Omega}\\ &+\bigg\|\bigg(-\mu_1\overset{1}{\overline{H}}\otimes\overset{1}{\overline{H}} + \mu_1\frac{\overset{1}{\overline{H}}{}^2}{2}I\bigg)\overline{n}_v\bigg\|^2_{3/2,S} + \|\overline{v}\|^2_{0,\Omega}\bigg).\end{aligned}$$

Then from

$$\frac{d}{dt}\|v\|^2_{2,\Omega_t} \le c(\|v\|^2_{2,\Omega_t} + \varepsilon\|v_t\|^2_{2,\Omega_t} + \|v\|^2_{3,\Omega_t}\|v\|^2_{1,\Omega_t})$$

and (3.3) we get (3.9).

LEMMA 3.8. *For a sufficiently smooth solution* (v,p',H) *of* (1.1)–(1.7), *we have*

$$\begin{aligned}(3.12)\qquad &\frac{d}{dt}\|v_t\|^2_{1,\Omega_t} + \frac{d}{dt}\|v_{tt}\|^2_{0,\Omega_t} + \frac{d}{dt}\|v_t\|^2_{0,\Omega_t} + \|v_t\|^2_{2,\Omega_t} + \|p'_t\|^2_{1,\Omega_t}\\ &\le c\Big[\gamma\Big(t\int_0^t\|v\|^2_{3,\Omega_\tau}\,d\tau\Big)(\|v\|^2_{2,\Omega_t}(\|v\|^2_{3,\Omega_t} + \|p'\|^2_{2,\Omega_t} + \|\overset{1}{H}\|^4_{2,\Omega_t})\\ &\quad + \|\overset{1}{H}_t\|^2_{1,\Omega_t}\|\overset{1}{H}\|^2_{2,\Omega_t} + \|\overset{1}{H}_t\|^2_{2,\Omega_t}\|\overset{1}{H}\|^2_{1,\Omega_t})\\ &\quad + \varepsilon(\|v_{tt}\|^2_{1,\Omega_t} + \|v_t\|^2_{2,\Omega_t}) + \|v_t\|^2_{2,\Omega_t}\|v\|^2_{1,\Omega_t} + X_1 + X_2\Big].\end{aligned}$$

Proof. Similarly to [1] we prove the inequality

$$\tag{3.12} \begin{aligned} &\|\overline{v}_t\|_{2,\Omega}^2+\|\overline{p}_t'\|_{1,\Omega}^2\\ &\le \alpha(a)\Big[\|\overline{v}\|_{2,\Omega}^2(\|\overline{v}\|_{3,\Omega}^2+\|\overline{p}'\|_{2,\Omega}^2+\|\overset{1}{\overline{H}}\|_{2,\Omega}^4)\\ &\quad+\|\overset{1}{\overline{H}}_t\|_{2,\Omega}^2\|\overset{1}{\overline{H}}\|_{2,\Omega}^2+\|\overset{1}{\overline{H}}_t\|_{2,\Omega}^2\|\overset{1}{\overline{H}}\|_{2,\Omega}^2+\|\overline{v}_{tt}\|_{0,\Omega}^2\\ &\quad+\Big\|\Big[\Big(-\mu_1\overset{1}{\overline{H}}\otimes\overset{1}{\overline{H}}-\mu_1\frac{\overset{1}{\overline{H}}{}^2}{2}I\Big)\overline{n}_v\Big]_t\Big\|_{1/2,S}^2\\ &\quad+\|(\overline{n}_v\mathbb{T}_v)_t(\overline{v},\overline{p}')\|_{1/2,S}^2+\|\overline{v}_t\|_{0,\Omega}^2\Big]. \end{aligned}$$

Then from

$$\frac{d}{dt}\|v_t\|_{1,\Omega_t}^2\le\|v_t\|_{1,\Omega_t}^2+\varepsilon\|v_{tt}\|_{1,\Omega_t}^2+\|v_t\|_{2,\Omega_t}^2\|v\|_{1,\Omega_t}^2$$

and (3.3), (3.5) we get (3.11).

LEMMA 3.9. *For a sufficiently smooth solution* (v,p',H) *of* (1.1)–(1.7), *we have*

$$\tag{3.13} \begin{aligned} &\frac{d}{dt}\|v\|_{1,\Omega_t}^2+\frac{d}{dt}\|v_t\|_{0,\Omega_t}^2+\|v\|_{2,\Omega_t}^2\\ &\le c\Big[\gamma\Big(t\int_0^t\|v\|_{3,\Omega_\tau}^2\,d\tau\Big)\|\overset{1}{H}\|_{2,\Omega_t}^4\\ &\quad+\varepsilon(\|v_t\|_{1,\Omega_t}^2+\|v\|_{2,\Omega_t}^2)+\|v\|_{2,\Omega_t}^2\|v\|_{0,\Omega_t}^2+X_1\Big]. \end{aligned}$$

Proof. Similarly to [1] we prove the inequality

$$\tag{3.14} \begin{aligned} \|\overline{v}\|_{2,\Omega}^2\le\alpha(a)\Big[&\|\overset{1}{\overline{H}}\|_{1,\Omega}^4+\|\overset{1}{\overline{H}}\|_{0,\Omega}^2\|\overset{1}{\overline{H}}\|_{2,\Omega}^2+\|\overline{v}_t\|_{0,\Omega}^2\\ &+\Big\|\Big(-\mu_1\overset{1}{\overline{H}}\otimes\overset{1}{\overline{H}}+\mu_1\frac{\overset{1}{\overline{H}}{}^2}{2}I\Big)\overline{n}_v\Big\|_{1/2,S}^2+\|\overline{v}\|_{0,\Omega}^2\Big]. \end{aligned}$$

Then from

$$\frac{d}{dt}\|v\|_{1,\Omega_t}^2\le c(\|v\|_{1,\Omega_t}^2+\varepsilon\|v_t\|_{1,\Omega_t}^2+\|v\|_{2,\Omega_t}^2\|v\|_{0,\Omega_t}^2)$$

and (3.3), we get (3.13).

LEMMA 3.10. *For a sufficiently smooth solution* (v,p',H) *of* (1.1)–(1.7), *we have*

$$\tag{3.15} \frac{d}{dt}\|H_t\|_{2,\Pi_t}^2+\frac{d}{dt}\|H_{tt}\|_{0,\Pi_t}^2+\frac{d}{dt}\|v_t\|_{0,\Omega_t}^2+\frac{d}{dt}\|v\|_{2,\Omega_t}^2+\|H_t\|_{2,\Pi_t}^2$$

$$\le c\gamma\Big(t\int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\Big)[\|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t} + \|H\|^2_{2,\Pi_t}$$
$$\cdot(\|v\|^2_{3,\Pi_t} + \|v_t\|^2_{2,\Pi_t} + \|v\|^4_{2,\Pi_t})$$
$$+ \|v\|^2_{3,\Pi_t}(\|H_t\|^2_{1,\Pi_t} + \|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t} + \|v\|^2_{2,\Pi_t} + \|v_t\|^2_{1,\Pi_t})$$
$$+ a^2\|H\|^2_{3,\Pi_t} + X_3 + X_4] \equiv X_5.$$

Proof. From the inequalities (see [1])

$$(3.16)\quad \begin{aligned} &\frac{d}{dt}\|\widetilde H_{t\tau}\|^2_{0,\widehat\Omega} + \|\widetilde H_t\|^2_{2,\widehat\Omega} \\ &\qquad \le \alpha(\widehat a)[\|\widetilde H_{tt}\|^2_{0,\widehat\Omega} + \|\widehat v\|^2_{3,\widehat\Omega}(\|\widehat H\|^2_{1,\widehat\Omega} + \|\widehat H_t\|^2_{1,\widehat\Omega} \\ &\qquad\quad + \|\widehat v_t\|^2_{1,\widehat\Omega} + \|\widehat H\|^2_{2,\widehat\Omega}\|\widehat v\|^2_{2,\widehat\Omega}\|\widehat v\|^2_{2,\widehat\Omega} + 1) \\ &\qquad\quad + \|\widehat v_t\|^2_{2,\widehat\Omega} + \|\widehat H\|^2_{2,\widehat\Omega}\|\widehat v_t\|^2_{2,\widehat\Omega}], \\ &\frac{d}{dt}\|\widetilde H_{\tau t}\|^2_{0,\widehat\Omega} \le c(\varepsilon\|\widehat H_t\|^2_{1,\widehat\Omega} + \|\widehat H_{tt}\|^2_{1,\widehat\Omega}), \\ &\frac{d}{dt}\|H_t\|^2_{1,\Pi_t} \le c(\varepsilon\|H_t\|^2_{1,\Pi_t} + \|H_{tt}\|^2_{1,\Pi_t} + \|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t}), \end{aligned}$$

we get

$$(3.17)\quad \begin{aligned} &\frac{d}{dt}\|H_t\|^2_{1,\Pi_t} + \|H_t\|^2_{2,\Pi_t} \\ &\le \gamma\Big(t\int_0^t \|v\|^2_{3,\Omega_\tau}d\tau\Big)[\|H_{tt}\|^2_{1,\Pi_t} + \|H_t\|^2_{2,\Pi_t}\|v\|^2_{1,\Pi_t} \\ &\quad + \|H\|^2_{2,\Pi_t}(\|v\|^2_{3,\Pi_t} + \|v_t\|^2_{2,\Pi_t} + \|v\|^4_{2,\Pi_t}) + \|v\|^2_{3,\Pi_t}(\|H_t\|^2_{1,\Pi_t} \\ &\quad + \|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t} + \|v\|^2_{2,\Pi_t} + \|v_t\|^2_{2,\Pi_t}) + \|v\|^2_{2,\Pi_t} + a^2\|H\|^2_{3,\Pi_t}]. \end{aligned}$$

Using (3.8), (3.9), we get (3.15).

LEMMA 3.11. *For a sufficiently smooth solution (v, p', H) of* (1.1)–(1.7), *we have*

$$(3.18)\quad \begin{aligned} &\frac{d}{dt}\|H\|^2_{2,\Pi_t} + \frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t} + \frac{d}{dt}\|v_t\|^2_{0,\Omega_t} + \frac{d}{dt}\|v\|^2_{2,\Omega_t} + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t} \\ &\qquad \le c\gamma\Big(t\int_0^t \|v\|^2_{3,\Omega_\tau}d\tau\Big)[\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t} + \|H\|^2_{2,\Pi_t}\|v\|^2_{3,\Pi_t} \\ &\qquad\quad + \|H\|^2_{3,\Pi_t}\|v\|^2_{2,\Pi_t} + X_4 + X_5] \equiv X_6. \end{aligned}$$

Proof. From the inequalities (see [1])

$$(3.19)\quad \begin{aligned} \frac{d}{dt}\|\widetilde H_\tau\|^2_{0,\widehat\Omega} + \|\widetilde H\|^2_{2,\widehat\Omega} &\le \alpha_1(\widehat a)[\|\widehat H\|^2_{1,\widehat\Omega}\|\widehat v\|^2_{2,\widehat\Omega} + \|\widehat v\|^2_{2,\widehat\Omega} + \|\widehat H\|^2_{1,\widehat\Omega} \\ &\quad + \|\widehat H_t\|^2_{0,\widehat\Omega} + \widehat a^2\|\widehat H\|^2_{3,\widehat\Omega}], \end{aligned}$$

$$
\begin{aligned}
&\frac{d}{dt}\|\widetilde{H}_{\tau\tau}\|^2_{0,\widehat{\Omega}} + \|\widetilde{H}\|^2_{3,\widehat{\Omega}} \le \alpha_2(\widehat{a})[\|\widehat{H}\|^2_{2,\widehat{\Omega}}\|\widehat{v}\|^2_{2,\widehat{\Omega}} + \|\widehat{v}\|^2_{2,\widehat{\Omega}} \\
&\qquad\qquad + \|\widehat{H}\|^2_{2,\widehat{\Omega}} + \|\widehat{H}\|^2_{1,\widehat{\Omega}}\|\widehat{v}\|^2_{3,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{1,\widehat{\Omega}}], \\
&\frac{d}{dt}\|\widetilde{H}_{\tau}\|^2_{0,\widehat{\Omega}} \le c(\varepsilon\|\widehat{H}\|^2_{1,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{1,\widehat{\Omega}}), \\
&\frac{d}{dt}\|\widetilde{H}_{\tau\tau}\|^2_{0,\widehat{\Omega}} \le c(\varepsilon\|\widehat{H}\|^2_{2,\widehat{\Omega}} + \|\widehat{H}_t\|^2_{2,\widehat{\Omega}}),
\end{aligned} \tag{3.20}
$$

we get

$$
\begin{aligned}
\|H\|^2_{3,\Pi_t} \le c\gamma\Big(t\int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\Big)[\|H\|^2_{2,\Pi_t}\|v\|^2_{2,\Pi_t} + \|v\|^2_{2,\Pi_t} \\
+ \|H\|^2_{2,\Pi_t}\|v\|^2_{3,\Pi_t} + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t}\|v\|^2_{2,\Pi_t}].
\end{aligned} \tag{3.21}
$$

Then from the inequality

$$
\frac{d}{dt}\|H\|^2_{2,\Pi_t} \le \varepsilon\|H\|^2_{2,\Pi_t} + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t}\|v\|^2_{1,\Pi_t}
$$

and (3.9), (3.15) we get (3.18).

LEMMA 3.12. *For a sufficiently smooth solution* (v, p', H) *of* (1.1)–(1.7), *we have*

$$
\begin{aligned}
\frac{d}{dt}\|H\|^2_{1,\Pi_t} + \frac{d}{dt}\|H\|^2_{2,\Pi_t} + \frac{d}{dt}\|H_{tt}\|^2_{0,\Pi_t} + \frac{d}{dt}\|v_t\|^2_{0,\Omega_t} \\
+ \frac{d}{dt}\|v\|^2_{2,\Omega_t} + \|H_t\|^2_{2,\Pi_t} + \|H\|^2_{3,\Pi_t} \le X_6.
\end{aligned} \tag{3.22}
$$

Proof. From the inequality

$$
\frac{d}{dt}\|H\|^2_{1,\Pi_t} \le \|H\|^2_{1,\Pi_t} + \varepsilon\|H_t\|^2_{1,\Pi_t} + \|H\|^2_{2,\Pi_t}\|v\|^2_{0,\Pi_t}
$$

and from (3.18) we get (3.22).

Now let $\overset{2}{H} = H_*$ on B; then from Lemmas 3.1–3.12 we get

LEMMA 3.13. *For a sufficiently smooth solution* (v, p', H) *of* (1.1)–(1.7), *we have*

$$
\begin{aligned}
\frac{d}{dt}\varphi + \phi \le c\Big[\Big(\gamma\Big(t\int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\Big) + 1\Big)\phi\varphi(1+\varphi) + \|H_*\|^2_{3,B} \\
+ \|H_*\|^4_{1,B} + \|H_{*t}\|^2_{2,B}(1 + \|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B}\Big],
\end{aligned} \tag{3.23}
$$

where γ is an increasing positive function, $t \in [0,T]$ and

$$
\begin{aligned}
\varphi(t) &= \sum_{i+k\le 2} (\|\partial_t^i v\|_{k,\Omega_t}^2 + \|\partial_t^i H\|_{k,\Pi_t}^2),\\
\phi(t) &= \sum_{\substack{i+k\le 3\\ i\le 2}} (\|\partial_t^i v\|_{k,\Omega_t}^2 + \|\partial_t^i H\|_{k,\Pi_t}^2) + \|p'\|_{2,\Omega_t}^2 + \|p'_t\|_{1,\Omega_t}^2.
\end{aligned}
\tag{3.24}
$$

4. Korn inequality

LEMMA 4.1. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let (v,p') be a solution of $(1.1)_1$, $(1.1)_2$, $(1.5)_1$ and $f = \int_\Omega v_0\,dx = \int_\Omega v_0 \cdot \varphi_i\,dx = 0$, where φ_i is defined by (4.4), $i = 1,2,3$, and*

$$
\mathbb{E}_{\Omega_t}(v_t) = \int_{\Omega_t} (\partial_{x_i} v_{jt} + \partial_{x_j} v_{it})^2\,dx < \infty. \tag{4.1}
$$

Then there exists a constant c such that

$$
\|v_t\|_{1,\Omega_t}^2 \le c(\mathbb{E}_{\Omega_t}(v_t) + \|v\|_{1,\Omega_t}^4). \tag{4.2}
$$

Proof. Introduce a function u by

$$
u = \sum_{i=1}^3 b_i \varphi_i(x) + v_t, \tag{4.3}
$$

where

$$
\begin{aligned}
\varphi_i &= (x - \overline{x}) \times e_i,\\
\overline{x} &= \frac{1}{|\Omega_t|}\Big(\int_{\Omega_t} x_1\,dx, \int_{\Omega_t} x_2\,dx, \int_{\Omega_t} x_3\,dx\Big),\\
e_i &= (\delta_{i1}, \delta_{i2}, \delta_{i3}), \qquad i = 1,2,3.
\end{aligned}
\tag{4.4}
$$

Define $b = (b_1, b_2, b_3)$ by

$$
b = \frac{1}{2|\Omega_t|} \int_{\Omega_t} \operatorname{rot} v_t\,dx. \tag{4.5}
$$

Since $\operatorname{rot}\varphi_i = 2e_i$ $i = 1,2,3$, equations (4.3) and (4.4) imply

$$
\int_{\Omega_t} \operatorname{rot} u\ dx = 0. \tag{4.6}
$$

From (4.4) we have $\int_{\Omega_t} \varphi_i\,dx = 0$, $i = 1,2,3$, so

$$
\int_{\Omega_t} u\,dx = \int_{\Omega_t} v_t\,dx, \quad \text{and also} \quad \mathbb{E}_{\Omega_t}(\varphi_i) = 0, \quad i = 1,2,3, \tag{4.7}
$$

hence

$$
\mathbb{E}_{\Omega_t}(u) = \mathbb{E}_{\Omega_t}(v_t). \tag{4.8}
$$

By Theorem 1 of [6] we have

$$\partial_{x_j} w_i = \varepsilon_{ikl}\partial_{x_k} S_{jl}, \quad i=1,2,3, \quad w = \operatorname{rot} u, \quad S_{ij} = \partial_{x_i} u_j + \partial_{x_j} u_i, \tag{4.9}$$

so by (4.6) and Lemma 2.4 of [3] it follows that

$$\|\operatorname{rot} u\|^2_{0,\Omega_t} \le c \sum_{i,j=1}^{3} \|S_{ij}\|^2_{0,\Omega_t} = c\,\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v_t). \tag{4.10}$$

Employing the identity

$$\partial_{x_j} u_i = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j) + \frac{1}{2}(\partial_{x_j} u_i - \partial_{x_i} u_j)$$

and (4.10) we have

$$\|\nabla u\|^2_{0,\Omega_t} \le c(\mathbb{E}_{\Omega_t}(u) + \|\operatorname{rot} u\|^2_{0,\Omega_t}) \le c\mathbb{E}_{\Omega_t}(u) = c\mathbb{E}_{\Omega_t}(v_t). \tag{4.11}$$

Using (4.3) we obtain

$$\|\nabla v_t\|^2_{0,\Omega_t} \le c(\mathbb{E}_{\Omega_t}(v_t) + |b|). \tag{4.12}$$

Integrating $(1.1)_1$ over Ω_t we get

$$\int_{\Omega_t} v_t\,dx = -\int_{\Omega_t} v\cdot\nabla v\,dx \tag{4.13}$$

and multiplying $(1.1)_1$ by φ_i, $i=1,2,3$, and integrating over Ω_t, from (4.3) we get systems of equations

$$\sum_{i=1}^{3} b_i \int_{\Omega_t} \varphi_i\cdot\varphi_j\,dx = \int_{\Omega_t} u\cdot\varphi_j\,dx + \int_{\Omega_t} v\cdot\nabla v\cdot\varphi_i\,dx, \quad j=1,2,3. \tag{4.14}$$

Since $\det\Gamma \ne 0$, where $\Gamma = \{\Gamma_{ij}\}$, $\Gamma_{ij} = \int_{\Omega_t} \varphi_i\cdot\varphi_j\,dx$, we can calculate b from (4.14), so

$$|b|^2 \le c(\|u\|^2_{0,\Omega_t} + \|v\|^4_{1,\Omega_t}). \tag{4.15}$$

Now by the Poincaré inequality and (4.8), (4.12) we obtain

$$\begin{aligned} \|u\|^2_{0,\Omega_t} &\le 2\bigg\|u - \frac{1}{|\Omega_t|}\int_{\Omega_t} u\,dx\bigg\|^2_{0,\Omega_t} + 2\bigg\|\frac{1}{|\Omega_t|}\int_{\Omega_t} u\,dx\bigg\|^2_{0,\Omega_t} \\ &\le c\bigg(\|\nabla u\|^2_{0,\Omega_t} + \bigg\|\frac{1}{|\Omega_t|}\int_{\Omega_t} v_t\,dx\bigg\|^2_{0,\Omega_t}\bigg) \\ &\le c(\mathbb{E}_{\Omega_t}(u) + \|v\|^4_{1,\Omega_t}) \\ &= c(\mathbb{E}_{\Omega_t}(v_t) + \|v\|^4_{1,\Omega_t}). \end{aligned} \tag{4.16}$$

From (4.3) we get

$$\|v_t\|^2_{0,\Omega_t} \le c(\|u\|^2_{0,\Omega_t} + |b|^2). \tag{4.17}$$

Then from (4.12), (4.15), (4.16) and (4.17) we get (4.2).

LEMMA 4.2. *Let $\Omega_t \subset \mathbb{R}^3$ be a bounded domain. Let (v,p') be a solution of* (1.1)$_1$, (1.1)$_2$, (1.5)$_1$ *and* $f = \int_\Omega v_0\,dx = \int_\Omega v_0 \cdot \varphi_i\,dx$, $i = 1,2,3$, *and*

$$\mathbb{E}_{\Omega_t}(v_{tt}) = \int_{\Omega_t} (\partial_{x_i} v_{jtt} + \partial_{x_j} v_{itt})^2 dx < \infty. \tag{4.18}$$

Then there exists a constant c such that

$$\|v_{tt}\|_{1,\Omega_t}^2 \le c[\mathbb{E}_{\Omega_t}(v_t) + \|v_t\|_{1,\Omega_t}^2 \|v\|_{1,\Omega_t}^2 + \|v\|_{2,\Omega_t}^4 (\|v\|_{1,\Omega_t}^2 + 1)]. \tag{4.19}$$

Proof. Let $u = \sum_{i=1}^3 b_i \varphi_i(x) + v_{tt}$, where φ_i are described by (4.4). The rest of the argument is as in Lemma 4.1.

5. Global existence. To prove the global existence we introduce the spaces

$$\mathcal{N}(t) = \{(v,p',H) : \varphi(t) < \infty\},$$
$$\mathcal{M}(t) = \Big\{(v,p',H) : \varphi(t) + \int_0^t \phi(\tau)\,d\tau < \infty\Big\},$$

where $\varphi(t), \phi(t)$ are defined by (3.24). From Theorem 2.1 we get

LEMMA 5.1. *Assume that $(v(0),p'(0),H(0)) \in \mathcal{N}(0)$ and $\varphi(0) < \varepsilon_1$. Then $(v(t),p'(t),H(t)) \in \mathcal{M}(t)$ for $t \le T$, where T is the time of local existence and*

$$\begin{aligned}\varphi(t) + \int_0^T \phi(\tau)\,d\tau &\le c\varepsilon_1 + c\int_0^T (\|E_*\|_{0,B}^2 + \|E_{*t}\|_{0,B}^2 \\ &\quad + \|H_*\|_{3,B}^2 + \|H_{*t}\|_{2,B}^2 + \|H_{*tt}\|_{0,B}^2)\,dt \\ &\equiv c(\varepsilon_1 + \beta).\end{aligned} \tag{5.1}$$

Proof. From the inequalities

$$\|\overline{v}\|_{2,\Omega}^2 \le c(\varepsilon \|\overline{v}_t\|_{2,2,2,\Omega^t}^2 + c(\varepsilon)\|\overline{v}\|_{2,2,2,\Omega^t}^2 + \|\overline{v}(0)\|_{2,\Omega}^2),$$
$$\|\overline{H}\|_{2,\Pi}^2 \le c(\varepsilon \|\overline{H}_t\|_{2,2,2,\Pi^t}^2 + c(\varepsilon)\|\overline{H}\|_{2,2,2,\Pi^t}^2 + \|\overline{H}(0)\|_{2,\Pi}^2),$$

and Theorem 2.1 we get (5.1).

LEMMA 5.2. *Assume that there exists a local solution of* (1.1)–(1.7) *in $\mathcal{M}(t)$, $0 \le t \le T$, with initial data in $\mathcal{N}(0)$ sufficiently small and*

$$\begin{aligned}\alpha(t) &= \|H_*\|_{3,B}^2 + \|H_*\|_{1,B}^4 + \|H_{*t}\|_{2,B}^2(1 + \|H_{*t}\|_{2,B}^2) + \|H_{*tt}\|_{1,B}^2 \\ &\le e^{-\mu t},\end{aligned} \tag{5.2}$$

for $0 \le t \le T$, where $\mu > 1/2$. Then

$$\varphi(t) \le e^{-t/2}\Big(\varphi(0) + \frac{c}{\mu - 1/2}\Big). \tag{5.3}$$

Proof. From (3.23) and (5.2) we get

$$\frac{d}{dt}\varphi + \phi \le c\Big(\gamma\Big(t\int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau\Big) + 1\Big)\phi\varphi(1+\varphi) + ce^{-\mu t}. \tag{5.4}$$

From Lemma 5.1 we have $c(\gamma(t\int_0^t \|v\|^2_{3,\Omega_\tau}\,d\tau)+1)\phi\varphi(1+\varphi) \le \frac{1}{2}\phi$ if β and ε_1 are sufficiently small. Then from (5.4) we get

$$\frac{d}{dt}\varphi + \frac{1}{2}\,\phi \le ce^{-\mu t}. \tag{5.5}$$

We have $\varphi \le \phi$. Then from (5.5),

$$\frac{d}{dt}\varphi + \frac{1}{2}\,\varphi \le ce^{-\mu t}. \tag{5.6}$$

From (5.6) we get (5.3).

LEMMA 5.3. *Let the assumptions of Lemma* 5.2 *be satisfied and* $\varphi(0) < \varepsilon_1$. *Then* $\varphi(T) \le \varepsilon_1$ *where* $T > 0$ *is the time of local existence.*

Proof. If T and $\mu > 0$ are sufficiently large, then from (5.3) we get

$$\varphi(T) \le e^{-T/2}\Big(\varphi(0) + \frac{c}{\mu - 1/2}\Big) \le \varphi(0).$$

Now we consider problem (1.1)–(1.7) for $t \in [kT,(k+1)T]$. Then similarly to (3.23) we obtain the inequality

$$\begin{aligned}\frac{d}{dt}\varphi + \phi \le c\Big[\Big(\gamma\Big((t-kT)\int_{kT}^t \|v\|^2_{3,\Omega_\tau}d\tau\Big) + 1\Big)\phi\varphi(1+\varphi) + \|H_*\|^2_{3,B} \\ + \|H_*\|^4_{1,B} + \|H_{*t}\|^2_{2,B}(1+\|H_{*t}\|^2_{2,B}) + \|H_{*tt}\|^2_{1,B}\Big],\end{aligned} \tag{5.7}$$

where $t \in [kT,(k+1)T]$.

Let $\varphi(kT) \le \varphi(0)$. Then from (5.7) similarly to (5.3) we get, for $t \in [kT,(k+1)T]$,

$$\varphi(t) \le c\frac{e^{(1/2-\mu)kT - t/2}}{\mu - 1/2} + \varphi(kT)e^{(kT-t)/2} \le c\frac{e^{-t/2}}{\mu-1/2} + \varphi(0)e^{(kT-t)/2}. \tag{5.8}$$

Therefore if T, $\mu > 0$ are sufficiently large we get $\varphi((k+1)T) \le \varphi(0)$. We also obtain the inequalities

$$\int_{kT}^t \phi\,d\tau \le \frac{2c}{\mu}\,e^{-\mu kT} + 2\varphi(kT) \quad \text{for } t \in [kT,(k+1)T]. \tag{5.9}$$

Hence

$$\int_{kT}^{(k+1)T} \|v\|^2_{3,\Omega_t}\,dt \le \int_{kT}^{(k+1)T} \phi\,dt \le \frac{c}{\mu}\,e^{-\mu kT} + 2\varphi(kT) \tag{5.10}$$

and

$$\int\limits_{kT}^{t} \varphi(\tau)\,d\tau \le \frac{2c}{\mu-1/2}e^{-\mu kT} + 2\varphi(kT) \quad \text{for } t\in[kT,(k+1)T]. \tag{5.11}$$

Then inequalities (5.9) and (5.11) imply

$$\begin{aligned} \Big|\int\limits_{kT}^{(k+1)T} v\,dt\Big| &\le cT^{1/2}\Big(\int\limits_{kT}^{(k+1)T}\|v\|^2_{2,\Omega_t}\,dt\Big)^{1/2} \\ &\le cT^{1/2}\Big(\frac{2c}{\mu-1/2}\,e^{-\mu kT} + 2\varphi(kT)\Big)^{1/2}. \end{aligned} \tag{5.12}$$

Proof of Main Theorem. The theorem is proved step by step using local existence in a fixed time interval. Under the assumptions that

$$(v(0),p'(0),H(0)) \in \mathcal{N}(0) \tag{5.13}$$

Theorem 2.1 and Lemma 5.1 yield local existence of solutions of (1.1)–(1.7).

By (5.13) and Lemma 5.1 the local solution belongs to $\mathcal{M}(t)$, $t\le T$. For small ε_1 and β the existence time T is correspondingly large, so we can assume it is a fixed positive number. To prove the last result we needed the Korn inequalities (see Section 4) and imbedding theorems. The constants in those theorems depend on Ω_t, the shape of S_t and $\int_0^t\|v\|^2_{3,\Omega_\tau}\,d\tau$, so generally they are functions of t.

But in view of (5.1) with sufficiently small ε_1,β we obtain

$$\Big|\int\limits_0^t v\,d\tau\Big| \le c(\varepsilon_1+\beta), \quad t\in[0,T]. \tag{5.14}$$

Hence from the relation

$$x = \xi + \int\limits_0^t v(x(\xi,\tau),\tau)\,d\tau, \qquad \xi\in S,\ t\le T, \tag{5.15}$$

for sufficiently small ε_1,β and fixed T, the shape of S_t, $t\le T$, does not change too much, so the constants from the imbedding theorems can be chosen independent of time. Now we wish to extend the solution to the interval $[T,2T]$. Using Lemma 5.3 and (5.8)–(5.12) we can prove the existence of a local solution in $\mathcal{M}(t)$, $T\le t\le 2T$. To prove

$$\varphi(2T)\le\varepsilon_1 \tag{5.16}$$

we need inequality (5.7), where the constants depend on the constants from the imbedding theorems and Korn inequalities for $t\in[T,\ 2T]$. Therefore we have to show that the shape of S_t and $\int_0^t\|v\|^2_{3,\Omega_\tau}\,d\tau$, $t\le 2T$, do not change more than for $t\le T$. Assume that there exists a local solution in the interval $[0,kT]$. Then in view of Lemma 5.2 and (5.8)–(5.12) we have, for $t\in[0,kT]$,

$$
(5.17)\qquad \Big|\int_0^t v\,d\tau\Big| \le \int_0^t \|v\|_{2,\Omega_\tau}\,d\tau \le c_1\sum_{i=0}^{k-1}\int_{iT}^{(i+1)T}\|v\|_{2,\Omega_t}\,dt
$$
$$
\le c_1T^{1/2}\sum_{i=0}^{k-1}\Big(\int_{iT}^{(i+1)T}\|v\|_{2,\Omega_t}^2\,dt\Big)^{1/2} \le c_1T^{1/2}\sum_{i=0}^{k-1}\Big(\int_{iT}^{(i+1)T}\varphi(t)\,dt\Big)^{1/2}
$$
$$
\le c_1T^{1/2}\sum_{i=0}^{k-1}\sqrt{2}\Big[\frac{c}{\mu-1/2}\,e^{-iT\mu}+\varphi(iT)\Big]^{1/2}
$$
$$
\le \frac{cT^{1/2}}{1-e^{-T/4}}\Big[\frac{1}{(\mu-1/2)^{1/2}}\Big(1+\frac{1}{1-e^{-T/4}}\Big)+\varphi^{1/2}(0)\Big]\equiv I.
$$

From (5.8),

$$
\varphi((i+1)T)\le\frac{c}{\mu-1/2}\,e^{-(i+1)T/2}+\varphi(iT)e^{-T/2},\qquad i=0,1,\dots,k-1,
$$

we have

$$
\sum_{i=0}^{k-1}\varphi(iT)\le\frac{\varphi(0)}{1-e^{-T/2}}+\frac{c}{\mu-1/2}\,\frac{e^{-T/2}}{1-e^{-T/2}}+\frac{c}{\mu-1/2}\,\frac{e^{-T}}{1-e^{-T/2}}+\cdots
$$
$$
+\frac{c}{\mu-1/2}\,\frac{e^{-nT/2}}{1-e^{-T/2}}+\cdots\le\frac{1}{1-e^{-T/2}}\Big(\varphi(0)+\frac{ce^{-T/2}}{(\mu-1/2)(1-e^{-T/2})}\Big).
$$

Therefore,

$$
(5.18)\qquad \int_0^t\|v\|_{3,\Omega_\tau}^2\,d\tau\le\sum_{i=0}^{k-1}\int_{iT}^{(i+1)T}\|v\|_{3,\Omega_\tau}^2\,d\tau\le\sum_{i=0}^{k-1}\int_{iT}^{(i+1)T}\phi\,dt
$$
$$
\le 2\sum_{i=0}^{k-1}\Big(\frac{c}{\mu}\,e^{-\mu iT}+\varphi(iT)\Big)
$$
$$
\le\frac{c}{1-e^{-T/2}}\Big(\frac{1}{\mu}+\varphi(0)+\frac{1}{(\mu-1/2)(1-e^{-T/2})}\Big).
$$

We have to underline the importance of the fact that the quantity $\int_0^t v\,d\tau$ is sufficiently small for all t. We have $\operatorname{dist}\{S_t,S_0\}\le|\int_0^t v\,d\tau|\le I$, where I is defined in (5.17), so for small I the domains Ω_t and Ω are close to each other, hence all the imbedding theorems applied and results for elliptic problems (3.10), (3.12), (3.14) are valid for all Ω_t, $t>0$.

Taking $k=2$, ε_1 sufficiently small and μ sufficiently large we see that $\int_0^t v(x(\xi,t),t)\,dt$ is small for any $t\in[0,2T]$, so (5.17) and (5.18) imply that the shape of S_t and $\int_0^t\|v\|_{3,\Omega_\tau}^2\,d\tau$ change no more than in $[0,T]$, and then the differential inequality (3.23) can also be shown for this interval with the same constants. Hence in view of Lemma 5.1 the solution of (1.1)–(1.7) belongs to $\mathcal{M}(t)$, $t \in [T, 2T]$. Next Lemmas 5.1–5.3 and (5.8)–(5.12) imply (5.17).

Repeating the above considerations for the intervals $[kT, (k+1)T]$, $k \geq 2$, we prove the existence for all $t \in \mathbb{R}_+$.

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