ALEKSANDRA ORPEŁ (Łódź)

SEMILINEAR ELLIPTIC PROBLEMS
IN UNBOUNDED DOMAINS

Abstract. We investigate the existence of positive solutions and their continuous dependence on functional parameters for a semilinear Dirichlet problem. We discuss the case when the domain is unbounded and the nonlinearity is smooth and convex on a certain interval only.

1. Introduction. In this paper we are dealing with the following boundary value problem for second order PDE of elliptic type:

$$
\begin{cases}
-\Delta x(y) = F_x(y, x(y)) & \text{for a.e. } y \in \Omega, \\
x \in W^{1,2}_0(\Omega, \mathbb{R}),
\end{cases}
$$

for $\Omega$ being an unbounded domain in $\mathbb{R}^n$ with boundary $\partial \Omega$ and $F_x$ denoting the derivative of $F$ with respect to $x$. We are looking for a nonnegative and nontrivial weak solution $x \in W^{1,2}_0(\Omega, \mathbb{R})$ of this problem such that $\Delta x(\cdot)$ belongs to $L^2(\Omega, \mathbb{R})$.

There are numerous papers concerning similar equations for a bounded domain $\Omega$ (see, among others, [1]–[5]). In the vast existing literature we can also find results on radial solutions for our problem in an exterior domain (see [9], [10], [17]–[19]). More precisely, [17] was devoted to both radial and nonradial cases for an exterior domain with sublinear nonlinearities. In the first part of [17], the authors presented the results for the radial case. Then they obtained sub- and supersolutions of (1.1) as radial solutions of a problem associated to (1.1). Finally, they derived the existence of positive nonradial solutions for (1.1) using the sub- and supersolution methods based on the theory due to Noussair ([11]) for $\Omega$ being the exterior of a ball.

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Here we do not impose any symmetry condition on $\Omega$, and we cover both sub- and superlinear cases. Similar boundary value problems on unbounded domains have been discussed e.g. in [11]–[14]. In [12]–[14] (for systems of equations) the authors investigated a semilinear elliptic problem of the form

$$(1.2) \quad \left\{ \begin{array}{ll} Lu = \lambda f(y, u) & \text{for } y \in \Omega, \\ u(y) = 0 & \text{for } y \in \partial \Omega, \end{array} \right.$$ 

where $L$ is a uniformly elliptic operator in $\Omega$, $n > 2$, $\lambda > 0$ and $\Omega$ is a smooth unbounded domain in $\mathbb{R}^n$. They obtained the existence and nonexistence results for (1.2) provided that, among other things, $f$ is locally Lipschitz continuous on $(\Omega \cup \partial \Omega) \times [0, \infty)$ and $f(x, t) < 0$ for all $t \in \Omega$ and sufficiently large $t$. Here we consider the case when the nonlinearity is increasing and smooth with respect to the second variable on a certain interval $\tilde{I}$ only. So there is no information concerning its behavior and smoothness outside $\tilde{I}$.

2. The existence results. We propose an approach based on the following assumptions:

(Ω) $\Omega$ is an unbounded domain in $\mathbb{R}^n$ with a locally Lipschitz boundary $\partial \Omega$.

(G1) There exist $M, M_0 \in W^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ such that $0 < M_0(y) < M(y)$ for a.e. $y \in \Omega$, $M_0|_{\partial \Omega}, M|_{\partial \Omega} \geq 0$, $\Delta M_0(\cdot) \in L^2(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ and for each bounded set $\Omega' \subset \Omega$,

$$(2.1) \quad -F_x(y, M(y)) \geq \Delta M_0(y) \quad \text{a.e. in } \Omega'.$$

(G2) $F(y, \cdot) \in C^1(\tilde{I})$ and is convex in $\tilde{I}$ for a.e. $y \in \Omega$, $F(\cdot, x)$ is measurable in $\tilde{I}$ for all $x \in \tilde{I}$, where $\tilde{I}$ is a certain neighborhood of $I := [0, a]$, with $a := \text{ess sup}_{y \in \Omega} M(y)$.

(G3) $F_x(y, \cdot)$ is nonnegative in $I$ for a.e. $y \in \Omega$, $F_x(\cdot, a) \in L^2(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$;

(G4) $\int_{\Omega} F_x(y, 0) \, dy \neq 0, \quad \left| \int_{\Omega} F(y, 0) \, dy \right| < \infty$.

Let us define

$$X := \{ x \in W^{1,2}_0(\Omega, \mathbb{R}) : 0 \leq x(y) \leq M(y) \text{ a.e. on } \Omega \}$$

and $\Delta x(\cdot) \in L^2(\Omega, \mathbb{R})$.

We will prove the existence of solutions to (1.1) in $X$ and their properties in two steps. First we shall construct a sequence of solutions of the corresponding problems in bounded domains. Then a solution of (1.1) will be obtained as the limit of this sequence (precisely, of a subsequence). Let us consider
the sequence of bounded sets
\[ \Omega_m := \{ y = (y_1, \ldots, y_n) \in \Omega : |y_i| < m \text{ for each } i = 1, \ldots, n \}, \quad m \in \mathbb{N}. \]
There exists an \( m_0 \in \mathbb{N} \) such that \( \Omega_m \neq \emptyset \) for all \( m \in N_0 := \{ m \in \mathbb{N} : m \geq m_0 \} \). For each \( m \in N_0 \), we will use the Schauder fixed point theorem to prove the existence of a solution \( x_m \in X_m \) of the problem
\[
\begin{cases}
-\Delta x(y) = F_x(y, x(y)) & \text{for a.e. } y \in \Omega_m, \\
x \in W^{1,2}_0(\Omega_m, \mathbb{R}),
\end{cases}
\]
with
\[ X_m = \{ x \in W^{1,2}_0(\Omega_m, \mathbb{R}) : 0 \leq x(y) \leq M(y) \text{ a.e. on } \Omega_m \}
\quad \text{and } \Delta x(y) \in L^2(\Omega_m, \mathbb{R}). \]
Thus, we fix \( m \in N_0 \) and consider a map \( T_m \) defined in \( X_m \) as follows:
\[ T_m x(y) = \int_{\Omega_m} G_m(y, z) \tilde{F}_x(z, x(z)) \, dz \quad \text{for } x \in X_m, \]
where \( G_m \) is the Green’s function corresponding to the linear homogeneous problem associated with (2.2), and
\[ \tilde{F}_x(z, x) := \begin{cases} F_x(z, 0) & \text{for } x < 0 \text{ and } z \in \Omega_m, \\
F_x(z, x) & \text{for } 0 \leq x \leq a \text{ and } z \in \Omega_m, \\
F_x(z, a) & \text{for } x > a \text{ and } z \in \Omega_m, \end{cases} \]
where \( a \) was given in (G2). By the above assumptions \( T_m \) is well defined on \( L^2(\Omega_m, \mathbb{R}) \) and is continuous and compact.

It is clear that our problem is equivalent to the existence of a fixed point of \( T_m \) in \( X_m \). So we have to show that \( T_m \) maps \( X_m \) into \( X_m \). To this end we prove the following lemma:

**Lemma 2.1.** For each \( m \in N_0 \) and each \( x_0 \in X_m \) there exists \( \bar{x} \in X_m \) such that
\[
\begin{cases}
-\Delta \bar{x}(y) = F_x(y, x_0(y)) & \text{for a.e. } y \in \Omega_m, \\
x \in W^{1,2}_0(\Omega_m, \mathbb{R}),
\end{cases}
\]

**Proof.** Since \( M_0|_{\Omega_m} \in X_m \) we get \( X_m \neq \emptyset \). Let us fix \( x_0 \in X_m \) and investigate the existence of solution for the linear problem
\[
\begin{cases}
-\Delta x(y) = F_x(y, x_0(y)) & \text{for a.e. } y \in \Omega_m, \\
x \in W^{1,2}_0(\Omega_m, \mathbb{R}).
\end{cases}
\]
From assumptions (G1)–(G3) we can derive that
\[
0 \leq F_x(y, x_0(y)) \leq F_x(y, M(y)) \leq -\Delta M_0(y)
\]
a.e. in \( \Omega_m \) and \( F_x(\cdot, x_0(\cdot)) \in L^2(\Omega_m, \mathbb{R}) \). It is well known that problem (2.3) has a unique solution \( \bar{x} \in W^{1,2}_0(\Omega_m, \mathbb{R}) \cap W^{2,2}_{\text{loc}}(\Omega_m, \mathbb{R}) \) (see e.g. [5, Th. 8.9]).
Our task is now to show that $\bar{x} \in X_m$. To this end we can observe that, by (G3), $\Delta \bar{x} \leq 0$ a.e. in $\Omega_m$. Applying the weak maximum principle (see e.g. [5, Th. 8.1]) we infer that $\bar{x} \geq 0$ a.e. in $\Omega_m$. On the other hand, taking into account (2.4), we obtain
\[ -\Delta \bar{x}(y) = F_x(y, x_0(y)) \leq -\Delta M_0(y) \]
a.e. in $\Omega_m$, so that
\[ \Delta (\bar{x}(y) - M_0(y)) \geq 0. \]
Moreover we know that $\bar{x} - M_0 \leq 0$ in $\partial \Omega_m$. Finally, using again the weak maximum principle, we find that $\bar{x} \leq M_0$ a.e. in $\Omega_m$ and further $0 \leq \bar{x} \leq M$ a.e. in $\Omega_m$. Thus $\bar{x} \in X_m$. 

By the above lemma, for each $m \in N_0$, the continuous and compact operator $T_m$ maps the convex set $X_m \subset L^2(\Omega_m, \mathbb{R})$ into itself. Now Schauder’s fixed point theorem gives the existence of a fixed point $x_m \in X_m$ of $T_m$. Thus we have proved the following result.

**Theorem 2.2.** If hypotheses (\(\Omega\)) and (G1)-(G4) are satisfied then for each $m \in N_0$, there exists a solution $x_m \in X_m$ for (2.2).

Now we define the sequence $\{\bar{x}_m\}_{m \in N_0}$ as follows: for each $m \in N_0$,
\[ \bar{x}_m(y) = \begin{cases} 
 x_m(y) & \text{for } y \in \Omega_m, \\
 0 & \text{for } y \in \Omega \setminus \Omega_m,
\end{cases} \]
where $x_m$ is a solution for (2.2). Its existence follows from Theorem 2.2. Our task is to prove that the weak limit of a certain subsequence of $\{\bar{x}_m\}_{m \in N_0}$ is a solution for (1.1). A similar approach was also used e.g. by Noussair, and Noussair and Swanson (see [11]-[13]). However, we shall consider a quite different class of elliptic problems.

Now we formulate our main result:

**Theorem 2.3.** Assume hypotheses (\(\Omega\)) and (G1)-(G4). Then there exists a solution $x_0 \in X$ of the problem
\[
\begin{cases} 
 -\Delta x(y) = F_x(y, x(y)) & \text{for a.e. } y \in \Omega, \\
 x \in W^{1,2}_0(\Omega, \mathbb{R}).
\end{cases}
\]  

**Proof.** For each $m \in N_0$, Theorem 2.2 yields the existence of $x_m \in X_m$ such that
\[
\begin{cases} 
 -\Delta x_m(y) = F_x(y, x_m(y)) & \text{for a.e. } y \in \Omega_m, \\
 x_m \in W^{1,2}_0(\Omega_m, \mathbb{R}).
\end{cases}
\]  
By the definitions of $X_m$ and $\bar{x}_m$ we have
\[ 0 \leq \bar{x}_m(y) \leq M(y) \quad \text{a.e. in } \Omega. \]
Moreover using (2.6), the monotonicity of $\tilde{T} \ni x \mapsto F_x(y, x)$ and the fact that $F_x(\cdot, M(\cdot)) \in L^2(\Omega, \mathbb{R})$, we can derive that for each $m \in N_0$,

\begin{equation}
\begin{aligned}
\int_\Omega |\nabla \varphi_m(y)|^2 \, dy &= \int_\Omega (\nabla \varphi_m(y), \nabla \varphi_m(y)) \, dy \\
&= \int_\Omega F_x(y, \varphi_m(y)) \varphi_m(y) \, dy \leq \left[ \int_\Omega (F_x(y, M(y))^2 \, dy \right]^{1/2} \left[ \int_\Omega (M(y))^2 \, dy \right]^{1/2}.
\end{aligned}
\end{equation}

Taking into account (2.8) we derive that the sequence $\{\nabla \varphi_m\}_{m \in N_0}$ is bounded in $L^2(\Omega, \mathbb{R}^n)$, so (up to a subsequence) $\{\nabla \varphi_m\}_{m \in N_0}$ tends weakly in $L^2(\Omega, \mathbb{R}^n)$ to a certain $v \in L^2(\Omega, \mathbb{R}^n)$. Thus we obtain the existence of $\varphi_1 \in W^{1,2}(\Omega, \mathbb{R})$ such that $v = \nabla \varphi_1$ in $L^2(\Omega, \mathbb{R}^n)$ and further (up to a subsequence again) $\{\varphi_m(y)\}_{m \in N_0}$ tends to $\varphi_1(y)$ a.e. in $\Omega$, so $\varphi_1(y) \leq M(y)$ a.e. in $\Omega$.

Now we claim that

$$\Delta \varphi_m \rightharpoonup p_1 \quad \text{(weakly) in } L^2(\Omega, \mathbb{R}).$$

Indeed, from (G2) and the definition of $\varphi_m$ one obtains the estimate

$$|\Delta \varphi_m(y)| \leq F_x(y, \varphi_m(y)) \leq F_x(y, M(y)) \quad \text{a.e. on } \Omega,$$

for each $m \in N_0$. Therefore $\{\Delta \varphi_m\}_{m \in N_0}$ is bounded in $L^2(\Omega, \mathbb{R})$, and consequently, passing to a subsequence if necessary, it tends weakly to a certain element $p_1$ in $L^2(\Omega, \mathbb{R})$. So for any $h \in C^\infty_c(\Omega, \mathbb{R})$,

\begin{equation}
\begin{aligned}
\int_\Omega (\nabla \varphi_1(y), \nabla h(y)) \, dy &= \lim_{m \to \infty} \int_\Omega (\nabla \varphi_m(y), \nabla h(y)) \, dy \\
&= - \lim_{m \to \infty} \int_\Omega \Delta \varphi_m(y) h(y) \, dy = - \int_\Omega p_1(y) h(y) \, dy,
\end{aligned}
\end{equation}

which means that $\Delta \varphi_1(y) = p_1(y)$ for a.e. $y \in \Omega$. On the other hand, by (2.6), we obtain, for $h \in C^\infty_c(\mathbb{R}^n, \mathbb{R})$,

\begin{equation}
\begin{aligned}
\int_\Omega -\Delta \varphi_1(y) h(y) \, dy &= \lim_{m \to \infty} \int_\Omega -\Delta \varphi_m(y) h(y) \, dy \\
&= \lim_{m \to \infty} \int_{\Omega_m} -\Delta \varphi_m(y) h(y) \, dy = \lim_{m \to \infty} \int_{\Omega_m} F_x(y, \varphi_m(y)) h(y) \, dy \\
&= \lim_{m \to \infty} \left[ \int_\Omega F_x(y, \varphi_m(y)) h(y) \, dy - \int_{\Omega \setminus \Omega_m} F_x(y, \varphi_m(y)) h(y) \, dy \right] \\
&= \lim_{m \to \infty} \left[ \int_\Omega F_x(y, \varphi_m(y)) h(y) \, dy - \int_{\Omega \setminus \Omega_m} F_x(y, 0) h(y) \, dy \right].
\end{aligned}
\end{equation}
Taking into account (G2)-(G3), the Lebesgue dominated convergence theorem leads to
\begin{equation}
\lim_{m \to \infty} \int_{\Omega} F(x,y, \varphi_m(y))h(y) \, dy = \int_{\Omega} F(x,y, \varphi_1(y))h(y) \, dy.
\end{equation}
Moreover, by the continuity of the integral as a function of a set, and the fact that \( \bigcup_{n=0}^{\infty} \Omega_n = \Omega \) and \( \Omega_n \subset \Omega_{n+1} \subset \Omega \) for all \( m \in N_0 \), we have
\begin{equation}
\lim_{m \to \infty} \int_{\Omega \setminus \Omega_n} F(x,y, 0)h(y) \, dy = 0.
\end{equation}
Combining (2.9) with (2.10) and (2.11) we obtain
\begin{equation}
\int_{\Omega} -\Delta \varphi_1(y)h(y) \, dy = \int_{\Omega} F(x,y, \varphi_1(y))h(y) \, dy.
\end{equation}
Since \( h \in C_c^\infty(\mathbb{R}^n, \mathbb{R}) \) was arbitrary we infer that \( \varphi_1 \in X \) satisfies (2.5).

3. Applications

Example 1. Let us consider (1.1) with \( \Omega = \{ y = (y_1, y_2) \in \mathbb{R}^2 : 1/10 < y_1 < 1/2 \ \text{and} \ y_2 < 6 \} \), and
\[ F(y, x) = \frac{25}{11} \ln |x + 5| - \frac{36}{11} \ln |6 - x| - x + \left( \frac{1}{4} x^4 + x \right) \frac{1}{y^4} \]
for \( y \in \Omega \) and all \( x \in \mathbb{R} \setminus \{-5, 6\} \). Then the problem
\begin{equation}
\begin{cases}
-\Delta x(y) = \frac{(x(y))^2}{(6-x(y))(x(y)+5)} + \frac{(x(y))^3+1}{(y_2)^4}, & \text{for a.e. } y \in \Omega, \\
x \in W_0^{1,2}(\Omega, \mathbb{R}),
\end{cases}
\end{equation}
has at least one positive solution \( x_0 \) such that \( x_0(y) \leq M \) a.e. on \( \Omega \).

Proof. Our task is to find \( 0 < M_0 \leq M \) a.e. on \( \Omega \) such that (2.1) holds. Let us consider
\[ M_0(y_1, y_2) = \frac{1}{2} \left[ \frac{y_1}{(y_1)^4 + 1/20} + \frac{1}{(y_2)^4} \right] \]
and \( M(y_1, y_2) = 1.1 M_0(y_1, y_2) \). It is easy to check that \( M_0 \in W^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \), \( \Delta M_0(\cdot) \in L^2(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \) and
\[ -F_x(y, M(y)) \geq \Delta M_0(y) \quad \text{a.e. in } \Omega, \]
where
\[ F_x(y, x) = \frac{x^2}{(6-x)(x+5)} + \frac{x^3+1}{(y_2)^4}. \]
Since \( 0 \leq M(y_1, y_2) \leq 3.5 \) on \( \Omega \) and \( F(y, \cdot) \) is smooth and convex, e.g. in \((-1, 4)\), assumptions (G2)-(G4) are satisfied. Thus, by Theorem 2.3 there exists a nonnegative, nontrivial and bounded solution of (3.1).
Of course our results can also be applied to sublinear problems.

**Example 2.** The sublinear elliptic BVP

\[
(3.2) \quad \begin{cases}
-\Delta x(y) = \frac{(x(y))^2}{(4-x(y))(5+x(y))} + \sqrt{x(y)} + \frac{y_1}{(y_2)^6} & \text{a.e. in } \Omega, \\
x \in W^{1,2}_0(\Omega, \mathbb{R}),
\end{cases}
\]

with \( \Omega \) given as in Example 1, has at least one positive solution.

**Proof.** One can easily check that for \( M_0, M \) from Example 1, assumption \((G1)\) is satisfied. Moreover

\[
F(y, x) = -x - \frac{16}{9} \ln |4-x| + \frac{25}{9} \ln |x+5| + \frac{2}{3} (x + 1)^{3/2} \frac{y_1}{(y_2)^6}
\]

is continuously differentiable and convex in \( x \), e.g. in \( \tilde{I} = (-\frac{1}{2}, 3\frac{1}{2}) \). Finally, \((G2)-(G4)\) hold. Thus Theorem 2.3 gives the existence of a nonnegative, nontrivial and bounded solution of (3.2). \(
\)

**4. Continuous dependence on parameters.** Continuous dependence of solutions for elliptic problems has been widely discussed by S. Walczak since the 1990’s (see e.g. [6]-[8], [20]-[22]). It was also studied in [15] (for bounded \( \Omega \)) and in [16] (for an exterior domain).

This section is devoted to the following PDE:

\[
(4.1) \quad \begin{cases}
-\Delta x(y) = F_x(y, x(y)) + u(y) & \text{for a.e. } y \in \Omega, \\
x \in W^{1,2}_0(\Omega, \mathbb{R}),
\end{cases}
\]

with functional parameters \( u \) from a certain subset \( U \) of \( L^2(\Omega, \mathbb{R}_+) \). We introduce the following assumption:

\[(G1u)\] there exists \( M_0 \in W^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \) such that for each \( u \in U \) there exist \( M_u, M_{0u} \in W^{1,2}(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \) such that

\[
0 < M_{0u}(y) < M_u(y) \leq M_0(y)
\]

for a.e. \( y \in \Omega \), and \( \Delta M_{0u}(\cdot) \in L^2(\Omega, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R}) \) and for each bounded set \( \Omega' \subset \Omega \),

\[
(4.2) \quad -F_x(y, M_u(y)) \geq \Delta M_{0u}(y)
\]

a.e. in \( \Omega' \), \( M_u|_{\partial \Omega}, M_{0u}|_{\partial \Omega} \geq 0 \).

We shall consider the case when \((\Omega), (G2)-(G4)\) hold for \( M = M_0 \) a.e. in \( \Omega \).

**Theorem 4.1.** Assume hypotheses \((\Omega), (G1u)\) and \((G2)-(G4)\). Suppose that \( \{u_m\}_{m \in \mathbb{N}} \subset U \) tends weakly to 0 in \( L^2(\Omega, \mathbb{R}_+) \). For each \( m \in \mathbb{N} \), denote by \( x_m \in X_{u_m} \) a solution of (4.1) corresponding to \( u_m \), namely

\[
(4.3) \quad -\Delta x_m(y) = F_x(y, x_m(y)) + u_m(y)
\]
for a.e. \( y \in \Omega \), with
\[
X_{u_m} = \{ x \in W^{1,2}_0(\Omega, \mathbb{R}) : 0 \leq x(y) \leq M_{u_m}(y) \text{ a.e. on } \Omega \}
\]
and \( \Delta x \in L^2(\Omega, \mathbb{R}) \).

Then \( \{ x_m \}_{m \in \mathbb{N}} \) (up to a subsequence) tends weakly to \( x_0 \) in \( W^{1,2}_0(\Omega, \mathbb{R}) \), where \( x_0 \in X_0 \) is a solution of the equation
\[
(4.4) \quad -\Delta x(y) = F_x(y, x(y)) \quad \text{for a.e. } y \in \Omega.
\]

**Proof.** We start with the observation that \((G1u)\), the properties of \( F_x \) and (4.3) yield
\[
(4.5) \quad \int\limits_\Omega |\nabla x_m(y)|^2 \, dy = \int\limits_\Omega (-\Delta x_m(y) x_m(y)) \, dy \\
= \int\limits_\Omega F_x(y, x_m(y)) x_m(y) \, dy + \int\limits_\Omega u_m(y) x_m(y) \, dy \\
\leq \left[ \int\limits_\Omega (F_x(y, M_0(y)))^2 \, dy \right]^{1/2} \left[ \int\limits_\Omega (M_0(y))^2 \, dy \right]^{1/2} + \int\limits_\Omega u_m(y) M_0(y) \, dy
\]
for each \( m \in \mathbb{N} \). Combining (4.5) with the weak convergence of \( \{ u_m \}_{m \in \mathbb{N}} \) to 0 in \( L^2(\Omega, \mathbb{R}_+) \) we infer that \( \{ \nabla x_m \}_{m \in \mathbb{N}} \) is bounded in \( L^2(\Omega, \mathbb{R}) \), and consequently, it is (up to a subsequence) weakly convergent in \( L^2(\Omega, \mathbb{R}) \) to a certain \( v \in L^2(\Omega, \mathbb{R}) \). This yields the existence of \( x_0 \in W^{1,2}_0(\Omega, \mathbb{R}) \) such that \( v = \nabla x_0 \) in \( L^2(\Omega, \mathbb{R}_n) \). We can also derive that some subsequence of \( \{ x_m \}_{m \in \mathbb{N}} \) (still denoted by \( \{ x_m \}_{m \in \mathbb{N}} \)) tends to \( x_0 \) a.e. on \( \Omega \), which implies that \( x_0 \leq M_0 \) a.e. in \( \Omega \).

Our task is to show that \( x_0 \) is a solution for (4.4). To see this, we use again (4.3), monotonicity of \( F_x(y, \cdot) \) and the fact that \( u_m \to 0 \) in \( L^2(\Omega, \mathbb{R}_+) \), and obtain the boundedness of \( \{ \Delta x_m \}_{m \in \mathbb{N}} \) in \( L^2(\Omega, \mathbb{R}) \). So (up to a subsequence) \( \{ \Delta x_m \}_{m \in \mathbb{N}} \) is weakly convergent to \( p \) in \( L^2(\Omega, \mathbb{R}) \). Analysis similar to that in the proof of Theorem 2.3 shows that \( p = \Delta x_0 \) a.e. on \( \Omega \). Taking into account (4.3) and the weak convergence of \( \{ u_m(\cdot) \}_{m \in \mathbb{N}} \) to 0 in \( L^2(\Omega, \mathbb{R}_+) \), and employing the scheme used in the proof of (2.9), we get, for any \( h \in C_c^\infty(\Omega, \mathbb{R}) \),
\[
(4.6) \quad \int\limits_\Omega -\Delta x_0(y) h(y) \, dy = \lim_{m \to \infty} \int\limits_\Omega -\Delta x_m(y) h(y) \, dy \\
= \lim_{m \to \infty} \int\limits_\Omega (F_x(y, x_m(y)) + u_m(y)) h(y) \, dy = \int F_x(y, x_0(y)) h(y) \, dy.
\]
Since \( h \in C_c^\infty(\Omega, \mathbb{R}) \) was arbitrary we conclude that \( x_0 \in X \) satisfies (4.4).

Summarizing we have proved that the sequence \( \{ x_m \}_{m \in \mathbb{N}} \) of solutions corresponding to the sequence \( \{ u_m \}_{m \in \mathbb{N}} \) of parameters tends weakly in \( W^{1,2}_0(\Omega, \mathbb{R}) \) (up to a subsequence) to \( x_0 \) provided that \( u_m(\cdot) \to 0 \) in \( L^2(\Omega, \mathbb{R}_+) \) as \( m \to \infty \).
Semilinear elliptic problems in unbounded domains

References


Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: orpela@math.uni.lodz.pl

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