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THE MAGNETIZATION AT HIGH TEMPERATURE FOR A p-SPIN INTERACTION MODEL WITH EXTERNAL FIELD

Abstract. This paper is devoted to a detailed and rigorous study of the magnetization at high temperature for a p-spin interaction model with external field, generalizing the Sherrington–Kirkpatrick model. In particular, we prove that $\langle \sigma_i \rangle$ (the mean of a spin with respect to the Gibbs measure) converges to an explicitly given random variable, and that $\langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle$ are asymptotically independent.

1. Introduction. We consider a spin glass model with the configuration space $\Sigma_N = \{-1, 1\}^N$ where the energy of a given configuration $\sigma \in \Sigma_N$ is represented by a Hamiltonian $H(\sigma)$. We are interested in the Gibbs measure G_N whose density with respect to the uniform measure μ_N on Σ_N is $Z_N^{-1} e^{-H}$, where Z_N is the normalization factor

$$Z_N = \sum_{\sigma \in \Sigma_N} \exp(-H(\sigma)).$$

In order to introduce our model we borrow the notations of Bardina *et al.* (2004). The Hamiltonian of the p-spin interaction model with external field is defined by

$$-H_{N,\beta,h}(\sigma) = \beta u_N \sum_{(i_1,\ldots,i_p) \in A_N^p} g_{i_1,\ldots,i_p} \sigma_{i_1} \ldots \sigma_{i_p} + h \sum_{i=1}^{N} \sigma_i,$$

with

$$u_N = \left(\frac{p!}{2N^{p-1}} \right)^{1/2}, \tag{1.1}$$
$$A_N^p = \{(i_1, \ldots, i_p) \in \mathbb{N}^p;\ 1 \le i_1 < \cdots < i_p \le N\},$$

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where the parameter β represents the inverse of the temperature and where $g = \{g_{i_1,\dots,i_p};\ (i_1,\dots,i_p) \in A_N^p\}$ is a family of independent standard Gaussian random variables. The strictly positive parameter h stands for the external magnetic field, under which the spins tend to take the same value $+1$.

In physics, this kind of model was introduced to study the spin distribution of atoms submitted to disordered long range interactions (see, for instance, the paper of Gardner (1985)). In mathematics, the p-spin interaction model is a natural generalization of the SK model (see Sherrington and Kirkpatrick (1975)). However, the mathematical papers devoted to this general kind of model are rare: see Talagrand (2000a) on low temperature regime; Bardina *et al.* (2004) and Cadel *et al.* (2004) on high temperature regime; and Bovier *et al.* (2002) for some fluctuation results for the free energy.

We will denote by $\langle f\rangle$ the average of a function $f : \Sigma_N \to \mathbb{R}$ with respect to G_N, as well as the average of a function $f : \Sigma_N^n \to \mathbb{R}$ with respect to $G_N^{\otimes n}$, without mentioning the number n of independent copies of the spin configurations, i.e.

$$\langle f\rangle = \frac{1}{Z_N^n} \sum_{(\sigma^1,\dots,\sigma^n)\in\Sigma_N^n} f(\sigma^1,\dots,\sigma^n) \exp\Big(-\sum_{l\le n} H_{N,\beta,h}(\sigma^l)\Big).$$

We write $\nu(f) = \mathbf{E}\langle f\rangle$, where $\mathbf{E}$ denotes expectation with respect to the randomness of the Hamiltonian.

The following assumption on β determines our high temperature region:

(H) The parameter $\beta > 0$ is smaller than a constant β_p defined by

$$8p^2\beta_p^2 \exp(16\beta_p^2 p) = \frac{1}{2}.$$

In statistical mechanics, Gibbs' measure represents the probability of observing a configuration σ after the system has reached equilibrium with an infinite heat bath at temperature $1/\beta$. For this reason, β small means high temperature.

Our aim is to prove the following theorem:

THEOREM 1.1. *Assume* (H). *Then*, *given a positive integer* m, *there exist independent standard Gaussian random variables* $z_1,\dots,z_m$ *such that*

$$\mathbf{E}\sum_{i=1}^m \bigg[\langle\sigma_i\rangle - \tanh\bigg(\beta\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2} z_i + h\bigg)\bigg]^2 \le \frac{C(m,h)}{N}. \tag{1.2}$$

Here the constant $q = q_p$ *is the unique solution of*

$$q = \mathbf{E}\bigg[\tanh^2\bigg(\beta\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2} Y + h\bigg)\bigg], \tag{1.3}$$

where Y *stands for a standard Gaussian random variable.*

The constant $q = q_p$ is directly connected with the behavior of the overlap of two configurations

$$R_{1,2} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^1 \sigma_i^2, \tag{1.4}$$

and with the Hamming distance

$$d(\sigma^1, \sigma^2) = \mathrm{card}\{i \leq N;\ \sigma_i^1 = \sigma_i^2\}.$$

More specifically, for β small enough, $R_{1,2}$ will self average into q (see Proposition 2.1) and the knowledge of behavior of the overlap gives us information on this well-known distance by means of the equality

$$d(\sigma^1, \sigma^2) = \frac{N}{2}\,(1 - R_{1,2}).$$

For more information about the parameters β_p and q_p we refer the reader to Bardina *et al.* (2004).

As a consequence of Theorem 1.1, we have the following result:

COROLLARY 1.2. *Assume* (H). *Then the mean of a spin* (*with respect to the randomness of the configuration space*) *converges in law to an explicitly given random variable*, *namely*

$$\langle \sigma_i \rangle \xrightarrow[N\to\infty]{\mathcal{L}} \tanh\left(\beta\left(\frac{p}{2}\right)^{1/2} q^{(p-1)/2} z_i + h\right).$$

Moreover, $\langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle$ *are asymptotically independent.*

In order to prove Theorem 1.1 we need the following important intermediate result.

PROPOSITION 1.3. *Given* $\beta \leq \beta_p$, *there exists a standard Gaussian random variable* z *such that*

$$\mathbf{E}\left[\langle \sigma_N \rangle - \tanh\left(\beta\left(\frac{p}{2}\right)^{1/2} q^{(p-1)/2} z + h\right)\right]^2 \leq \frac{C(h)}{N}, \tag{1.5}$$

where z *depends only on* $\{g_J : J \in A_N^p\}$ *but is probabilistically independent of* $\{g_J : J \in A_{N-1}^p\}$, *with*

$$A_{N-1}^p = \{(i_1, \ldots, i_p) \in \mathbb{N}^p : 1 \leq i_1 < \cdots < i_p \leq N-1\}.$$

The paper is organized as follows: some preliminary results on the cavity method for our model are given in Section 2; Section 3 contains some intermediate results (Lemma 3.1) for the proof of Theorem 1.1, and the definition of the Gaussian path which will be used later on; the proofs of Lemma 3.1, Proposition 1.3 and Theorem 1.1 are given in Sections 4, 5 and 6, respectively. In the following, the size of a given finite set D will be denoted by $|D|$. Let C denote a constant which may vary from line to line.

2. The cavity method. This method allows us, in some sense, to measure the difference between our original system and a system where the last spin is independent of the others. The cavity method for our model is already described in Bardina *et al.* (2004, Section 2.3), and it is given here only for the convenience of the reader.

For $\beta > 0$, we define β_- that plays the role of β in the new reduced system:

$$\beta_- = \left(\frac{N-1}{N}\right)^{(p-1)/2} \beta.$$

Set

$$Q_N^p = \{J = (i_1, \dots, i_{p-1}, N) \in \mathbb{N}^p;\ 1 \le i_1 < \dots < i_{p-1} \le N-1\},$$

and recall that

$$A_N^p = \{(i_1, \dots, i_p) \in \mathbb{N}^p;\ 1 \le i_1 < \dots < i_p \le N\}.$$

Lemmas A.2 and A.4 in Bardina *et al.* (2004) prove that

$$|A_N^p| = \binom{N}{p} = \frac{N^p}{p!} + P_{p-1}(N),$$
$$|Q_N^p| = \binom{N-1}{p-1} = \frac{N^{p-1}}{(p-1)!} + P_{p-2}(N),$$

where $P_m(N)$ denotes some polynomial of degree m in N. Moreover, as a consequence of Lemma A.4 in Bardina *et al.* (2004), it is not difficult to prove another deterministic result about the size of Q_N^p:

$$\left| u_N^2 |Q_N^p| q^{p-1} - \frac{p}{2}\, q^{p-1} \right| \le \frac{C}{N} \tag{2.1}$$

for some positive constant C.

We use the following notation: $\varrho = (\sigma_1, \dots, \sigma_{N-1})$ is a configuration of Σ_{N-1}, $\eta_J = \sigma_{i_1} \cdots \sigma_{i_{p-1}}$ for $J \in Q_N^p$, and $\varepsilon = \sigma_N$. The basic idea of the cavity method is to regroup the Hamiltonian as follows:

$$-H_{N,\beta,h}(\sigma) = -H_{N-1,\beta_-,h}(\varrho) + \varepsilon[g(\varrho) + h],$$

where

$$-H_{N-1,\beta_-,h}(\varrho) = \beta_- u_{N-1} \sum_{(i_1,\dots,i_p) \in A_{N-1}^p} g_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p} + h \sum_{i=1}^{N-1} \sigma_i,$$
$$g(\varrho) = \beta u_N \sum_{J \in Q_N^p} g_J \eta_J.$$

Let $\langle \cdot \rangle_-$ denote the average with respect to Gibbs measure on Σ_{N-1} relative to the reduced Hamiltonian $H_{N-1,\beta_-,h}$. In the spin glass theory, the cavity method becomes a powerful tool through the construction of a continuous path from the original configuration to a configuration where the last spin is independent of the others.

Set, for $t \in [0,1]$ and the constant $q \in [0,1]$ defined in (1.3),

$$g_t(\varrho) = t^{1/2} g(\varrho) + \beta u_N q^{(p-1)/2}(1-t)^{1/2} \sum_{J \in Q_N^p} z_J, \tag{2.2}$$

where $\{z_J;\ J \in Q_N^p\}$ is a family of independent standard Gaussian random variables, also independent of all the disorders g.

For $n \geq 1$ and n independent copies of an N-spin configuration $\sigma^1, \ldots, \sigma^n$, we write

$$\mathcal{E}_{n,t} = \exp\Big\{ \sum_{l=1}^{n} \varepsilon^l [g_t(\varrho^l) + h] \Big\}, \tag{2.3}$$

$$Z_t = \langle \mathbf{Av}\, \mathcal{E}_{1,t} \rangle_- = \langle \cosh[g_t(\varrho) + h] \rangle_-, \tag{2.4}$$

where $\varepsilon^l = \sigma_N^l$ and $\mathbf{Av}$ means the average over $\{\varepsilon^l;\ l = 1, \ldots, n\}$. For $f : \Sigma_N^n \to \mathbb{R}$, we can define

$$\langle f \rangle_t = \frac{\langle \mathbf{Av}\, f \mathcal{E}_{n,t} \rangle_-}{Z_t^n}, \qquad \nu_t(f) = \mathbf{E}\, \langle f \rangle_t.$$

Note that $\nu(f) = \nu_1(f)$.

The idea is that $\nu_0(f)$ (or a slight modification of it) should be simpler to compute than $\nu_1(f)$ in some interesting cases of functions f. On the other hand, we will relate these two quantities by means of

$$\nu_1(f) - \nu_0(f) = \int_0^1 \nu_t'(f)\, dt. \tag{2.5}$$

Let us summarize some results proved in Bardina *et al.* (2004) that will be useful in our proofs.

- For $t \in [0,1]$ and $f : \Sigma_N^n \to \mathbb{R}$, we have

$$\begin{aligned} \nu_t'(f) = \beta^2 u_N^2 \sum_{J \in Q_N^p} \Big[&\nu_t\Big(f \sum_{1 \leq l < l' \leq n} (\eta_J^l \eta_J^{l'} - q^{p-1})\ \varepsilon^l \varepsilon^{l'} \Big) \\ &- n \nu_t\Big(f \sum_{l=1}^{n} (\eta_J^l \eta_J^{n+1} - q^{p-1}) \varepsilon^l \varepsilon^{n+1} \Big) \\ &+ \frac{n(n+1)}{2} \nu_t(f(\eta_J^{n+1} \eta_J^{n+2} - q^{p-1}) \varepsilon^{n+1} \varepsilon^{n+2}) \Big]. \end{aligned} \tag{2.6}$$

- If $\tau_1, \tau_2 > 0$ are such that $1/\tau_1 + 1/\tau_2 = 1$, then, for any $t \in [0,1]$,

$$|\nu_t(f_1 f_2)| \leq \nu_t(|f_1|^{\tau_1})^{1/\tau_1} \nu_t(|f_2|^{\tau_2})^{1/\tau_2}. \tag{2.7}$$

PROPOSITION 2.1. *Assume that* β *satisfies* (H). *Then, for* $q \in [0,1]$ *defined in* (1.3) *and for any* $l \geq 1$,

$$\nu((R_{1,2}-q)^{2l}) = \mathbf{E}\,\langle (R_{1,2}-q)^{2l}\rangle \leq \left(\frac{Cl}{N}\right)^{l}, \tag{2.8}$$

$$|\nu(R_{1,2}^{l}-q^{l})| \leq \frac{C(l)}{N}, \tag{2.9}$$

where $R_{1,2}$ *has been defined in* (1.4); *and, for a function* f *on* Σ_N^n,

$$|\nu(f)-\nu_0(f)| \leq \frac{C}{N^{1/2}}\,\nu^{1/2}(f^2), \tag{2.10}$$

$$|\nu(f)-\nu_0(f)-\nu_0'(f)| \leq \frac{C}{N}\,\nu^{1/2}(f^2). \tag{2.11}$$

Proof. See Proposition 3.2, Corollary 3.10 and Corollary 3.8 in Bardina *et al.* (2004). ■

3. Continuous path. The first and crucial step in the proof of Proposition 1.3 is the verification of the following two facts:

1. The average of σ_N with respect to the Hamiltonian $H_{N,\beta,h}$ behaves asymptotically as the hyperbolic tangent of a quantity depending on $\{g_J;\ J\in A_N^p\}$ but probabilistically independent of $\{g_J;\ J\in A_{N-1}^p\}$.
2. The average of σ_1 with respect to the Hamiltonian $H_{N,\beta,h}$ behaves asymptotically as the average of the same spin σ_1 but only with respect to the Hamiltonian $H_{N-1,\beta_-,h}$.

These two facts can be deduced from the following lemma.

LEMMA 3.1. *Assume that* β *satisfies* (H). *Then, for* $a\in\{0,1\}$,

$$\Delta := \mathbf{E}\Big[\langle\sigma_1^a\varepsilon^{1-a}\rangle - \langle\sigma_1^a\rangle_-\tanh^{1-a}\Big(\beta u_N\sum_{J\in Q_N^p} g_J\langle\eta_J\rangle_- + h\Big)\Big]^2 \leq \frac{C}{N}. \tag{3.1}$$

We start by giving the definition of the Gaussian path we will use: let

$$\widetilde{g}(c) = \beta u_N \sum_{J\in Q_N^p} g_J\ \langle\eta_J\rangle_-,$$

and, for $t\in[0,1]$,

$$\widetilde{g}_t(c) = t^{1/2}\widetilde{g}(c) + \beta u_N q^{(p-1)/2}(1-t)^{1/2}\sum_{J\in Q_N^p} z_J,$$

where $\{z_J;\ J\in Q_N^p\}$ is as in (2.2). As in (2.3) and (2.4), for $n\geq 1$ and n independent copies of an N-spin configuration $\sigma^1,\dots,\sigma^n$, we can define

$$\widetilde{\mathcal{E}}_{n,t}=\exp\Big\{\sum_{l=1}^{n}\varepsilon^l[\widetilde{g}_t(c)+h]\Big\}, \tag{3.2}$$

$$\widetilde{Z}_t=\langle\mathbf{Av}\,\widetilde{\mathcal{E}}_{1,t}\rangle_-=\langle\cosh[\widetilde{g}_t(c)+h]\rangle_-. \tag{3.3}$$

Then, for $t\in[0,1]$, we consider the function

$$\Theta(t)=\mathbf{E}\,[(\Phi(t)-\Psi(t))^2],$$

where, for $a\in\{0,1\}$,

$$\Phi(t):=\langle\sigma_1^a\varepsilon^{1-a}\rangle_t=\frac{\langle\mathbf{Av}\,\sigma_1^a\varepsilon^{1-a}\mathcal{E}_{1,t}\rangle_-}{Z_t},$$

$$\Psi(t):=\frac{\langle\mathbf{Av}\,\sigma_1^a\varepsilon^{1-a}\widetilde{\mathcal{E}}_{1,t}\rangle_-}{\widetilde{Z}_t}=\langle\sigma_1^a\rangle_-\tanh^{1-a}[\widetilde{g}_t(c)+h].$$

We can decompose Θ into three terms

$$\Theta(t)=\Theta_1(t)+\Theta_2(t)+\Theta_3(t),$$

with

$$\begin{aligned}\Theta_1(t)&=\mathbf{E}[\Phi(t)^2],\\ \Theta_2(t)&=\mathbf{E}[\Psi(t)^2],\\ \Theta_3(t)&=-2\mathbf{E}[\Phi(t)\Psi(t)].\end{aligned}$$

Since it is easy to check that $\Phi(0)=\Psi(0)$, it follows that Δ, defined in (3.1), satisfies

$$\Delta=|\Theta(1)|=|\Theta(1)-\Theta(0)|\leq\sum_{j=1}^{3}[|\Theta_j(1)-\Theta_j(0)-\Theta_j'(0)|+|\Theta_j'(0)|].$$

Thus, (3.1) in Lemma 3.1 will be achieved as soon as we can show that

$$|\Theta_j(1)-\Theta_j(0)-\Theta_j'(0)|\vee|\Theta_j'(0)|\leq C/N\qquad\text{for any } j=1,2,3. \tag{3.4}$$

4. Proof of Lemma 3.1

4.1. *Study of Θ_1.* Using two replicas of σ, we obtain

$$\Theta_1(t)=\mathbf{E}[\Phi(t)^2]=\mathbf{E}\,\langle\sigma_1^a\varepsilon^{1-a}\rangle_t^2=\nu_t((\sigma_1^1\sigma_1^2)^a(\varepsilon^1\varepsilon^2)^{1-a}),$$

where the measure ν_t is defined in Section 2; recall that $a\in\{0,1\}$.

First of all, since $|(\sigma_1^1\sigma_1^2)^a(\varepsilon^1\varepsilon^2)^{1-a}|\leq 1$, by (2.11) in Proposition 2.1, we have

$$|\Theta_1(1)-\Theta_1(0)-\Theta_1'(0)|\leq C/N.$$

Thus, if we check that $|\Theta_1'(0)|\leq C/N$, we will have proved (3.4) when $j=1$ and concluded the study of Θ_1. From (2.6), the symmetry and independence yield

$$
\begin{aligned}
\Theta_1'(0) &= \nu_0'((\sigma_1^1\sigma_1^2)^a(\varepsilon^1\varepsilon^2)^{1-a}) \\
&= \beta^2 u_N^2 \sum_{J\in Q_N^p} [\nu_0((\sigma_1^1\sigma_1^2)^a(\eta_J^1\eta_J^2 - q^{p-1}))\nu_0((\varepsilon^1\varepsilon^2)^{2-a}) \\
&\quad - 4\nu_0((\sigma_1^1\sigma_1^2)^a(\eta_J^1\eta_J^3 - q^{p-1}))\nu_0((\varepsilon^1)^{2-a}(\varepsilon^2)^{1-a}\varepsilon^3) \\
&\quad + 3\nu_0((\sigma_1^1\sigma_1^2)^a(\eta_J^3\eta_J^4 - q^{p-1}))\nu_0((\varepsilon^1\varepsilon^2)^{1-a}\varepsilon^3\varepsilon^4)].
\end{aligned}
$$

So, in order to bound $|\Theta_1'(0)|$, since $|\varepsilon| \le 1$, we only need to check that, for any couple $(i,j) \in \{(1,2),(1,3),(3,4)\}$,

$$
\Upsilon := \Big|\beta^2 u_N^2 \sum_{J\in Q_N^p} \nu_0((\sigma_1^1\sigma_1^2)^a(\eta_J^i\eta_J^j - q^{p-1}))\Big| \le C/N. \tag{4.1}
$$

The quantity Υ can be bounded by three terms as follows:

$$
\Upsilon \le \beta^2[\Upsilon_1 + \Upsilon_2 + \Upsilon_3],
$$

with

$$
\begin{aligned}
\Upsilon_1 &= \Big|u_N^2 \sum_{J\in Q_N^p} \nu_0((\sigma_1^1\sigma_1^2)^a\eta_J^i\eta_J^j) - \frac{p}{2}\,\nu_0((\sigma_1^1\sigma_1^2)^a R_{i,j}^{p-1})\Big|, \\
\Upsilon_2 &= \frac{p}{2}\,|\nu_0((\sigma_1^1\sigma_1^2)^a R_{i,j}^{p-1}) - \nu_0((\sigma_1^1\sigma_1^2)^a q^{p-1})|, \\
\Upsilon_3 &= \Big|\frac{p}{2}\,\nu_0((\sigma_1^1\sigma_1^2)^a q^{p-1}) - u_N^2 \sum_{J\in Q_N^p} \nu_0((\sigma_1^1\sigma_1^2)^a q^{p-1})\Big|.
\end{aligned}
$$

Recall that $R_{1,2}$ has been defined in (1.4). On the one hand, Lemma 5.11 in Talagrand (2000a) gives

$$
\Big|u_N^2 \sum_{J\in Q_N^p} \eta_J^i\eta_J^j - \frac{p}{2}\,R_{i,j}^{p-1}\Big| \le C/N, \tag{4.2}
$$

which together with the estimate (2.1) implies

$$
\beta^2(\Upsilon_1 + \Upsilon_3) \le C/N. \tag{4.3}
$$

On the other hand, we have

$$
\Upsilon_2 = \frac{p}{2}\,|\nu_0((\sigma_1^1\sigma_1^2)^a(R_{i,j}^{p-1} - q^{p-1}))| \le \frac{p}{2}\,[\Upsilon_{2,1} + \Upsilon_{2,2}], \tag{4.4}
$$

where

$$
\begin{aligned}
\Upsilon_{2,1} &= |\nu_0((\sigma_1^1\sigma_1^2)^a(R_{i,j}^{p-1} - q^{p-1})) - \nu((\sigma_1^1\sigma_1^2)^a(R_{i,j}^{p-1} - q^{p-1}))|, \\
\Upsilon_{2,2} &= |\nu((\sigma_1^1\sigma_1^2)^a(R_{i,j}^{p-1} - q^{p-1}))|.
\end{aligned}
$$

Applying the estimates (2.10) and (2.8) for $l = 1$, and using the fact that

$|(\sigma_1^1\sigma_1^2)^a| \vee |R_{i,j}| \vee q \leq 1$, we obtain

$$\begin{aligned}\Upsilon_{2,1} &\leq \frac{C}{N^{1/2}}\,[\nu((R_{i,j}^{p-1}-q^{p-1})^2)]^{1/2} \\ &\leq \frac{C}{N^{1/2}}\,[\nu((R_{i,j}-q)^2)]^{1/2} \leq \frac{C}{N}.\end{aligned} \tag{4.5}$$

Using the symmetry, the Cauchy–Schwarz inequality (2.7) and Proposition 2.1 (in particular, the bounds (2.8) and (2.9)), we get

$$\begin{aligned}\Upsilon_{2,2} &= |\nu(R_{1,2}^a(R_{i,j}^{p-1}-q^{p-1}))| \\ &\leq |\nu((R_{1,2}-q)^a(R_{i,j}^{p-1}-q^{p-1}))| + q^a|\nu(R_{i,j}^{p-1}-q^{p-1})| \\ &\leq C/N.\end{aligned} \tag{4.6}$$

Putting together (4.3)–(4.6) provides (4.1), which concludes the study of Θ_1.

4.2. *Study of* Θ_2. For $t \in [0,1]$ and $f : \Sigma_N^n \to \mathbb{R}$, consider the new measure $\widetilde{\nu}_t$ defined by

$$\widetilde{\nu}_t(f) = \mathbf{E}\bigg(\frac{\langle \mathbf{Av}\, f\widetilde{\mathcal{E}}_{n,t}\rangle_-}{\widetilde{Z}_t^n}\bigg),$$

where $\widetilde{\mathcal{E}}_{n,t}$ and $\widetilde{Z}_t$ are given in (3.2) and (3.3), respectively.

Working as in Proposition 2.1 of Bardina *et al.* (2004), we can express the derivative of this new measure as

$$\begin{aligned}\widetilde{\nu}_t'(f) = \beta^2 u_N^2 \sum_{J\in Q_N^p} \bigg[&\widetilde{\nu}_t\Big(f(\langle\eta_J\rangle_-^2 - q^{p-1}) \sum_{1\leq l<l'\leq n} \varepsilon^l\varepsilon^{l'}\Big) \\ &- n\widetilde{\nu}_t\Big(f(\langle\eta_J\rangle_-^2 - q^{p-1}) \sum_{l=1}^{n} \varepsilon^l\varepsilon^{n+1}\Big) \\ &+ \frac{n(n+1)}{2}\,\widetilde{\nu}_t(f(\langle\eta_J\rangle_-^2 - q^{p-1})\ \varepsilon^{n+1}\varepsilon^{n+2})\bigg].\end{aligned} \tag{4.7}$$

First of all, taking two replicas of σ allows us to write Θ_2, for $a \in \{0,1\}$, as

$$\Theta_2(t) = \mathbf{E}[\Psi(t)^2] = \widetilde{\nu}_t((\sigma_1^1\sigma_1^2)^a(\varepsilon^1\varepsilon^2)^{1-a}). \tag{4.8}$$

Then, in order to bound $|\Theta_2'(0)|$, we will use (4.7) with $f = (\sigma_1^1\sigma_1^2)^a(\varepsilon^1\varepsilon^2)^{1-a}$. So, by symmetry and independence, using the fact that $|\varepsilon^i\varepsilon^j| \leq 1$, the definition of $\widetilde{\nu}_t$ for $t=0$, and taking new replicas of σ, we obtain

$$
\begin{aligned}
(4.9)\qquad |\Theta_2'(0)| &\le 8\Big|\beta^2 u_N^2 \sum_{J\in Q_N^p} \widetilde{\nu}_0((\sigma_1^1\sigma_1^2)^a(\langle\eta_J\rangle_-^2 - q^{p-1}))\Big| \\
&= 8\Big|\beta^2 u_N^2 \sum_{J\in Q_N^p} \mathbf{E}\,\langle(\sigma_1^1\sigma_1^2)^a(\langle\eta_J^3\eta_J^4 - q^{p-1}\rangle_-\rangle_-\Big| \\
&= 8\Big|\beta^2 u_N^2 \sum_{J\in Q_N^p} \nu_0((\sigma_1^1\sigma_1^2)^a(\eta_J^3\eta_J^4 - q^{p-1}))\Big|.
\end{aligned}
$$

We now proceed as for the study of (4.1) to prove that

$$
(4.10)\qquad |\Theta_2'(0)| \le C/N.
$$

It remains to analyze the other term of (3.4) for $j=2$. Taylor expansion applied to (4.8) yields

$$
|\Theta_2(1)-\Theta_2(0)-\Theta_2'(0)| = \big|\widetilde{\nu}_1(f)-\widetilde{\nu}_0(f)-\widetilde{\nu}_0'(f)\big| \le \frac{1}{2}\int_0^1 |\widetilde{\nu}_t''(f)|\,dt
$$

for $f=(\sigma_1^1\sigma_1^2)^a(\varepsilon^1\varepsilon^2)^{1-a}$. Bounding accurately the derivative of (4.7) we obtain

$$
|\widetilde{\nu}_t''(f)| \le C\beta^4 u_N^4\,\Big|\sum_{J_1,J_2\in Q_N^p} \widetilde{\nu}_t(f[\langle\eta_{J_1}\rangle_-^2 - q^{p-1}][\langle\eta_{J_2}\rangle_-^2 - q^{p-1}]\widehat{\varepsilon})\Big|
$$

with $\widehat{\varepsilon}=\varepsilon^i\varepsilon^j\varepsilon^{i'}\varepsilon^{j'}$. Then, considering different replicas of σ, using the fact that $|f\widehat{\varepsilon}|\vee|R_{1,2}|\vee q\le 1$ and applying (4.2) and (2.1) (as in (4.3) for Υ), we get

$$
\begin{aligned}
(4.11)\qquad |\nu_t''(f)| &\le C\beta^4\Big|\widetilde{\nu}_t\Big(f\widehat{\varepsilon}\,\Big\langle u_N^2\sum_{J\in Q_N^p}[\eta_J^1\eta_J^2 - q^{p-1}]\Big\rangle_-^2\Big)\Big| \\
&\le C\beta^4\mathbf{E}\Big(\frac{1}{\widetilde{Z}_t^2}\Big\langle\,\mathbf{Av}\,\Big\langle u_N^2\sum_{J\in Q_N^p}[\eta_J^1\eta_J^2 - q^{p-1}]\Big\rangle_-^2\widetilde{\mathcal{E}}_{2,t}\Big\rangle_-\Big) \\
&= C\beta^4\mathbf{E}\Big(\Big\langle u_N^2\sum_{J\in Q_N^p}[\eta_J^1\eta_J^2 - q^{p-1}]\Big\rangle_-^2\Big) \\
&\le C\beta^4\nu_0(|(R_{1,2}^{p-1}-q^{p-1})(R_{3,4}^{p-1}-q^{p-1})|) + C/N,
\end{aligned}
$$

and now we proceed as in (4.4) for Υ_2 to conclude that

$$
|\Theta_2(1)-\Theta_2(0)-\Theta_2'(0)| \le C/N.
$$

This estimate together with (4.10) ends the study of Θ_2.

4.3. *Study of* Θ_3. Here the term Θ_3 is, in some sense, a mixture between Θ_1 and Θ_2. For $t\in[0,1]$ and

$$f:\Sigma_N^n\times\Sigma_N^{\widetilde{n}}\to\mathbb{R},\qquad(\sigma,\widetilde{\sigma})\mapsto f(\sigma,\widetilde{\sigma}),$$

we define

$$\widehat{\nu}_t(f)=\mathbf{E}\bigg(\frac{1}{Z_t^n(\sigma)\widetilde{Z}_t^{\widetilde{n}}(\widetilde{\sigma})}\,\langle\widehat{\mathbf{Av}}\,f(\sigma,\widetilde{\sigma})\mathcal{E}_{n,t}(\sigma)\widetilde{\mathcal{E}}_{\widetilde{n},t}(\widetilde{\sigma})\rangle_-\bigg),$$

where $\widehat{\mathbf{Av}}$ means the average over $\{\varepsilon^l,\widetilde{\varepsilon}^{\widetilde{l}};\ l=1,\dots,n,\ \widetilde{l}=1,\dots,\widetilde{n}\}$.

It is long and tedious but not difficult to deduce that the derivative of $\widehat{\nu}_t(f)$ is composed of three kinds of terms, namely

$$\Xi_{1,t}(f)=\beta^2u_N^2\sum_{J\in Q_N^p}\widehat{\nu}_t(f(\sigma,\widetilde{\sigma})[\eta_J^l\eta_J^{l'}-q^{p-1}]\varepsilon^l\varepsilon^{l'}),$$

$$\Xi_{2,t}(f)=\beta^2u_N^2\sum_{J\in Q_N^p}\widehat{\nu}_t(f(\sigma,\widetilde{\sigma})[\langle\widetilde{\eta}_J\rangle_-^2-q^{p-1}]\widetilde{\varepsilon}^{\widetilde{l}}\widetilde{\varepsilon}^{\widetilde{l}'}),$$

$$\Xi_{3,t}(f)=\beta^2u_N^2\sum_{J\in Q_N^p}\widehat{\nu}_t(f(\sigma,\widetilde{\sigma})[\eta_J^l\langle\widetilde{\eta}_J\rangle_--q^{p-1}]\varepsilon^l\widetilde{\varepsilon}^{\widetilde{l}}),$$

where $l,l'\in\{1,\dots,n+2\}$, $\widetilde{l},\widetilde{l}'\in\{1,\dots,\widetilde{n}+2\}$. As in the previous sections, we also have, for $a\in\{0,1\}$,

$$\Theta_3(t)=-2\mathbf{E}[\Phi(t)\Psi(t)]=-2\widehat{\nu}_t((\sigma_1\widetilde{\sigma}_1)^a(\varepsilon\widetilde{\varepsilon})^{1-a}).$$

In order to check that $|\Theta_3(0)|\le C/N$, the cases $\Xi_{1,0}(f)$ and $\Xi_{2,0}(f)$ (with $f=(\sigma_1\widetilde{\sigma}_1)^a(\varepsilon\widetilde{\varepsilon})^{1-a}$) are handled as in the subsections devoted to Θ_1 and Θ_2, respectively. In the remaining case, by symmetry and independence we have

$$\begin{aligned}(4.12)\qquad|\Xi_{3,0}(f)|&=\beta^2u_N^2\Big|\sum_{J\in Q_N^p}\widehat{\nu}_0((\sigma_1^1\widetilde{\sigma}_1^1)^a(\varepsilon^1\widetilde{\varepsilon}^1)^{1-a}[\eta_J^l\langle\widetilde{\eta}_J\rangle_--q^{p-1}]\varepsilon^l\widetilde{\varepsilon}^{\widetilde{l}})\Big|\\&\le\beta^2u_N^2\Big|\sum_{J\in Q_N^p}\widehat{\nu}_0((\sigma_1^1\widetilde{\sigma}_1^1)^a[\eta_J^l\langle\widetilde{\eta}_J\rangle_--q^{p-1}])\Big|\\&=\beta^2u_N^2\Big|\sum_{J\in Q_N^p}\mathbf{E}\,\langle(\sigma_1^1\sigma_1^2)^a[\eta_J^k\eta_J^3-q^{p-1}]\rangle_-\Big|\\&=\beta^2u_N^2\Big|\sum_{J\in Q_N^p}\nu_0((\sigma_1^1\sigma_1^2)^a[\eta_J^k\eta_J^3-q^{p-1}])\Big|,\end{aligned}$$

where k is equal to 1 or 4. Now, since $|\Xi_{3,0}(f)|$ is bounded by the same type of factor as Υ in (4.1), we proceed as in the study of Υ in Section 4.1.

Finally, we can conclude that

$$|\Theta_3(1)-\Theta_3(0)-\Theta_3'(0)|\le\frac{1}{2}\int_0^1\widehat{\nu}_t''(f)\,dt\quad\text{with}\quad f=(\sigma_1\widetilde{\sigma}_1)^a(\varepsilon\widetilde{\varepsilon})^{1-a}.$$

Since the terms of this second derivative are of the same type as Θ_1 or Θ_2 or a mixture between Θ_1 and Θ_2, they can be dealt with as in Sections 4.1, 4.2 or as in (4.12).

5. Proof of Proposition 1.3.

Let

$$z = \frac{1}{\|c\|} \sum_{J \in Q_N^p} g_J \langle \eta_J \rangle_- \quad \text{with} \quad \|c\|^2 = \sum_{J \in Q_N^p} \langle \eta_J \rangle_-^2. \tag{5.1}$$

It will be observed later that this z is the random variable appearing in Proposition 1.3. Let us start with an easy but curious property of z that will be used in the proof of this proposition.

LEMMA 5.1. *The law of z is standard Gaussian. This random variable depends only on $\{g_J;\ J \in A_N^p\}$ but is independent of $\{g_J;\ J \in A_{N-1}^p\}$.*

Proof. Since $A_N^p = A_{N-1}^p \,\dot\cup\, Q_N^p$, it is obvious that z depends on $\{g_J;\ J \in A_N^p\}$. Moreover, conditionally upon $\{\langle \eta_J \rangle_-;\ J \in Q_N^p\}$, the law of $(1/\|c\|) g_J \langle \eta_J \rangle_-$ is trivially centered Gaussian with variance $(1/\|c\|^2)\langle \eta_J \rangle_-^2$. So, denoting by E_- the conditional expectation upon $\{\langle \eta_J \rangle_-;\ J \in Q_N^p\}$, by conditional independence we can get

$$\begin{aligned}
\mathbf{E}(e^{ivz}) = \mathbf{E}[E_-(e^{ivz})] &= \mathbf{E}\bigg[E_-\bigg(\prod_{J \in Q_N^p} \exp\bigg\{\frac{1}{\|c\|}\, iv g_J \langle \eta_J \rangle_-\bigg\}\bigg)\bigg] \\
&= \mathbf{E}\bigg[\prod_{J \in Q_N^p} E_-\bigg(\exp\bigg\{\frac{1}{\|c\|}\, iv g_J \langle \eta_J \rangle_-\bigg\}\bigg)\bigg] \\
&= \mathbf{E}\bigg[\prod_{J \in Q_N^p} \exp\bigg\{-\frac{v^2 \langle \eta_J \rangle_-^2}{2\|c\|^2}\bigg\}\bigg] = e^{-v^2/2},
\end{aligned}$$

which implies that z is a standard Gaussian random variable. Finally, z is independent of $\{g_J;\ J \in A_{N-1}^p\}$ since we can check that $\mathbf{E}[z g_{\widetilde{J}}] = 0$ for any $g_{\widetilde{J}}$, $\widetilde{J} \in A_{N-1}^p$. ■

Proof of Proposition 1.3. We want to show that

$$\Lambda := \mathbf{E}\bigg[\langle \sigma_N \rangle - \tanh\bigg(\beta\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2} z + h\bigg)\bigg]^2 \le \frac{C(h)}{N},$$

where z is defined in (5.1).

We can write

$$\Lambda \le 2(\Lambda_1 + \Lambda_2),$$

with

$$\Lambda_1 = \mathbf{E}[\langle\sigma_N\rangle - \tanh(\widetilde{g}(c)+h)]^2,$$

$$\Lambda_2 = \mathbf{E}\bigg[\tanh\bigg(\beta\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2}z + h\bigg) - \tanh(\widetilde{g}(c)+h)\bigg]^2.$$

We only need to study Λ_2 because Lemma 3.1 for $a=0$ implies $\Lambda_1 \le C/N$. Using the inequality $|\tanh a - \tanh b| \le |a-b|$, the definitions of $\widetilde{g}(c)$, z and $\|c\|$, and the conditional expectation E_- defined in Lemma 5.1, we obtain

$$\begin{aligned}
(5.2)\qquad \Lambda_2 &\le \beta^2\mathbf{E}\bigg[\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2}z - u_N \sum_{J\in Q_N^p} g_J\langle\eta_J\rangle_-\bigg]^2 \\
&= \beta^2\mathbf{E}\bigg[E_-\bigg\{\bigg(\frac{1}{\|c\|}\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2} - u_N\bigg)\sum_{J\in Q_N^p} g_J\langle\eta_J\rangle_-\bigg\}^2\bigg] \\
&= \beta^2\mathbf{E}\bigg[\bigg(\frac{1}{\|c\|}\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2} - u_N\bigg)^2\sum_{J\in Q_N^p} \langle\eta_J\rangle_-^2\bigg] \\
&= \beta^2\mathbf{E}\bigg(\Big(\frac{p}{2}\Big)^{1/2} q^{(p-1)/2} - u_N\sqrt{\sum_{J\in Q_N^p} \langle\eta_J\rangle_-^2}\bigg)^2.
\end{aligned}$$

When $h=0$, we have $q=0$, hence the result. Assume now that $h>0$. Then, since the lower bound of q (solution of (1.3)) is uniform in $\beta\le\beta_p$, by means of (2.1) we have

$$\begin{aligned}
\Lambda_2 &\le \frac{2\beta^2}{pq^{p-1}}\,\mathbf{E}\bigg(\frac{p}{2}\,q^{p-1} - u_N^2\sum_{J\in Q_N^p}\langle\eta_J\rangle_-^2\bigg)^2 \\
&\le \frac{2\beta^2 u_N^4}{pq^{p-1}}\,\mathbf{E}\bigg(\sum_{J\in Q_N^p}[q^{p-1} - \langle\eta_J\rangle_-^2]\bigg)^2 + \frac{C}{N}.
\end{aligned}$$

This last term can be bounded as in (4.11). ∎

6. Proof of Theorem 1.1.

A last result will be needed to be able to prove this theorem.

Lemma 6.1. *Let q be the unique solution of* (1.3) *and q_- the unique solution of*

$$q_- = \mathbf{E}\bigg[\tanh^2\bigg(\beta_-\Big(\frac{p}{2}\Big)^{1/2} q_-^{(p-1)/2}Y + h\bigg)\bigg]$$

with $\beta_- = ((N-1)/N)^{(p-1)/2}\beta$ and Y as in (1.3). *Then, if $\beta\le\beta_p$, we have*

$$|q-q_-| \le C/N.$$

Proof. Lemma 2.4.15 in Talagrand (2000b) proves the case $p = 2$. Assume $p \geq 3$. For $s > 0$, set $\lambda(s) = \mathbf{E} \tanh^2(X_s + h)$, where X_s is a centered Gaussian random variable with variance s^2. It is not difficult to check that $|\lambda'(s)| \leq C$. Then, by using the mean value theorem and the fact that $|q \vee q_-| \leq 1$, we obtain

$$\begin{aligned} |q - q_-| &= \left|\lambda\left(\beta\left(\frac{p}{2}\right)^{1/2} q^{(p-1)/2}\right) - \lambda\left(\beta_-\left(\frac{p}{2}\right)^{1/2} q_-^{(p-1)/2}\right)\right| \\ &\leq C|\beta q^{(p-1)/2} - \beta_- q_-^{(p-1)/2}| \leq C[|\beta - \beta_-| + \beta|q^{(p-1)/2} - q_-^{(p-1)/2}|] \\ &\leq C/N + C\beta|q - q_-|. \end{aligned}$$

Taking β small enough, we have

$$|q - q_-| \leq \frac{C}{(1 - C\beta)N} \leq \frac{C}{2N}. \blacksquare$$

Proof of Theorem 1.1. We argue by induction. We assume that the random variables $\{z_1, \ldots, z_m\}$ depend on $\{g_{i_1,\ldots,i_p}; (i_1, \ldots, i_p) \in A_N^p\}$ as part of the induction hypothesis. The case $m = 1$ is a consequence of the symmetry applied to Proposition 1.3.

We now assume that Theorem 1.1 is true for m and we will check it for $m + 1$. In order to show the independence of the random variables $\{z_1, \ldots, z_m, z_{m+1}\}$ of Theorem 1.1, we need to apply the induction hypothesis to the $N - 1$-spin system with Hamiltonian $H_{N-1,\beta_-,h}$. First of all, we make the following decomposition:

$$\mathbf{E} \sum_{i=1}^{m+1} \left[\langle\sigma_i\rangle - \tanh\left(\beta\left(\frac{p}{2}\right)^{1/2} q^{(p-1)/2} z_i + h\right)\right]^2 \leq C \sum_{j=1}^{4} \Gamma_j$$

with

$$\begin{aligned} \Gamma_1 &= \mathbf{E} \sum_{i=1}^{m} [\langle\sigma_i\rangle - \langle\sigma_i\rangle_-]^2, \\ \Gamma_2 &= \mathbf{E} \sum_{i=1}^{m} \left[\langle\sigma_i\rangle_- - \tanh\left(\beta_-\left(\frac{p}{2}\right)^{1/2} q_-^{(p-1)/2} z_i + h\right)\right]^2, \\ \Gamma_3 &= \mathbf{E} \sum_{i=1}^{m} \left[\tanh\left(\beta_-\left(\frac{p}{2}\right)^{1/2} q_-^{(p-1)/2} z_i + h\right)\right. \\ &\qquad \left. - \tanh\left(\beta\left(\frac{p}{2}\right)^{1/2} q^{(p-1)/2} z_i + h\right)\right]^2, \\ \Gamma_4 &= \mathbf{E}\left[\langle\sigma_{m+1}\rangle - \tanh\left(\beta\left(\frac{p}{2}\right)^{1/2} q^{(p-1)/2} z_{m+1} + h\right)\right]^2. \end{aligned}$$

Lemma 3.1 for $a = 1$ and symmetry yield $\Gamma_1 \leq C/N$. The induction hypothesis implies the existence of independent standard Gaussian random variables $z_1, \dots, z_m$ depending on $\{g_{i_1,\dots,i_p}; (i_1, \dots, i_p) \in A^p_{N-1}\}$ such that $\Gamma_2 \leq C(m,h)/N$. Using the inequality $|\tanh a - \tanh b| \leq |a-b|$ and Lemma 6.1 we obtain

$$\Gamma_3 \leq C[|\beta - \beta_-| + |q^{(p-1)/2} - q_-^{(p-1)/2}|] \leq C/N.$$

Finally, Proposition 1.3 gives the existence of a standard Gaussian random variable $z = z_{m+1}$ such that $\Gamma_4 \leq C(h)/N$ and z_{m+1} is independent of $\{z_1, \dots, z_m\}$ because these random variables depend only on $\{g_{i_1,\dots,i_p}; (i_1, \dots, i_p) \in A^p_{N-1}\}$. ■

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