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Samir Benaissa (Sidi Bel Abbès)

BERNSTEIN INEQUALITY FOR THE PARAMETER OF THE pTH ORDER AUTOREGRESSIVE PROCESS AR(p)

Abstract. The autoregressive process takes an important part in predicting problems leading to decision making. In practice, we use the least squares method to estimate the parameter $\widetilde{\theta}$ of the first-order autoregressive process taking values in a real separable Banach space B ($ARB(1)$), if it satisfies the following relation:

$$\widetilde{X}_t = \widetilde{\theta}\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t.$$

In this paper we study the convergence in distribution of the linear operator $I(\widetilde{\theta}_T, \widetilde{\theta}) = (\widetilde{\theta}_T - \widetilde{\theta})\widetilde{\theta}^{T-2}$ for $\|\widetilde{\theta}\| > 1$ and so we construct inequalities of Bernstein type for this operator.

1. Introduction. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and let $(B, \|\cdot\|)$ be a real separable Banach space endowed with its Borel σ-field $\mathcal{B}$. We denote by $\|\cdot\|$ the norm of bounded linear operators. A sequence of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $(B, \mathcal{B})$ is a sequence of measurable maps $\Omega \to B$ with respect to the σ-fields $\mathcal{A}$ and $\mathcal{B}$.

We consider a stochastic process $(X_t,\ t \in \mathbb{Z})$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which takes its values in a real separable Banach space $(B, \|\cdot\|)$, of the following general form:

$$X_t = \theta_1 X_{t-1} + \theta_2 X_{t-2} + \cdots + \theta_{p-1} X_{t-p+1} + \theta_p X_{t-p} + \varepsilon_t. \tag{1}$$

The equation (1) is the representation of the pth order autoregressive process AR(p). Here $\theta_1, \ldots, \theta_p$ are fixed constants and $(\varepsilon_t, t \in \mathbb{Z})$ is a sequence of independent identically distributed (i.i.d.) random variables with mean zero

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and variance $\mathbb{E}\varepsilon_t^2 = \sigma^2$. Write (1) as

$$\begin{pmatrix} X_t \\ X_{t-1} \\ \vdots \\ X_{t-p+1} \end{pmatrix} = \begin{pmatrix} \theta_1 & \theta_2 & \cdots & \theta_{p-1} & \theta_p \\ & & I_{p-1} & & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} X_{t-1} \\ X_{t-2} \\ \vdots \\ X_{t-p} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which can be written as a first-order autoregressive process in p variables in the following way:

$$\widetilde{X}_t = \widetilde{\theta}\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t, \tag{2}$$

where $\widetilde{X}_t$ denotes the transpose vector $(X_t, X_{t-1}, \ldots, X_{t-p+1})' \in B^p$, $\widetilde{\varepsilon}_t = (\varepsilon_t, 0, 0, \ldots, 0)' \in B^p$ $(\mathbb{E}\|\widetilde{\varepsilon}_t\|^2 = \mathbb{E}\varepsilon_t^2)$, $\widetilde{\theta}$ is a linear operator on B^p (an arbitrary $p \times p$ matrix) and I_{p-1} is the identity operator on B^{p-1}.

Definition 1.1. Consider the nonstationary first-order autoregressive process which takes values in B defined by

$$\begin{cases} \widetilde{X}_t = \widetilde{\theta}\widetilde{X}_{t-1} + \widetilde{\varepsilon}_t, \\ \|\widetilde{\theta}\| > 1, \quad \widetilde{\beta} = \widetilde{\theta}^{-1}, \end{cases} \tag{3}$$

where $\widetilde{\theta}$ is the autoregressive operator. Considering $\widetilde{X}_t$ as an exogenous variable, the ordinary least squares method gives an estimator of $\widetilde{\theta}$ which is the sequence defined by

$$\begin{cases} \widetilde{\theta}_T = \Big(\sum_{t=1}^{T} \widetilde{X}_t\widetilde{X}'_{t-1}\Big)\Big(\sum_{t=1}^{T} \widetilde{X}_{t-1}\widetilde{X}'_{t-1}\Big)^{-1}, \\ I(\widetilde{\theta}_T, \widetilde{\theta})\widetilde{\theta}^{-(T-2)} = \widetilde{\theta}_T - \widetilde{\theta} = \widetilde{\mathcal{A}}_T\widetilde{\mathcal{B}}_T^{-1}, \end{cases} \tag{4}$$

where

$$\widetilde{\mathcal{A}}_T = \sum_{t=1}^{T} \widetilde{X}_t\widetilde{X}'_{t-1} - \widetilde{\theta}\sum_{t=1}^{T} \widetilde{X}_{t-1}\widetilde{X}'_{t-1}, \tag{5}$$

$$\widetilde{\mathcal{B}}_T = \sum_{t=1}^{T} \widetilde{X}_{t-1}\widetilde{X}'_{t-1}. \tag{6}$$

2. Preliminaries. In this section, we introduce notations, definitions and show the invariance of the set of initial data.

Let

$$\widetilde{Z}_T = \widetilde{\theta}^{-(T-2)}\widetilde{X}_{T-1} = \widetilde{\varepsilon}_1 + \widetilde{\theta}^{-1}\widetilde{\varepsilon}_2 + \cdots + \widetilde{\theta}^{-(T-2)}\widetilde{\varepsilon}_{T-1} + \widetilde{\theta}\widetilde{X}_0, \tag{7}$$

$$\widetilde{Z} = \sum_{t=1}^{\infty} \widetilde{\theta}^{-(t-1)}\widetilde{\varepsilon}_t + \widetilde{\theta}\widetilde{X}_0, \tag{8}$$

$$F_T = \widetilde{Z}_T\widetilde{Z}'_T + \widetilde{\theta}^{-1}\widetilde{Z}_T\widetilde{Z}'_T(\widetilde{\theta}^{-1})' + \cdots + \widetilde{\theta}^{-(T-1)}\widetilde{Z}_T\widetilde{Z}'_T(\widetilde{\theta}^{-(T-1)})', \tag{9}$$

$$G_T = \widetilde{\varepsilon}_T\widetilde{Z}'_T + \widetilde{\varepsilon}_{T-1}\widetilde{Z}'_T(\widetilde{\theta}^{-1})' + \cdots + \widetilde{\varepsilon}_1\widetilde{Z}'_T(\widetilde{\theta}^{-(T-1)})'. \tag{10}$$

Definition 2.1 ([3]–[5]). Let $(Y, Y_T, T \geq 1)$ be a family of B-valued random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then Y_T *converges to* Y *in probability* ($Y_T \xrightarrow{\mathbb{P}} Y$) as $T \to \infty$ if for each $\varepsilon > 0$,

$$\mathbb{P}\{\|Y_T - Y\| > \varepsilon\} \to 0 \quad \text{as } T \to \infty,$$

and Y_T *converges to* Y *in distribution* ($Y_T \xrightarrow{\mathcal{D}} Y$) if $\mathbb{P}_{Y_T} \to \mathbb{P}_Y$, where $\mathbb{P}_{Y_T}$ (resp. $\mathbb{P}_Y$) is the distribution of Y_T (resp. Y).

Remark 2.1. The following properties of convergence in distribution will be used in the remainder of the text:

- If Y_T converges to Y in probability then Y_T converges to Y in distribution.
- If Y_T converges to Y in distribution and $Y_T - Z_T$ converges to 0 in probability then Z_T converges to Y in distribution.
- Let B' be a separable Banach space and $h : B \to B'$ be a continuous function. If Y_T converges to Y in distribution then $h(Y_T)$ converges to $h(Y)$ in distribution.

Theorem 2.1 (see [6]). *Let $I = [a, b]$ and suppose that f is J-convex on I. For any points $x_1, \dots, x_T \in I$ and any rational nonnegative numbers $r_1, \dots, r_T$ such that $r_1 + \cdots + r_T = 1$, we have*

$$f\Big(\sum_{t=1}^{T} r_t x_t\Big) \leq \sum_{t=1}^{T} r_t f(x_t). \tag{11}$$

In particular, if $r_t = 1/T$ $(t = 1, \dots, T)$, then

$$f\bigg(\frac{1}{T}\sum_{t=1}^{T} x_t\bigg) \leq \frac{1}{T}\sum_{t=1}^{T} f(x_t). \tag{12}$$

Lemma 2.1 (Bernstein's inequality, see [4], [7]). *Let W be a random variable. Then for all $\zeta, \tau > 0$,*

$$\mathbb{P}\{W \geq \zeta\} \leq \exp(-\zeta\tau)\mathbb{E}[\exp(\tau W)].$$

3. Asymptotic distributions in the unstable case. In this section we assume that the $(\varepsilon_t,\ t \in \mathbb{Z})$ are independent and identically distributed and that $\|\widetilde{\theta}\| > 1$.

In order to state the main results of the paper, we introduce the following conditions:

(H_0) $\|\widetilde{\beta}\| < 1$, $\widetilde{\beta} = \widetilde{\theta}^{-1}$.
(H_1) $\mathbb{E}\|\widetilde{\varepsilon}_t\|^4 < \infty$, $\mathbb{E}\|\widetilde{X}_0\|^4 < \infty$.

THEOREM 3.1. *Suppose that the hypotheses* (H_0)–(H_1) *are satisfied and* $\varepsilon > 0$. *Then*

$$\mathbb{P}\{\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T\| > \varepsilon\} \le \exp\left(-\frac{1}{2}\,\frac{(\varepsilon - \|\widetilde{\beta}\|^{T-1}T\sigma^2)^2}{\|\widetilde{\beta}\|^{2(T-1)}T^4\sigma^4 + o(1)}\right), \tag{13}$$

that is,

$$\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T \to 0 \quad \text{in probability as } T \to \infty, \tag{14}$$

where G_T *is defined in the preliminaries.*

Proof. Using (5) and (10), we obtain

$$\begin{aligned}\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T &= \sum_{t=1}^{T} \widetilde{\varepsilon}_t \widetilde{X}'_{t-1}(\widetilde{\beta}^{T-2})' - \sum_{t=1}^{T} \widetilde{\varepsilon}_t \widetilde{Z}'_T(\widetilde{\beta}^{T-t})' \\ &= \sum_{t=1}^{T} \widetilde{\varepsilon}_t(\widetilde{\beta}^{-(t-2)}\widetilde{Z}_t)'(\widetilde{\beta}^{T-2})' - \sum_{t=1}^{T} \widetilde{\varepsilon}_t \widetilde{Z}'_T(\widetilde{\beta}^{T-t})' \\ &= \sum_{s=0}^{T-1} \widetilde{\varepsilon}_{T-s}\widetilde{Z}'_{T-s}(\widetilde{\beta}^{s})' - \sum_{s=0}^{T-1} \widetilde{\varepsilon}_{T-s}\widetilde{Z}'_T(\widetilde{\beta}^{s})' \\ &= \sum_{s=0}^{T-1} \widetilde{\varepsilon}_{T-s}(\widetilde{Z}'_{T-s} - \widetilde{Z}'_T)(\widetilde{\beta}^{s})'\end{aligned} \tag{15}$$

with

$$\widetilde{Z}_T = \widetilde{\beta}^{T-2}\widetilde{\varepsilon}_{T-1} + \widetilde{\beta}^{T-3}\widetilde{\varepsilon}_{T-2} + \cdots + \widetilde{\beta}^{T-s-1}\widetilde{\varepsilon}_{T-s} + \widetilde{Z}_{T-s}.$$

Hence

$$\widetilde{Z}_T - \widetilde{Z}_{T-s} = \widetilde{\beta}^{T-2}\widetilde{\varepsilon}_{T-1} + \widetilde{\beta}^{T-3}\widetilde{\varepsilon}_{T-2} + \cdots + \widetilde{\beta}^{T-s-1}\widetilde{\varepsilon}_{T-s}.$$

From (15) it is easy to deduce that

$$\begin{aligned}\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T = \sum_{s=0}^{T-1} \widetilde{\varepsilon}_{T-s}(\widetilde{\varepsilon}'_{T-1}(\widetilde{\beta}^{T-2})' &+ \cdots \\ &+ \widetilde{\varepsilon}'_{T-s}(\widetilde{\beta}^{T-s-1})')(\widetilde{\beta}^{s})'.\end{aligned} \tag{16}$$

Using the elementary inequality

$$\mathbb{1}_{\{\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T\| - \varepsilon > 0\}} \le \exp(\lambda(\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T\| - \varepsilon)) \quad \text{for all } \lambda > 0,$$

where $\mathbb{1}_A$ is the indicator of the set A, by Bernstein's inequality (Lemma 2.1) we obtain

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\{\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})'-G_T\|-\varepsilon>0\}}) &\le \mathbb{E}\exp(\lambda(\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})'-G_T\|-\varepsilon))\\
&= e^{-\lambda\varepsilon}\mathbb{E}\exp(\lambda(\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})'-G_T\|))\\
&\le e^{-\lambda\varepsilon}\mathbb{E}\exp\Big(\lambda\Big\|\sum_{s=0}^{T-1}\widetilde{\varepsilon}_{T-s}(\widetilde{\varepsilon}'_{T-1}(\widetilde{\beta}^{T-2})'+\cdots+\widetilde{\varepsilon}'_{T-s}(\widetilde{\beta}^{T-s-1})')(\widetilde{\beta}^{s})'\Big\|\Big)\\
&\le e^{-\lambda\varepsilon}\mathbb{E}\exp\lambda\sum_{s=0}^{T-1}\|\widetilde{\varepsilon}_{T-s}\widetilde{\varepsilon}'_{T-1}+\cdots+\widetilde{\varepsilon}_{T-s}\widetilde{\varepsilon}'_{T-s}\|\,\|\beta\|^{T-1}
\end{aligned}$$

because

$$\|\widetilde{\beta}\|<1,\quad \|\widetilde{\beta}\|^{T-s-1}>\|\widetilde{\beta}\|^{T-2},\ \ldots,\ \|\widetilde{\beta}\|^{T-s-1}>\|\widetilde{\beta}\|^{T-s}.$$

Using the Taylor expansion near $a=0$, $f(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+o((x-a)^2)$, we obtain

$$\begin{aligned}
(17)\quad &\mathbb{E}(\mathbb{1}_{\{\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})'-G_T\|-\varepsilon>0\}})\\
&\qquad\le e^{-\lambda\varepsilon}\Big[1+\lambda\sum_{s=0}^{T-1}\mathbb{E}I_0+\frac{\lambda^2}{2}\mathbb{E}\Big(\sum_{s=0}^{T-1}I_0\Big)^2+o\Big(\frac{\lambda^2}{2}\Big)\Big]
\end{aligned}$$

where

$$I_0=\|\widetilde{\varepsilon}_{T-s}\widetilde{\varepsilon}'_{T-1}+\cdots+\widetilde{\varepsilon}_{T-s}\widetilde{\varepsilon}'_{T-s}\|\,\|\widetilde{\beta}\|^{T-1}.$$

Now we have

$$\begin{aligned}
&\Big(\Big\|\sum_{s=0}^{T-1}(\widetilde{\varepsilon}_{T-s}\widetilde{\varepsilon}'_{T-1}+\cdots+\widetilde{\varepsilon}_{T-s}\widetilde{\varepsilon}'_{T-s})\Big\|\Big)^2\\
&\qquad\le\Big(\sum_{s=0}^{T-1}(\|\widetilde{\varepsilon}_{T-s}\|\,\|\widetilde{\varepsilon}'_{T-1}\|+\cdots+\|\widetilde{\varepsilon}_{T-s}\|\,\|\widetilde{\varepsilon}'_{T-s}\|)\Big)^2.
\end{aligned}$$

From Theorem 2.1 it follows that

$$\begin{aligned}
(18)\quad &\Big(\sum_{s=1}^{T-1}\|\widetilde{\varepsilon}_{T-s}\|\,\|\widetilde{\varepsilon}'_{T-1}\|+\cdots+\|\widetilde{\varepsilon}_{T-s}\|\,\|\widetilde{\varepsilon}'_{T-s}\|\Big)^2\\
&=\Big(\sum_{s=1}^{T-1}\sum_{r=1}^{s}\|\widetilde{\varepsilon}_{T-s}\|\,\|\widetilde{\varepsilon}'_{T-r}\|\Big)^2\\
&\le T\Big[\Big(\sum_{r=1}^{1}\|\widetilde{\varepsilon}_{T-1}\|\,\|\widetilde{\varepsilon}'_{T-r}\|\Big)^2+\cdots+\Big(\sum_{r=1}^{T-1}\|\widetilde{\varepsilon}_{1}\|\,\|\widetilde{\varepsilon}'_{T-r}\|\Big)^2\Big]\\
&\le T\Big[\sum_{r=1}^{1}\|\widetilde{\varepsilon}_{T-1}\|^2\|\widetilde{\varepsilon}'_{T-r}\|^2+\cdots+\sum_{r=1}^{T-1}(T-1)^2\|\widetilde{\varepsilon}_{1}\|^2\|\widetilde{\varepsilon}'_{T-r}\|^2\Big].
\end{aligned}$$

From (17) and (18) we obtain

$$\begin{aligned}
&\mathbb{P}\{\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T\| - \varepsilon > 0\} \\
&\quad \le e^{-\lambda\varepsilon}\bigg[1 + \lambda\|\widetilde{\beta}\|^{T-1}T\sigma^2 + \frac{\lambda^2}{2}\|\widetilde{\beta}\|^{2(T-1)}T\mathbb{E}\Big(\sum_{r=1}^{1}\|\widetilde{\varepsilon}_{T-1}\|^2\|\widetilde{\varepsilon}'_{T-r}\|^2 \\
&\qquad + \cdots + \sum_{r=1}^{T-1}(T-1)^2\|\widetilde{\varepsilon}_1\|^2\|\widetilde{\varepsilon}'_{T-r}\|^2\Big) + o\Big(\frac{\lambda^2}{2}\Big)\bigg] \\
&\quad = e^{-\lambda\varepsilon}\bigg[1 + \lambda\|\widetilde{\beta}\|^{T-1}T\sigma^2 + \frac{\lambda^2}{2}\,\|\widetilde{\beta}\|^{2(T-1)}T^4\sigma^4 + o\Big(\frac{\lambda^2}{2}\Big)\bigg] \\
&\quad = e^{-\lambda\varepsilon}\bigg[1 + \lambda\|\widetilde{\beta}\|^{T-1}T\sigma^2 + \frac{\lambda^2}{2}\,(\|\widetilde{\beta}\|^{2(T-1)}T^4\sigma^4 + o(1))\bigg] \\
&\quad = e^{-\lambda\varepsilon}[1 + K(T,\lambda)] \le e^{-\lambda\varepsilon}\exp(K(T,\lambda)).
\end{aligned}$$

Note that $K(T,\lambda)$ is an even function of T and λ. Hence we have

$$\begin{aligned}
(19)\quad &\mathbb{P}\{\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T\| - \varepsilon > 0\} \\
&\quad \le \exp\Big(-\lambda(\varepsilon - \|\widetilde{\beta}\|^{T-1}T\sigma^2) + \frac{\lambda^2}{2}\,(\|\widetilde{\beta}\|^{2(T-1)}T^3(T\sigma^4) + o(1))\Big) \\
&\quad =: \exp(\Phi(\lambda,T)).
\end{aligned}$$

The equation $\partial\Phi(\lambda,T)/\partial\lambda = 0$ has the unique solution

$$\lambda = \frac{\varepsilon - \|\widetilde{\beta}\|^{T-1}T\sigma^2}{\|\widetilde{\beta}\|^{2(T-1)}T^4\sigma^4 + o(1)} \tag{20}$$

which minimizes $\varPhi(\lambda,T)$. Then from (19) and (20) it follows that

$$\mathbb{P}\{\|\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T\| > \varepsilon\} \le \exp\Big(-\frac{1}{2}\,\frac{(\varepsilon - \|\widetilde{\beta}\|^{T-1}T\sigma^2)^2}{\|\widetilde{\beta}\|^{2(T-1)}T^4\sigma^4 + o(1)}\Big).$$

The proof of Theorem 3.1 is complete.

Theorem 3.2. *Suppose that the hypotheses* (H_0)–(H_1) *are satisfied and* $\varepsilon > 0$. *Then*

$$\begin{aligned}
&\mathbb{P}\{\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| > \varepsilon\} \\
&\quad \le \exp\left[-\frac{1}{4}\frac{\Big(\varepsilon - 2\Big[\Big(\frac{\sigma}{1-\|\widetilde{\beta}\|^2} + C_1\Big)^{1/2}\sigma T^{3/2}\,\frac{\|\widetilde{\beta}\|^{T-1}}{1-\|\widetilde{\beta}\|}\Big]\Big)^2}{\psi(T,\|\widetilde{\beta}\|)}\right]
\end{aligned}$$

where

$$\psi(T,\|\widetilde{\beta}\|) = T^{9/2} C_2^{1/2} \frac{\|\widetilde{\beta}\|^{2(T-1)}}{1-\|\widetilde{\beta}\|^2}\left(C_2 \frac{1}{1-\|\widetilde{\beta}\|^4} + K\right)^{1/2} + o(1)$$

with

$$K = \|\widetilde{\theta}\|^4 \mathbb{E}\|\widetilde{X}_0\|^4, \qquad C_2 = \mathbb{E}\|\widetilde{\varepsilon}_t\|^4, \qquad C_1 = \|\widetilde{\theta}\|^2 \mathbb{E}\|\widetilde{X}_0\|^2,$$

i.e. $\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T \to 0$ *in probability as* $T \to \infty$.

Proof. Using (7), (8) and (10), we obtain

$$\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| = \Big\|\sum_{s=0}^{T-1} \widetilde{\beta}^s(\widetilde{Z}_{T-s}\widetilde{Z}'_{T-s} - \widetilde{Z}_T\widetilde{Z}'_T)(\widetilde{\beta}^s)'\Big\| \le \sum_{s=0}^{T-1} \|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_{T-s}\widetilde{Z}'_{T-s} - \widetilde{Z}_T\widetilde{Z}'_T\| \tag{21}$$

with

$$\widetilde{Z}_{T-s}\widetilde{Z}'_{T-s} - \widetilde{Z}_T\widetilde{Z}'_T = \widetilde{Z}_{T-s}(\widetilde{Z}'_{T-s} - \widetilde{Z}'_T) + (\widetilde{Z}_{T-s} - \widetilde{Z}_T)\widetilde{Z}'_T.$$

Hence

$$\|\widetilde{Z}_{T-s}\widetilde{Z}'_{T-s} - \widetilde{Z}_T\widetilde{Z}'_T\| \le \|\widetilde{Z}_{T-s}\|\,\|\widetilde{Z}'_{T-s} - \widetilde{Z}'_T\| + \|\widetilde{Z}_{T-s} - \widetilde{Z}_T\|\,\|\widetilde{Z}'_T\| \le 2\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s} - \widetilde{Z}_T\|. \tag{22}$$

From (21) and (22) we get

$$\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| \le \sum_{s=0}^{T-1} \|\widetilde{\beta}\|^{2s} 2\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s} - \widetilde{Z}_T\|.$$

Using the elementary inequality above, for all $\lambda > 0$ we have

$$\mathbb{1}_{\{\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| > \varepsilon\}} \le \exp(\lambda(\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| - \varepsilon)) \le e^{-\lambda\varepsilon}\exp\Big(2\lambda\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}2\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s} - \widetilde{Z}_T\|\Big).$$

Then

$$\mathbb{E}(\mathbb{1}_{\{\|\widetilde{\beta}^{(T-2)}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| > \varepsilon\}}) \le e^{-\lambda\varepsilon}\mathbb{E}\exp\Big[2\lambda\Big(\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}2\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s} - \widetilde{Z}_T\|\Big)\Big].$$

Using the Taylor formula

$$e^u = 1 + u + \frac{u^2}{2} + o(u^2),$$

we obtain

$$
\begin{aligned}
&\mathbb{E}(\mathbb{1}_{\{\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})'-F_T\|>\varepsilon\}}) \\
&\qquad \le \exp(-\lambda\varepsilon)\mathbb{E}\Big(1+2\lambda\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\| \\
&\qquad\qquad + \frac{4\lambda^2}{2}\Big(\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|\Big)^2+o(\lambda^2)\Big) \\
&\qquad \le \exp(-\lambda\varepsilon)\Big(1+2\lambda\mathbb{E}\Big(\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|\Big) \\
&\qquad\qquad + 2\lambda^2\mathbb{E}\Big(\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_T\|\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|\Big)^2+o(\lambda^2)\Big).
\end{aligned}
$$

Set

$$
\begin{aligned}
I_1 &= \mathbb{E}\Big(\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_T\|\,\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|\Big),\\
I_2 &= \mathbb{E}\Big(\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}\|\widetilde{Z}_T\|\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|\Big)^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{P}\{\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})'-F_T\|>\varepsilon\} &\le \exp(-\lambda\varepsilon+2\lambda I_1+2\lambda^2 I_2+o(\lambda^2))\\
&= \exp(-\lambda\varepsilon+2\lambda I_1+\lambda^2(2I_2+o(1))).
\end{aligned}
$$

We have

$$
\|\widetilde{Z}_T\| \le \|\widetilde{\varepsilon}_1\|+\|\widetilde{\beta}\|\,\|\widetilde{\varepsilon}_2\|+\cdots+\|\widetilde{\beta}^{T-2}\|\,\|\widetilde{\varepsilon}_{T-1}\|+\|\widetilde{\theta}\|\,\|\widetilde{X}_0\|.
$$

Hence

$$
\|\widetilde{Z}_T\|^2 \le T(\|\widetilde{\varepsilon}_1\|^2+\|\widetilde{\beta}\|^2\|\widetilde{\varepsilon}_2\|^2+\cdots+\|\widetilde{\beta}\|^{2(T-2)}\|\widetilde{\varepsilon}_{T-1}\|^2+\|\widetilde{\theta}\|^2\|\widetilde{X}_0\|^2)
$$

and

$$
\begin{aligned}
\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|^2 &\le (\|\widetilde{\beta}\|^{T-2}\|\widetilde{\varepsilon}_{T-1}\|+\cdots+\|\widetilde{\beta}\|^{T-s-1}\|\widetilde{\varepsilon}_{T-s}\|)\\
&\le s(\|\widetilde{\beta}\|^{2(T-2)}\|\widetilde{\varepsilon}_{T-1}\|^2+\cdots+\|\widetilde{\beta}\|^{2(T-s-1)}\|\widetilde{\varepsilon}_{T-s}\|^2)\\
&\le T(\|\beta\|^{2(T-2)}\|\widetilde{\varepsilon}_{T-1}\|^2+\cdots+\|\widetilde{\beta}\|^{2(T-s-1)}\|\widetilde{\varepsilon}_{T-s}\|^2).
\end{aligned}
$$

We get

$$
\begin{aligned}
I_1 &\le \sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}(\mathbb{E}\|\widetilde{Z}_T\|^2)^{1/2}(\mathbb{E}\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|^2)^{1/2}\\
&= \sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}T\sigma(\sigma(1+\|\widetilde{\beta}\|^2+\cdots+\|\widetilde{\beta}\|^{2(T-2)})+\|\widetilde{\theta}\|^2\mathbb{E}\|\widetilde{X}_0\|^2)^{1/2}\\
&\quad \times(\|\widetilde{\beta}\|^{2(T-2)}+\cdots+\|\widetilde{\beta}\|^{2(T-s-1)})^{1/2}
\end{aligned}
$$

$$= \sigma T\left(\frac{1-\|\widetilde{\beta}\|^{2T}}{1-\|\widetilde{\beta}\|^2}\,\sigma + C_1\right)^{1/2}\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{2s}(\|\widetilde{\beta}\|^{2(T-2)}+\cdots+\|\widetilde{\beta}\|^{2(T-s-1)})^{1/2}$$

$$\le T\left(\frac{\sigma}{1-\|\widetilde{\beta}\|^2}+C_1\right)^{1/2}\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{T+s-1}$$

$$= \sigma T^{3/2}\left(\frac{\sigma}{1-\|\widetilde{\beta}\|^2}+C_1\right)^{1/2}\frac{\|\widetilde{\beta}\|^T}{1-\|\widetilde{\beta}\|}$$

where $C_1 = \|\widetilde{\theta}\|^2\mathbb{E}\|\widetilde{X}_0\|^2$ and

$$\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|^2 \le s(\|\widetilde{\beta}\|^{2(T-2)}\|\widetilde{\varepsilon}_{T-1}\|^2+\cdots+\|\widetilde{\beta}\|^{2(T-s-1)}\|\widetilde{\varepsilon}_{T-s}\|^2)$$
$$\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|^4 \le T^3(\|\widetilde{\beta}\|^{4(T-2)}\|\widetilde{\varepsilon}_{T-1}\|^4+\cdots+\|\widetilde{\beta}\|^{4(T-s-1)}\|\widetilde{\varepsilon}_{T-s}\|^4).$$

Hence

$$\begin{aligned}\mathbb{E}\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|^4 &\le T^3C_2(\|\widetilde{\beta}\|^{4(T-2)}+\cdots+\|\widetilde{\beta}\|^{4(T-s-1)})\\ &\le T^3C_2 s\|\widetilde{\beta}\|^{4(T-s-1)}\end{aligned}$$

where $C_2 = \mathbb{E}\|\widetilde{\varepsilon}_t\|^4$, and we have

$$(\mathbb{E}\|\widetilde{Z}_T\|^4)^{1/2} \le T^{3/2}\left(C_2\,\frac{1}{1-\|\widetilde{\beta}\|^4}+\|\widetilde{\theta}\|^4\mathbb{E}\|\widetilde{X}_0\|^4\right)^{1/2}, \tag{23}$$

$$(\mathbb{E}\|\widetilde{Z}_{T-s}-\widetilde{Z}_T\|^4)^{1/2} \le T^{3/2}C_2T^{1/2}\|\widetilde{\beta}\|^{2(T-s-1)}. \tag{24}$$

From (23) and (24) we obtain

$$I_1 \le \sigma T^{3/2}\left(\frac{\sigma}{1-\|\widetilde{\beta}\|^2}+C_1\right)^{1/2}\frac{\|\widetilde{\beta}\|^{T-1}}{1-\|\widetilde{\beta}\|},$$

$$I_2 \le T\sum_{s=0}^{T-1}\|\widetilde{\beta}\|^{4s}T^{3/2}\left(C_2\,\frac{1}{1-\|\widetilde{\beta}\|^4}+\|\widetilde{\theta}\|^4\mathbb{E}\|\widetilde{X}_0\|^4\right)^{1/2}T^{3/2}C_2T^{1/2}\|\widetilde{\beta}\|^{2(T-s-1)}$$

$$\le T^{9/2}C_2^{1/2}\left(C_2\,\frac{1}{1-\|\widetilde{\beta}\|^4}+\|\widetilde{\theta}\|^4\mathbb{E}\|\widetilde{X}_0\|^4\right)^{1/2}\frac{\|\widetilde{\beta}\|^{2(T-1)}}{1-\|\widetilde{\beta}\|^2}.$$

Finally, we obtain

$$\mathbb{P}\{\|\widetilde{\beta}^{(T-2)}\widetilde{B}_T(\widetilde{\beta}^{T-2})'-F_T\|>\varepsilon\} \tag{25}$$

$$\begin{aligned}&\le \exp\left(-\lambda\varepsilon+2\lambda\left[\sigma T^{3/2}\left(\frac{\sigma}{1-\|\widetilde{\beta}\|^2}+C_1\right)^{1/2}\frac{\|\widetilde{\beta}\|^{T-1}}{1-\|\widetilde{\beta}\|}\right]\right.\\ &\qquad\left.+\,2\lambda^2\left(T^{9/2}C_2^{1/2}\left(C_2\,\frac{1}{1-\|\widetilde{\beta}\|^4}+\|\widetilde{\theta}\|^4\mathbb{E}\|\widetilde{X}_0\|^4\right)^{1/2}\frac{\|\widetilde{\beta}\|^{2(T-1)}}{1-\|\widetilde{\beta}\|^2}+o(1)\right)\right)\\ &=: \exp(\Phi(\lambda,T)).\end{aligned}$$

The equation $\partial\Phi(\lambda,T)/\partial\lambda = 0$ has the unique solution

$$\lambda = \frac{\varepsilon - 2\Big[\sigma T^{3/2}\Big(\dfrac{\sigma}{1-\|\widetilde{\beta}\|^2} + C_1\Big)^{1/2}\dfrac{\|\widetilde{\beta}\|^{T-1}}{1-\|\widetilde{\beta}\|}\Big]}{4\Big(T^{9/2}C_2^{1/2}\Big(C_2\,\dfrac{1}{1-\|\widetilde{\beta}\|^4} + \|\widetilde{\theta}\|^4\mathbb{E}\|\widetilde{X}_0\|^4\Big)^{1/2}\dfrac{\|\widetilde{\beta}\|^{2(T-1)}}{1-\|\widetilde{\beta}\|^2} + o(1)\Big)} \tag{26}$$

which minimizes $\Phi(\lambda,T)$. Then from (25) and (26) it follows that

$$\mathbb{P}\{\|\widetilde{\beta}^{T-2}\widetilde{B}_T(\widetilde{\beta}^{T-2})' - F_T\| > \varepsilon\} \le \exp\left(-\frac{1}{4}\frac{\Big(\varepsilon - 2\Big[\Big(\dfrac{\sigma}{1-\|\widetilde{\beta}\|^2} + C_1\Big)^{1/2}\sigma T^{3/2}\dfrac{\|\widetilde{\beta}\|^{T-1}}{1-\|\widetilde{\beta}\|}\Big]\Big)^2}{\psi(T,\|\widetilde{\beta}\|)}\right).$$

This completes the proof of the theorem.

Corollary 3.1 (see [1]).

$$F_T - \widetilde{C}^{-1}\widetilde{\mathbb{Z}}_T\Gamma\widetilde{\mathbb{Z}}_T\widetilde{C}^{-1} \to 0 \quad \textit{in probability as } T\to\infty.$$

Here:

- F_T is defined in (9).
- The matrix $\widetilde{\theta}$ decomposes as

$$\widetilde{\theta} = \widetilde{C}^{-1}\widetilde{\eta}\widetilde{C}, \tag{27}$$

  where $\widetilde{C} = (c_{ij})_{p\times p}$ is invertible and $\widetilde{\eta}$ is a diagonal matrix.
- $\widetilde{\mathbb{Z}}_T$ is a diagonal matrix with ith diagonal element equal to the ith element of $\widetilde{C}\widetilde{Z}_T$.
- $\widetilde{\gamma} = \widetilde{\eta}^{-1} = (1/\eta_i)_{p\times p} = (\gamma_i)_{p\times p}$ ($\eta_i \neq 0$ and γ_i is the ith diagonal element of $\widetilde{\gamma}$).
- $\Gamma = \left(\dfrac{1}{1-\gamma_i\gamma_j}\right)_{p\times p}$.

We have

$$\left(\frac{1-(\gamma_i\gamma_j)^T}{1-\gamma_i\gamma_j}\right)_{p\times p} \to \Gamma \quad \text{as } T\to\infty. \tag{28}$$

From (9), (27) and ($\widetilde{\gamma} = \widetilde{\eta}^{-1}$) we can write

$$\begin{aligned} F_T &= \widetilde{Z}_T\widetilde{Z}_T' + \widetilde{C}^{-1}\widetilde{\gamma}\widetilde{C}\widetilde{Z}_T\widetilde{Z}_T'\widetilde{C}'\widetilde{\gamma}(\widetilde{C}^{-1})' + \cdots + \widetilde{C}^{-1}\widetilde{\gamma}^{T-1}\widetilde{C}\widetilde{Z}_T\widetilde{Z}_T'\widetilde{C}'\widetilde{\gamma}^{T-1}(\widetilde{C}^{-1})' \\ &= \widetilde{C}^{-1}(\widetilde{C}\widetilde{Z}_T\widetilde{Z}_T'\widetilde{C}' + \cdots + \widetilde{\gamma}^{T-1}\widetilde{C}\widetilde{Z}_T\widetilde{Z}_T'\widetilde{C}'\widetilde{\gamma}^{T-1})(\widetilde{C}^{-1})'. \end{aligned}$$

The (i,j)th element of $\widetilde{C}F_T\widetilde{C}'$ is asymptotically the (i,j)th element of $\widetilde{C}\widetilde{Z}\widetilde{Z}'\widetilde{C}'$ divided by $1-\gamma_i\gamma_j$.

COROLLARY 3.2 (see [1]).

$$G_T - Y_T\widetilde{\mathbb{Z}}_T(\widetilde{C}^{-1})' \to 0 \quad \textit{in probability as } T \to \infty.$$

where

$$G_T = \widetilde{\varepsilon}_T\widetilde{Z}'_T + \widetilde{\varepsilon}_{T-1}\widetilde{Z}'_T\widetilde{C}'\widetilde{\gamma}(\widetilde{C}^{-1})' + \cdots + \widetilde{\varepsilon}_1\widetilde{Z}'_T\widetilde{C}'\widetilde{\gamma}^{T-1}(\widetilde{C}^{-1})'$$
$$= (\widetilde{\varepsilon}_T\widetilde{Z}'_T\widetilde{C}' + \widetilde{\varepsilon}_{T-1}\widetilde{Z}'_T\widetilde{C}'\widetilde{\gamma} + \cdots + \widetilde{\varepsilon}_1\widetilde{Z}'_T\widetilde{C}'\widetilde{\gamma}^{T-1})(\widetilde{C}^{-1})'.$$

The jth column of the matrix in parentheses is the jth element of $\widetilde{Z}'\widetilde{C}'$ times

(29) $$\widetilde{\varepsilon}_T + \gamma_j\widetilde{\varepsilon}_{T-1} + \cdots + \gamma_j^{T-1}\widetilde{\varepsilon}_1.$$

Let (29) be the jth element of a matrix Y_T.

COROLLARY 3.3 (see [1]).

$$\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' - G_T \to 0 \quad \textit{in probability as } T \to \infty$$

and

$$G_T \to Y\widetilde{\mathbb{Z}}(\widetilde{C}^{-1})' \quad \textit{in distribution as } T \to \infty$$

imply

$$\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' \to Y\widetilde{\mathbb{Z}}(\widetilde{C}^{-1})' \quad \textit{in distribution as } T \to \infty.$$

Proof. This is a consequence of Definition 2.1 and Corollary 3.2.

COROLLARY 3.4.

$$\widetilde{\beta}^{T-2}\widetilde{\mathcal{B}}_T(\widetilde{\beta}^{T-2})' - F_T \to 0 \quad \textit{in probability as } T \to \infty$$

and

$$F_T \to \widetilde{C}^{-1}\widetilde{\mathbb{Z}}\mathit{\Gamma}\widetilde{\mathbb{Z}}(\widetilde{C}^{-1})' \quad \textit{in distribution as } T \to \infty$$

imply

$$\widetilde{\beta}^{(T-2)}\widetilde{\mathcal{B}}_T(\widetilde{\beta}^{T-2})' \to \widetilde{C}^{-1}\widetilde{\mathbb{Z}}\mathit{\Gamma}\widetilde{\mathbb{Z}}(\widetilde{C}^{-1})' \quad \textit{in distribution as } T \to \infty.$$

Proof. This is a consequence of Definition 2.1 and Corollary 3.1.

COROLLARY 3.5.

$$\widetilde{\beta}^{T-2}\widetilde{\mathcal{B}}_T(\widetilde{\beta}^{T-2})' \to \widetilde{C}^{-1}\widetilde{\mathbb{Z}}\mathit{\Gamma}\widetilde{\mathbb{Z}}(\widetilde{C}^{-1})' \quad \textit{in distribution as } T \to \infty$$

and

$$\widetilde{\mathcal{A}}_T(\widetilde{\beta}^{T-2})' \to Y\widetilde{\mathbb{Z}}(\widetilde{C}^{-1})' \quad \textit{in distribution as } T \to \infty$$

then

$$(\widetilde{\theta}_T - \widetilde{\theta})\widetilde{\theta}^{T-2} \to Y\gamma^{-1}\widetilde{\mathbb{Z}}^{-1}C \quad \textit{in distribution as } T \to \infty.$$

Proof. Use Corollaries 3.3 and 3.4 and $(\widetilde{\theta}_T - \widetilde{\theta})\widetilde{\theta}^{T-2} = \widetilde{\mathcal{A}}_T\widetilde{\mathcal{B}}_T^{-1}\widetilde{\theta}^{T-2}$.

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Laboratoire de Mathématiques
Université Djillali Liabès
B.P. 89, Sidi Bel Abbès 22000, Algeria
E-mail: benaissamir@yahoo.fr

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