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# EVOLUTION IN A MIGRATING POPULATION MODEL

*Abstract.* We consider a model of migrating population occupying a compact domain $\Omega$ in the plane. We assume the Malthusian growth of the population at each point $x \in \Omega$ and that the mobility of individuals depends on $x \in \Omega$. The evolution of the probability density $u(x,t)$ that a randomly chosen individual occupies $x \in \Omega$ at time $t$ is described by the nonlocal linear equation $u_t = \int_\Omega \varphi(y)u(y,t)\,dy - \varphi(x)u(x,t)$, where $\varphi(x)$ is a given function characterizing the mobility of individuals living at $x$. We show that the asymptotic behaviour of $u(x,t)$ as $t \to \infty$ depends on the properties of $\varphi$ in the vicinity of its zeros.

**1. Introduction.** Evolution problems of the form

$$\partial_t u(x,t) = G(u,x,t) - u(x,t)Lu(x,t), \tag{1}$$

where $G$ is in general a nonlinear operator, which depends on $u$ in nonlocal way, and $L$ is a linear operator, have been considered in many papers; see e.g. [1], [2], [3], [5], [6] and the references therein.

For example, the equation

$$\partial_t u(x,t) = \int\limits_{\mathbb{R}^n} J(x-y)u(y,t)\,dy - u(x,t)$$

has been applied in [1] to describe the evolution of the density $u(x,t)$ of a population. Here the function $J(x-y)$ is interpreted as the migration probability from location $y$ to location $x$ and $-u(x,t)$ is the rate at which individuals leave $x$ to move to any other site.

The authors of [5] give many other examples of biological phenomena which can be modelled by equation (1).

In this paper we will consider a particular form of equation (1). A population of density $m(x,t)$ occupies a compact domain $\Omega \subset \mathbb{R}^2$ so that

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$\int_\omega m(x,t)\,dx$ is the mass of the population in a subdomain $\omega \subset \Omega$ at time $t$. Assume that the density of newborn individuals is proportional to $m(x,t)$ (the Malthusian law of growth of population), and moreover, that the individuals living at $x$ can move to other points in $\Omega$. Let $m_o(x,t)$ be the density of outgoing individuals, and $m_i(x,t)$ the density of incoming individuals. The mobility of individuals living at $x$ is characterized by a nonnegative, continuous function $\varphi$ on $\Omega$, $0 \le \varphi(x) \le 1$, i.e. the density has the form $m_o(x,t) = \varphi(x)m(x,t)$. Assume that $m_i(x,t)$ depends only on $t$, i.e. individuals can move from a point $x$ in $\Omega$ to another one with probability independent of the destination. This assumption and the fact that the mass of outgoing individuals equals the mass of incoming individuals lead to

$$\int_\Omega m_o(x,t)\,dx = \int_\Omega \varphi(x)m(x,t)\,dx = \int_\Omega m_i(t)\,dx = m_i(t)|\Omega|,$$

where $|\Omega|$ denotes the volume of the domain $\Omega$. Below we assume, for simplicity, that $|\Omega| = 1$. Hence $m_i(t) = \int_\Omega \varphi(y)m(y,t)\,dy$.

The change of $m(x,t)$ during the time interval $\Delta t$ depends on:

(i) the mass of newborn individuals, which according to the Malthusian law equals $am(x,t)\Delta t$, where $a > 0$ is a constant;
(ii) the growth of the mass of incoming individuals, which is $m_i(t)\Delta t$;
(iii) the growth of the mass of outgoing individuals, which is $m_o(x,t)\Delta t$.

The above assumptions lead to the following continuity equation:

$$\partial_t m(x,t) = am(x,t) + \int_\Omega \varphi(y)m(y,t)\,dy - \varphi(x)m(x,t). \tag{2}$$

Equation (2) is supplemented with the initial condition

$$m(x,0) = m_0(x), \tag{3}$$

where $m_0(x) \ge 0$ is a given initial density. Integrating (2) over $\Omega$ we get an equation for the total mass $M(t)$ of the population, $M'(t) = aM(t)$. Hence $M(t) = M_0 \exp(at)$, where $M_0 = \int_\Omega m_0(x)\,dx$.

The function $u(x,t) = m(x,t)/M(t)$ is the probability density that a randomly chosen individual at time $t$ lives at a point $x \in \Omega$. The evolution of $u(x,t)$ is described by the linear nonlocal problem

$$\partial_t u(x,t) = \int_\Omega \varphi(y)u(y,t)\,dy - \varphi(x)u(x,t) =: Au(t) - \varphi(x)u(x,t), \tag{4}$$

$$u(x,0) = u_0(x), \tag{5}$$

where $u_0(x) = m_0(x)/M_0$. Problem (4)–(5) can be considered for $\Omega \subset \mathbb{R}^n$ with $n > 2$, but this does not lead to more interesting mathematical phenomena. Indeed, (4) is a family of ordinary differential equations indexed by the parameter $x$ and the dimension of the space of parameters is not important for our considerations. The choice $n = 2$ is motivated by a biological interpretation of the problem.

By a *solution* of problem (4)–(5) we mean a continuous function of $x$ and $t$ on $\Omega \times [0, T]$, differentiable with respect to $t$, which satisfies (4) pointwise on $\Omega \times [0, T]$ and satisfies the initial condition (5).

For a fixed $x$, $u(x, t)$ as a function of $t$ satisfies a linear differential equation with the initial data $u_0(x)$, hence by the variation of parameters formula

$$u(x,t) = u_0(x)e^{-\varphi(x)t} + e^{-\varphi(x)t}\int_0^t e^{\varphi(x)s} Au(s)\,ds. \tag{6}$$

Equation (6) is the integral form of the differential problem (4)–(5), which is quite convenient to prove the existence of solution of (4)–(5) as well as to study its properties.

LEMMA 1.1. *If* $\lim_{t\to\infty} Au(t) = 0$ *and* $\varphi(x) \ge a > 0$ *on* $B \subset \Omega$, *then* $u(x, t) \to 0$ *as* $t \to \infty$ *uniformly on* $B$.

*Proof.* We rewrite (6) in the form

$$u(x,t) = u_0(x)e^{-\varphi(x)t} + e^{-\varphi(x)t}\int_0^T e^{\varphi(x)s} Au(s)\,ds + e^{-\varphi(x)t}\int_T^t e^{\varphi(x)s} Au(s)\,ds.$$

For a given $\varepsilon > 0$ we choose $T$ such that $Au(t) < \epsilon$ for $t > T$. Then for $x \in B$, we obtain

$$u(x,t) \le u_0(x)e^{-at} + e^{-at}\int_0^T e^{\varphi(x)s} Au(s)\,ds + \varepsilon\frac{1 - e^{T-t}}{a}.$$

Letting $t \to \infty$, we see that the solution $u(x, t)$ converges to 0 uniformly on $B$.

**2. Existence of solution and its properties.** Problem (4)–(5) features properties characteristic for diffusion problems. The reason is that equation (4), after some transformation of time, becomes the nonlocal diffusion equation related to the functional (see [4] for the definition of nonlocal diffusion)

$$\mathcal{F}(v) = \int_\Omega\int_\Omega (\varphi(x)v(x) - \varphi(y)v(y))^2\,dx\,dy.$$

One of the characteristic properties of diffusion problems is the existence of an entropy or, in other words, of a Lyapunov functional.

PROPERTY 1. *The square of the* $L^2$*-norm of the solution of* (4)–(5) *with respect to the measure* $\varphi(x)\,dx$, $\mathcal{L}u(t) = \int_\Omega u^2(x,t)\varphi(x)\,dx$, *is a Lyapunov functional for that problem.*

*Proof.* Multiplying (4) by $\varphi u$ we get

$$\frac{1}{2}\frac{d}{dt}\varphi(u^2) = \varphi u A u - \varphi^2 u^2. \tag{7}$$

Integrating (7) over $\Omega$, and using the Jensen inequality, we obtain

$$\begin{aligned}\frac{1}{2}(\mathcal{L}u)'(t) &= \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \varphi(x)u^2(x,t)\,dx\Big) \\ &= (Au(t))^2 - \int_\Omega \varphi^2(x)u^2(x,t)\,dx \le 0.\end{aligned} \tag{8}$$

PROPERTY 2. *The unique solution with homogeneous initial data is* $u(x,t) \equiv 0$.

*Proof.* We have $\mathcal{L}u(t) \equiv 0$, so the Lebesgue measure of the intersection of the supports of $u(\cdot,t)$ and $\varphi$ is zero. Hence (4) takes the form $\partial_t u(x,t) = -\varphi(x)u(x,t)$, which implies that $u(x,t) = u_0(x)e^{-\varphi(x)t} \equiv 0$.

The next property says that, under some assumption, the solution $u$ becomes instantaneously positive, so $u$ diffuses at infinite speed.

PROPERTY 3. *If the intersection of the supports of* $u_0$ *and* $\varphi$ *has a positive Lebesgue measure, then* $u(x,t) > 0$ *for all* $t > 0$.

*Proof.* Our asumption implies that $Au(t) > 0$ for small $t$. Now the positivity of the solution is a consequence of the integral equation (6) satisfied by any solution of (4)–(5).

If the assumption of Property 3 is not satisfied, we can write the solution of our problem in an explicit form.

PROPERTY 4. *If the supports of* $u_0$ *and* $\varphi$ *do not meet in a set of positive Lebesgue measure, then* $u(x,t) = u_0(x)$ *is the unique solution of our problem.*

*Proof.* It is easy to check that $u(x,t) = u_0(x)$ is a solution. Assume that there exists another solution $v$. Because our problem is linear, $w = u - v$ is a solution of (4) with homogeneous initial data. Using Property 2 we have $w \equiv 0$, and so $u(x,t) = v(x,t)$.

The problem (4)–(5) describes the evolution of density, hence its solution must satisfy

PROPERTY 5. *The solution* $u(x,t)$ *is nonnegative and* $\int_\Omega u(x,t)\,dx = 1$.

*Proof.* The conservation of the $L^1$-norm is obvious, it is enough to integrate (4) over $\Omega$. The nonnegativity of $u$ is a consequence of Properties 3 and 4.

For the proof of existence of such a solution we use the Banach contraction principle.

THEOREM 2.1. *There exists a unique solution of problem* (4)–(5).

*Proof.* It follows from Property 5 that it is enough to look for nonnegative solutions. For fixed $T > 0$ and $\lambda > 0$ we define

$$X = \{u : [0,T] \to C^0(\Omega) : u \geq 0,\ u(0) = u_0\}.$$

The set $X$ equipped with the distance function

$$d_\lambda(u,v) = \sup_{t\in[0,T]} e^{-\lambda t}\|u(t) - v(t)\|_{C^0(\Omega)}$$

is a complete metric space. The operator $W$ defined on $(X, d_\lambda)$ by

$$Wv(t)(x) = u_0(x)e^{-\varphi(x)t} + e^{-\varphi(x)t}\int_0^t e^{\varphi(x)s}\int_\Omega \varphi(x)v(s)(y)\,dy\,ds$$

is a contraction on $X$ for $\lambda > 1$. In fact, we have

$$\begin{aligned}
d_\lambda(Wu, Wv) &= \sup_{t\in[0,T]} e^{-\lambda t}\|Wu(t)(x) - Wv(t)(x)\|_{C^0(\Omega)} \\
&\leq \sup_{t\in[0,T]} \Big\| e^{-(\lambda+\varphi(x))t}\int_0^t e^{\varphi(x)s}\int_\Omega e^{\lambda s}e^{-\lambda s}|u(s)(y) - v(s)(y)|\,dy\,ds\Big\|_{C^0(\Omega)} \\
&\leq \sup_{t\in[0,T]} \Big\| e^{-(\lambda+\varphi(x))t}\int_0^t e^{\varphi(x)s}\int_\Omega e^{\lambda s}d_\lambda(u,v)\,dy\,ds\Big\|_{C^0(\Omega)} \\
&= d_\lambda(u,v)\sup_{t\in[0,T]} \Big\| e^{-(\lambda+\varphi(x))t}\int_0^t e^{(\lambda+\varphi(x))s}\,ds\Big\|_{C^0(\Omega)} \\
&= d_\lambda(u,v)\sup_{t\in[0,T]} \Big\| \frac{1 - e^{-(\lambda+\varphi(x))t}}{\lambda + \varphi(x)}\Big\|_{C^0(\Omega)} \leq \frac{1}{\lambda}d_\lambda(u,v).
\end{aligned}$$

**3. Asymptotic behaviour of solutions.** The asymptotic properties of solutions of (4)–(5) depend on the behaviour of the function $\varphi$ in the vicinity of the set of its zeros, $B = \{x \in \Omega : \varphi(x) = 0\}$. If $\varphi > 0$, the solution tends as $t \to \infty$ uniformly to the unique stationary solution, thus we may say that the steady state coincides with this stationary solution. If $B \neq \emptyset$, the situation is more complicated. The steady state is a measure with a density or a singular measure, depending on the properties of $1/\varphi$.

This kind of asymptotic behaviour is characteristic for semigroups of Markov operators for which the Foguel alternative is satisfied, i.e. sweeping occurs or the semigroup is asymptotically stable (for a detailed presentation see [7] and the references therein).

First, we consider the case $B = \emptyset$, i.e. $\varphi > 0$ on $\Omega$. The stationary solution $U$ satisfies $AU(x) = \varphi(x)U(x)$, hence $U(x) = A(U)/\varphi(x)$. Here $A(U)$ is a constant, which depends on the unknown density $U$. We have $\int_\Omega U(x)\,dx = 1$, so $A(U) = (\int_\Omega (\varphi(x))^{-1}\,dx)^{-1}$. This implies that the stationary solution is of the form

$$U(x) = \Big(\int_\Omega \frac{1}{\varphi(y)}\,dy\Big)^{-1} \frac{1}{\varphi(x)}. \tag{9}$$

THEOREM 3.1. *If $\varphi(x) > 0$ on $\Omega$, then the solution of* (4)–(5) *tends uniformly to the stationary solution $U$ defined in* (9).

*Proof.* The function $v(x,t) = u(x,t) - U(x)$ satisfies equation (4) and $\int_\Omega v(x,t)\,dx = 0$. Using the Cauchy inequality we get

$$Av(t) \le \Big(\int_\Omega \varphi(x)\,dx\Big)^{1/2} (\mathcal{L}v(t))^{1/2}. \tag{10}$$

The $L^2$-norm of $v$ satisfies

$$\begin{aligned}(\|v\|_2^2)_t &= \frac{d}{dt}\Big(\int_\Omega v^2\,dx\Big) = 2\int_\Omega v(x,t)\,dx\,Av(t) - 2\int_\Omega v^2(x,t)\varphi(x)\,dx \\ &= -2\mathcal{L}v(t).\end{aligned} \tag{11}$$

Now, it follows from (11) that $\mathcal{L}v$ tends to 0 as $t\to\infty$. Hence (10) implies that $Av(t)\to 0$ as $t\to\infty$. Using Lemma 1.1, we get the uniform convergence of solutions to the steady state $U$.

If $B \neq \emptyset$ and $\int_\Omega 1/\varphi(x)\,dx < \infty$, we define $U(x)$ by (9) for $x \notin B$ and $U(x) = \infty$ on $B$, and we call such a function the steady state.

THEOREM 3.2. *If $\int_\Omega 1/\varphi(x)\,dx < \infty$, then the solution of* (4)–(5) *converges to the steady state as $t\to\infty$ uniformly on each compact subset of $\Omega \setminus B$.*

*Proof.* Again we consider the difference $v(x,t) := u(x,t) - U(x)$. Evidently $v$ satisfies (4) on $\Omega\setminus B$ and $\int_\Omega v(x,t)\,dx = 0$. We cannot, as before, differentiate the $L^2$-norm of $v$ (it is not even clear if $v$ is in $L^2(\Omega)$ or not).

However, we can still use the Lyapunov functional which is well defined for $v$, and inequality (10) holds. We show that $Av(t)\to 0$ as $t\to\infty$. First, we prove that there exists a sequence $t_n\to\infty$ such that $Av(t_n)\to 0$. Suppose that $Av(t) > a > 0$ for all $t > 0$. Then

$$\partial_t v(x,t) \ge a - \varphi(x)v(x,t) \tag{12}$$

and therefore

$$v(x,t) \ge \frac{a}{\varphi(x)} - \Big(\frac{a}{\varphi(x)} - v_0(x)\Big)e^{-\varphi(x)t}. \tag{13}$$

For each neighbourhood $K_r = \{x\in\Omega : \operatorname{dist}(x,B) < r\}$ of $B$, and for sufficiently large $t$, we have

$$\int_{\Omega\setminus K_r} v(x,t)\,dx \ge \frac{1}{2}\int_{\Omega\setminus K_r} \frac{a}{\varphi(x)}\,dx. \tag{14}$$

Thus, for sufficiently small $r$ we get

$$\begin{aligned}\int_{K_r} v(x,t)\,dx &= \int_{K_r} u(x,t)\,dx - \int_{K_r} U(x)\,dx \ge -\int_{K_r} U(x)\,dx \\ &> -\frac{1}{4}\int_{\Omega\setminus K_r} \frac{a}{\varphi(x)}\,dx.\end{aligned} \tag{15}$$

Inequalities (14)–(15) imply $\int_\Omega v(x,t)\,dx > 0$ for sufficiently large $t$, which leads to a contradiction.

Suppose that there exists a sequence $\bar{t}_n \to \infty$ such that $Av(\bar{t}_n) > \bar{a}$ for some $\bar{a} > 0$. Note that

$$(Av)'(t) = \int_\Omega \varphi(x)\,dx\,Av(t) - \int_\Omega v(x,t)\varphi^2(x)\,dx. \tag{16}$$

It follows from (10) that $Av(t)$ is bounded. Moreover we have

$$\Big|\int_\Omega v(x,t)\varphi(x)\,dx\Big| \le \max\varphi^2(x)\int_\Omega |u(x,t)-U(x)|\,dx \le 2\max\varphi^2(x)$$

and hence $(Av)'(t)$ is bounded on $\mathbb{R}^+$.

To show that $Av(t)\to 0$, it is enough to prove that $\mathcal{L}v(t)\to 0$. Assume to the contrary that $\mathcal{L}v(t) > b > 0$ for all $t > 0$ and some $b$.

Note that

$$\frac{d}{dt}\Big(\int_\Omega u^2(x,t)\,dx\Big) = Av(t) - \mathcal{L}v(t). \tag{17}$$

As shown above, there exists $t_n \to \infty$ such that $Av(t_n) \to 0$, and the derivative of $Av(t)$ is bounded. Hence for some $\delta > 0$ the intervals $\varDelta_n := (t_n - \delta, t_n + \delta)$ are such that on $\varDelta_n$,

$$\frac{d}{dt}\Big(\int_\Omega u^2(x,t)\,dx\Big) < -\frac{1}{2}b. \tag{18}$$

This implies that $\int_\Omega u^2(x,t)\,dx$ becomes negative in finite time, which is absurd. In this way we proved that $\mathcal{L}v(t)\to 0$, so $Av(t)\to 0$.

If $1/\varphi$ is not integrable, we have only partial results about the asymptotic behaviour of solutions. If $B$ is a single point, $B = \{x_0\}$, and $1/\varphi$ is not integrable, we call the measure $\delta_{x_0}$ the *stationary state*. This is justified by the next theorem.

THEOREM 3.3. *If $B = \{x_0\}$ and $1/\varphi$ is not integrable, then the solution of problem (4)–(5) tends to $\delta_{x_0}$ as $t \to \infty$ in the sense of weak convergence of measures.*

*Proof.* We show that $Au(t) \to 0$ as $t \to \infty$. First, we prove that there exists a sequence $t_n$ tending to $\infty$ such that $Au(t_n) \to 0$. Assume that $Au(t) \geq a > 0$, hence

$$u(x,t) \geq \frac{a}{\varphi(x)} - \left(\frac{a}{\varphi(x)} - u_0(x)\right)e^{-\varphi(x)t}. \tag{19}$$

For $\delta > 0$ we define $E_\delta = \{x \in \Omega : \|x - x_0\| > \delta\}$. Now we choose $\delta > 0$ such that $\frac{1}{2}a\int_{E_\delta} 1/\varphi(x)\,dx > 2$. For each sufficiently large $t$,

$$\left(\frac{a}{\varphi(x)} - u_0(x)\right)e^{-\varphi(x)t} < 1, \tag{20}$$

so we get $\int_{E_\delta} u(x,t)\,dx > 1$, a contradiction. Note that

$$(Au)'(t) = Au(t)\int_\Omega \varphi(x)\,dx - \mathcal{L}u(t),$$

and the derivative $(Au)'(t)$ is bounded. To prove that $\mathcal{L}u(t)$ tends to 0, we proceed as in the preceding proof. Having this convergence and using (10) (with $u$ instead of $v$), we know that $Au(t) \to 0$ and Lemma 1.1 implies that $u(\cdot,t)$ goes to 0 uniformly on each compact subset of $\Omega \setminus \{x_0\}$.

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