APPLICATIONES MATHEMATICAE
39,3 (2012), pp. 315–328

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# LOGARITHMICALLY IMPROVED REGULARITY CRITERIA FOR THE MICROPOLAR FLUID EQUATIONS

*Abstract.* We discuss the 3D incompressible micropolar fluid equations, and give logarithmically improved regularity criteria in terms of both the velocity field and the pressure in Morrey–Campanato spaces, BMO spaces and Besov spaces.

**1. Introduction.** In this paper, we consider the regularity of the following three-dimensional (3D) micropolar fluid equations with the incompressibility condition:

$$\begin{cases} \partial_t v - \Delta v + v\cdot\nabla v + \nabla P - \nabla\times\omega = 0, \\ \partial_t\omega - \Delta\omega - \nabla \operatorname{div}\omega + 2\omega + v\cdot\nabla\omega - \nabla\times v = 0, \\ \operatorname{div} v = 0, \\ v(x,0) = v_0(x), \quad \omega(x,0) = \omega_0(x), \end{cases} \tag{1.1}$$

where $v = (v_1(t,x), v_2(t,x), v_3(t,x))$ denotes the velocity of the fluid at a point $x\in\mathbb{R}^3$, $t\in[0,T)$, and $\omega = (\omega_1(t,x), \omega_2(t,x), \omega_3(t,x))$ and $P = P(t,x)$ denote the microrotational velocity and the hydrostatic pressure, respectively. The functions $v_0$ and $\omega_0$ are prescribed initial data for the velocity and angular velocity with $\operatorname{div} v_0 = 0$. The theory of micropolar fluids was first proposed by Eringen [9] to consider some physical phenomena that cannot be treated by the classical Navier–Stokes equations for viscous incompressible fluids, for example, the motion of animal blood, liquid crystals and dilute aqueous polymer solutions etc. When the microrotation is neglected ($\omega = 0$), the micropolar fluid equations reduce to the classical Navier–Stokes

2010 *Mathematics Subject Classification*: 35Q35, 76W05, 35B65.
*Key words and phrases*: micropolar fluid equations, regularity criterion, Morrey–Campanato space, Besov space, BMO space.

DOI: 10.4064/am39-3-6  © Instytut Matematyczny PAN, 2012

equations which have been extensively analyzed: see, for example, the classical books by Ladyzhenskaya [19], Lions [23] or Lemarié-Rieusset [21].

There is a vast literature on the mathematical theory of micropolar fluid equations (1.1) (see, for example, [20, 32, 14, 12, 8, 31, 4, 5]). First of all, for results on the uniqueness and existence of local smooth solutions, we refer the reader to [9]. The existence and uniqueness of global solutions were extensively studied by Lange [20], Galdi and Rionero [14], Yamaguchi [32], and Chen–Miao [6]. Recently, Ferreira and Villamizar-Roa [12] considered the existence and stability of solutions to the micropolar fluid equations in exterior domains. Villamizar-Roa and Rodríguez-Bellido [31] studied the micropolar system in a bounded domain using the semigroup approach in $L^p$, showing the global existence of strong solutions for small data and the asymptotic behavior and stability of solutions. Concerning the dynamic behavior of solutions to (1.1) we refer the reader to [4, 5, 8] and the references therein.

As in the case of the classical 3D Navier–Stokes system, the problem of either the global regularity or finite time singularity for weak solutions of the 3D micropolar model (1.1) with large initial data remains unsolved. Thus, several regularity criteria have been developed. For the Navier–Stokes equations, Serrin [28], Prodi [27] and Beirão da Veiga [1] established classical regularity criteria for weak solutions in terms of $u$ or its gradient $\nabla u$ in $L^p$ spaces. Later on, improvements and extensions were found (see for example, [26, 18, 16, 17] and the references therein). Moreover, Berselli and Galdi [2], Chae and Lee [3] and Fan and Ozawa [11] obtained regularity criteria for weak solutions in terms of the pressure $P$ or its gradient $\nabla P$. Recently, Zhou and Gala [34] and Fan et al. [10] obtained logarithmically improved regularity criteria for the Navier–Stokes system in terms of the velocity field, the vorticity field and the pressure respectively.

For the micropolar fluid equations (1.1), Gala [13] and Yuan [33] established some regularity criteria in terms of both the velocity field and the pressure in Morrey–Campanato spaces and Lorentz spaces, respectively.

Motivated by the above results, the purpose of this paper is to establish logarithmically improved regularity criteria in terms of both the velocity field and the pressure field for the 3D micropolar fluid equations (1.1).

Our main results read as follows.

THEOREM 1.1. *Let* $v_0, \omega_0 \in H^3(\mathbb{R}^3)$. *Let* $(v, \omega)$ *be a smooth solution to equations* (1.1) *on some interval* $[0, T)$. *If the velocity field* $v$ *satisfies one of the following conditions:*

$$\int_0^T \frac{\|v(t,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|v(t,\cdot)\|_{L^\infty})}\,dt < \infty, \quad 0 < r < 1, \tag{1.2}$$

$$\int_0^T \frac{\|\nabla v(t,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|v(t,\cdot)\|_{L^\infty})}\,dt<\infty,\qquad 0<r\le 1,\tag{1.3}$$

*then the smooth solution $(v,\omega)$ can be extended beyond $t=T$.*

THEOREM 1.2. *Let $v_0,\omega_0\in H^3(\mathbb{R}^3)$. Let $(v,\omega)$ be a smooth solution to equations* (1.1) *on some interval* $[0,T)$. *If the gradient of the pressure $\nabla P$ satisfies one of the following conditions:*

$$\int_0^T \frac{\|\nabla P(t,\cdot)\|_{\dot{B}^0_{\infty,\infty}}^{2/3}}{(1+\ln(1+\|\nabla P(t,\cdot)\|_{\dot{B}^0_{\infty,\infty}}))^{2/3}}\,dt<\infty,\tag{1.4}$$

$$\int_0^T \frac{\|\nabla P(t,\cdot)\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P(t,\cdot)\|_{\mathrm{BMO}})}\,dt<\infty,\tag{1.5}$$

*then the smooth solution $(v,\omega)$ can be extended beyond $t=T$.*

REMARK 1.1. Theorem 1.1 contains a result which is new for the 3D incompressible Navier–Stokes equations. This is an improvement and extension of results reported in [26]–[28], [1] and [34].

REMARK 1.2. Since the critical Morrey–Campanato space $\dot{\mathcal{M}}_{2,3/r}$ is much wider than the Lebesgue space $L^{3/r}$ and the Lorentz space $L^{3/r,\infty}$, our Theorem 1.1 covers the recent results in [13] and [33]. Moreover, our result shows that the velocity field $v$ plays a more important role than the microrotation vector field $\omega$ in the regularity theory of solutions to the micropolar equations.

REMARK 1.3. The regularity criterion stated in (1.3) improves slightly a condition in [11], where the authors assumed

$$\int_0^T \|\nabla P(t,\cdot)\|_{\dot{B}^0_{\infty,\infty}}^{2/3}\,dt<\infty.$$

**2. Preliminaries and lemmas.** First, we recall the definitions and properties of some function spaces, which play an important role in studying the regularity of solutions to partial differential equations (see [21, 25, 30]).

DEFINITION 2.1. For $1<p\le q\le\infty$, the *Morrey–Campanato space* $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)$ is defined as

$$\dot{\mathcal{M}}_{p,q}(\mathbb{R}^3)=\Big\{f\in L^p_{\mathrm{loc}}(\mathbb{R}^3):\|f\|_{\dot{\mathcal{M}}_{p,q}}=\sup_{x\in\mathbb{R}^3}\sup_{R>0}R^{3/q-3/p}\|f\|_{L^p(B(x,R))}<\infty\Big\},$$

where $B(x,R)$ denotes the ball of center $x$ with radius $R$.

It is easy to check that

$$\|f(\lambda\cdot)\|_{\dot{\mathcal{M}}_{p,q}} = \frac{1}{\lambda^{3/q}}\|f\|_{\dot{\mathcal{M}}_{p,q}} \quad \text{for all } \lambda > 0, \tag{2.1}$$

$$\dot{\mathcal{M}}_{p,\infty}(\mathbb{R}^3) = L^\infty(\mathbb{R}^3) \quad \text{for each } 1 \le p \le \infty. \tag{2.2}$$

Additionally, for $2 \le p \le 3/r$ and $0 \le r < 3/2$, we have the following embedding relations:

$$L^{3/r}(\mathbb{R}^3) \hookrightarrow L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{p,3/r}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \hookrightarrow \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3), \tag{2.3}$$

where $L^{p,\infty}$ denotes the weak $L^p$-space, and $\dot{X}_r(\mathbb{R}^3)$ is a multiplier space (see [21]).

Definition 2.2. $\dot{B}^s_{p,q}$ denotes the homogeneous Besov space with the norm

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \Big(\sum_{j=-\infty}^{\infty} 2^{jsq}\|\dot{\Delta}_j f\|_p^q\Big)^{1/q} & \text{for } 1 \le p \le \infty,\ q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_p & \text{for } 1 \le p \le \infty,\ q = \infty, \end{cases}$$

where $\Delta_j$ is a Littlewood–Paley operator.

Remark 2.1. Recall that for $0 \le r < 3/2$, the space $\dot{Z}_r(\mathbb{R}^3)$ is defined in [22] as the set of all $f \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \le 1} \|fg\|_{L^2} < \infty. \tag{2.4}$$

It is proved in [22] that $f \in \dot{\mathcal{M}}_{2,3/r}(\mathbb{R}^3)$ if only if $f \in \dot{Z}_r(\mathbb{R}^3)$ with equivalence of norms.

Remark 2.2. In the following, we use the inequality

$$\|f\|_{\dot{B}^r_{2,1}} \le C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^r \quad \text{for } 0 < r < 1,$$

where $C$ is independent of $f$, which was proved in [24] and is vital to our proof.

Remark 2.3. Notice that if $\nabla u \in \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3)$, then $u \in \mathrm{BMO}(\mathbb{R}^3)$, where BMO is the space of functions of bounded mean oscillation of John and Nirenberg, with the norm

$$\|u\|_{\mathrm{BMO}}^2 = \sup_{x\in\mathbb{R}^3} \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |u(y) - m_{B(x,R)}u(y)|^2\,dy.$$

Indeed, by the classical Poincaré inequality, we have

$$\int_{B(x,R)} |u(y) - m_{B(x,R)}u(y)|^2\,dy \le CR^2 \int_{B(x,R)} |\nabla u(y)|^2\,dy \le CR^3\|\nabla u\|_{\dot{\mathcal{M}}_{2,3}}^2$$

for every ball $B(x,R)$ of any radius $R$.

## 3. Proofs of theorems

*Proof of Theorem 1.1.* We first show that Theorem 1.1 holds under the condition (1.2). Multiplying both sides of the first equation in (1.1) by $-\Delta v$ and the second equation by $-\Delta\omega$, and integrating by parts over $\mathbb{R}^3$, we get

(3.1)
$$\frac{1}{2}\frac{d}{dt}(\|\nabla v\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)+\|\Delta v\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+\|\nabla\operatorname{div}\omega\|_{L^2}^2+2\|\nabla\omega\|_{L^2}^2$$
$$=\int_{\mathbb{R}^3}(v\cdot\nabla)v\cdot\Delta v dx+\int(v\cdot\nabla)\omega\cdot\Delta\omega\,dx-\int_{\mathbb{R}^3}\operatorname{curl}\omega\cdot\Delta v\,dx-\int\operatorname{curl}v\cdot\Delta\omega\,dx$$
$$\triangleq I_1+I_2+I_3+I_4.$$

For $I_1$, by using Hölder's inequality, Young's inequality, (2.4), (2.5) and integration by parts, we obtain

$$\begin{aligned}I_1&\le\|v\cdot\nabla v\|_{L^2}\|\nabla^2 v\|_{L^2}\le C\|v\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla v\|_{\dot{B}^r_{2,1}}\|\nabla^2 v\|_{L^2}\\&\le C\|v\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla v\|_{L^2}^{1-r}\|\nabla^2 v\|_{L^2}^{1+r}\\&\le C(\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}\|\nabla v\|_{L^2}^2)^{(1-r)/2}(\|\nabla^2 v\|^2)_{L^2}^{(1+r)/2}\\&\le C\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}\|\nabla v\|_{L^2}^2+\tfrac{1}{4}\|\nabla^2 v\|_{L^2}^2.\end{aligned}\tag{3.2}$$

Similarly, for $I_2$, we deduce

$$I_2\le C\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}\|\nabla\omega\|_{L^2}^2+\tfrac{1}{4}\|\nabla^2\omega\|_{L^2}^2.\tag{3.3}$$

Finally, for $I_3$ and $I_4$, with the use of Hölder's inequality, Young's inequality, and integration by parts, we have

$$I_3+I_4\le 2\|\nabla\omega\|_{L^2}^2+\tfrac{1}{2}\|\Delta v\|_{L^2}^2.\tag{3.4}$$

Inserting (3.2)–(3.4) into (3.1), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla v\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)\le C\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}(\|\nabla v\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)\\&\le C\frac{\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)(1+\ln(e+\|v\|_{L^\infty}))\\&\le C\frac{\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2)(1+\ln(e+\|\Lambda^3 v\|_{L^2}+\|\Lambda^3\omega\|_{L^2})),\end{aligned}\tag{3.5}$$

where the Sobolev embedding has been used and $\Lambda^s=(-\Delta)^{s/2}$.

For any $T_0<t\le T$, we let

$$y(t)=\sup_{T_0\le s\le t}(\|\Lambda^3 v\|_{L^2}+\|\Lambda^3\omega\|_{L^2}).\tag{3.6}$$

Coming back to (3.5), we get

$$\tag{3.7} \begin{aligned} &\frac{d}{dt}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2) \\ &\quad \le C \frac{\|v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)(1+\ln(e+y(t))). \end{aligned}$$

Applying Gronwall's inequality to (3.7) on the interval $[T_0, t]$, one has

$$\tag{3.8} \begin{aligned} \|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 &\le C_0 \exp(C\varepsilon(1+\ln(e+y(t)))) \\ &\le C_0 \exp(2C\varepsilon \ln(e+y(t))) \le C_0(e+y(t))^{2C\varepsilon} \end{aligned}$$

where $\varepsilon > 0$ (to be chosen later) satisfies

$$\tag{3.8$_a$} \int_{T_0}^{t} \frac{\|v(s,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(1-r)}}{1+\ln(e+\|v(s,\cdot)\|_{L^\infty})}\, ds < \varepsilon,$$

and where $C_0$ is a positive constant depending on $T_0$.

Next, we turn to the estimate for the $H^3$-norm of $v$ and $\omega$. In the following calculations, we will use the following commutator estimate due to Kato and Ponce [15]:

$$\tag{3.9} \|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le (\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}),$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$. Applying $\Lambda^3$ to both sides of (1.1), then multiplying by $\Lambda^3 v$ and $\Lambda^3 \omega$, respectively, after integrating by parts over $\mathbb{R}^3$, we have

$$\tag{3.10} \begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} (|\Lambda^3 v|^2 + |\Lambda^3 \omega|^2)\, dx + \int_{\mathbb{R}^3} |\Lambda^4 v|^2\, dx + \int_{\mathbb{R}^3} |\Lambda^4 \omega|^2\, dx \\ &\qquad + \int_{\mathbb{R}^3} |\Lambda^3 \operatorname{div} v|^2\, dx + 2\int_{\mathbb{R}^3} |\Lambda^3 \omega|^2\, dx \\ &\quad = -\int_{\mathbb{R}^3} [\Lambda^3(v\cdot\nabla v) - v\cdot\nabla\Lambda^3 v]\Lambda^3 v\, dx + \int_{\mathbb{R}^3} \Lambda^3 \operatorname{curl}\omega \cdot \Lambda^3 v\, dx \\ &\qquad - \int_{\mathbb{R}^3} [\Lambda^3(v\cdot\nabla \omega) - v\cdot\nabla\Lambda^3 \omega]\Lambda^3 \omega\, dx + \int_{\mathbb{R}^3} \Lambda^3 \operatorname{curl} v \cdot \Lambda^3 \omega\, dx \\ &\quad \equiv A_1 + A_2 + A_3 + A_4. \end{aligned}$$

Hence

$$\tag{3.11} \begin{aligned} A_1 &\le C\|\nabla v\|_{L^3} \|\Lambda^3 v\|_{L^3}^2 \le C\|\nabla v\|_{L^2}^{13/12} \|\Lambda^3 v\|_{L^2}^{1/4} \|\Lambda^4 v\|_{L^2}^{5/3} \\ &\le \tfrac{1}{6}\|\Lambda^4 v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^{13/2} \|\Lambda^3 v\|_{L^2}^{3/2}, \end{aligned}$$

where we used (3.9) with $s = 3$, $p = 3/2$, $p_1 = q_1 = p_2 = q_2 = 3$, and the inequalities

$$\|\nabla v\|_{L^3} \le C\|\nabla v\|_{L^2}^{3/4}\|\Lambda^3 v\|_{L^2}^{1/4}, \tag{3.12}$$

$$\|\Lambda^3 v\|_{L^3} \le C\|\nabla v\|_{L^2}^{1/6}\|\Lambda^4 v\|_{L^2}^{5/6}. \tag{3.13}$$

If we use estimate (3.8) for $T_0 < t < T$, inequality (3.11) reduces to

$$A_1 \le \tfrac{1}{6}\|\Lambda^4 v\|_{L^2}^2 + CC_0(e+y(t))^{\frac{3}{2}+\frac{13}{2}C\varepsilon}. \tag{3.14}$$

Using (3.12) and (3.13) again, we have

$$\begin{aligned} A_3 &\le C(\|\nabla v\|_{L^3}\|\Lambda^3\omega\|_{L^3} + \|\nabla\omega\|_{L^3}\|\Lambda^3 v\|_{L^3})\|\Lambda^3\omega\|_{L^3} \\ &\le C(\|\nabla v\|_{L^3} + \|\nabla\omega\|_{L^3})(\|\Lambda^3 v\|_{L^3}^2 + \|\Lambda^3\omega\|_{L^3}^2) \\ &\le \tfrac{1}{6}(\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^4\omega\|_{L^2}^2) + CC_0(e+y(t))^{\frac{3}{2}+\frac{13}{2}C\varepsilon}. \end{aligned} \tag{3.15}$$

In the same way, for the terms $A_2$ and $A_4$ in (3.10) we have

$$\begin{aligned} A_2 + A_4 &\le \tfrac{1}{6}(\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^4\omega\|_{L^2}^2) + C(\|\Lambda^3 v\|_{L^2}^2 + \|\Lambda^3\omega\|_{L^2}^2) \\ &\le \tfrac{1}{6}(\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^4\omega\|_{L^2}^2) + CC_0(e+y(t))^2. \end{aligned} \tag{3.16}$$

Inserting (3.14)–(3.16) into (3.10), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3}(|\Lambda^3 v|^2 + |\Lambda^3\omega|^2)\,dx \le CC_0(e+y(t))^{\frac{3}{2}+\frac{13}{2}C\varepsilon} + CC_0(e+y(t))^2. \tag{3.17}$$

Now, we choose $t > T_0$ so close to $T_0$ such that $\frac{13}{2}C\varepsilon < \frac{1}{2}$, which can be achieved by the absolute continuity of integral (1.2). Thus, Gronwall's inequality implies the boundedness of the $H^3$-norms of $v$ and $\omega$.

Next, we turn to the proof of Theorem 1.1 under condition (1.3). First, we assume $0 < r < 1$. For $I_1$, by Hölder's inequality, Young's inequality, integrating by parts over $\mathbb{R}^3$ and (2.6), we get

$$\begin{aligned} I_1 &\le \|\nabla v\cdot\nabla v\|_{L^2}\|\nabla v\|_{L^2} \le C\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla v\|_{\dot{B}^r_{2,1}}\|\nabla v\|_{L^2} \\ &\le C\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}\|\nabla v\|_{L^2}^{1-r}\|\nabla^2 v\|_{L^2}^r\|\nabla v\|_{L^2} \\ &\le C(\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\nabla v\|_{L^2}^2)^{(2-r)/2}(\|\nabla^2 v\|_{L^2}^2)^{r/2} \\ &\le C\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\nabla v\|_{L^2}^2 + \tfrac{1}{4}\|\nabla^2 v\|_{L^2}^2. \end{aligned} \tag{3.18}$$

Similarly, for $I_2$, we have

$$I_2 \le C\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}\|\nabla\omega\|_{L^2}^2 + \tfrac{1}{4}\|\nabla^2\omega\|_{L^2}^2. \tag{3.19}$$

Inserting (3.18), (3.19) as well as (3.4) into (3.1), we obtain

$$
\begin{aligned}
(3.20)\qquad &\frac{d}{dt}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2) \le C\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)\\
&\le C\frac{\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)(1+\ln(e+\|v\|_{L^\infty}))\\
&\le C\frac{\|\nabla v\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)(1+\ln(e+y(t))),
\end{aligned}
$$

where $y(t)$ is defined by (3.6).

Applying Gronwall's inequality to inequality (3.20) on the interval $[T_0, t]$, one has

$$
\begin{aligned}
(3.21)\qquad \|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 &\le C_0 \exp(C\varepsilon(1+\ln(e+y(t))))\\
&\le C_0\exp(2C\varepsilon\ln(e+y(t))) \le C_0(e+y(t))^{2C\varepsilon}
\end{aligned}
$$

provided that

$$
\int_{T_0}^{t} \frac{\|\nabla v(s,\cdot)\|_{\dot{\mathcal{M}}_{2,3/r}}^{2/(2-r)}}{1+\ln(e+\|v(s,\cdot)\|_{L^\infty})}\,ds < \varepsilon,
$$

and where $C_0$ is a positive constant depending on $T_0$.

Now, in (3.21) we use a similar method to estimate the $H^3$-norms of $v$ and $\omega$.

In the remaining case $r = 1$ in (1.3), by the Coifman–Lions–Meyer–Semmes inequality [7] and Remark 2.3 we modify the above estimates to obtain

$$
\begin{aligned}
(3.22)\qquad &\frac{d}{dt}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2) \le C\|v\|_{\mathrm{BMO}}^2(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)\\
&\le C\|\nabla v\|_{\dot{\mathcal{M}}_{2,3}}^2(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)\\
&\le C\frac{\|\nabla v\|_{\dot{\mathcal{M}}_{2,3}}^2}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)(1+\ln(e+\|v\|_{L^\infty}))\\
&\le C\frac{\|\nabla v\|_{\dot{\mathcal{M}}_{2,3}}^2}{1+\ln(e+\|v\|_{L^\infty})}(\|\nabla v\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2)(1+\ln(e+y(t))).
\end{aligned}
$$

The remaining estimate is analogous to that for $r < 1$. This completes the proof of Theorem 1.1.

*Proof of Theorem 1.2.* We show that Theorem 1.2 holds under condition (1.4). Computing the inner product of the first equation of (1.1) with $|v|^2 v$

and the second with $|\omega|^2\omega$, and integrating by parts over $\mathbb{R}^3$, one shows that

$$\frac{1}{4}\frac{d}{dt}\|v\|_4^4 + \int_{\mathbb{R}^3} |\nabla v|^2|v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla |v|^2|^2\,dx \le \int_{\mathbb{R}^3} |\nabla P|\,|v|^2|v|\,dx + \int_{\mathbb{R}^3} \operatorname{curl} w\,|v|^2 v\,dx, \tag{3.23}$$

$$\frac{1}{4}\frac{d}{dt}\|\omega\|_4^4 + \int_{\mathbb{R}^3} |\nabla \omega|^2|\omega|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{div}\omega|^2|\omega|^2\,dx + 2\int_{\mathbb{R}^3}|\omega|^4\,dx \le \int_{\mathbb{R}^3} \operatorname{curl} v\,|\omega|^2\omega\,dx. \tag{3.24}$$

Combining these, we arrive at

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}(\|v\|_4^4 + \|\omega\|_4^4) &+ \int_{\mathbb{R}^3} |\nabla v|^2|v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla |v|^2|^2\,dx \\ &+ \int_{\mathbb{R}^3} |\nabla \omega|^2|\omega|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{div}\omega|^2|\omega|^2\,dx + 2\int_{\mathbb{R}^3}|\omega|^4\,dx \\ &\le \int_{\mathbb{R}^3} |\nabla P|\,|v|^2|v|\,dx + \int_{\mathbb{R}^3} \operatorname{curl} w\,|v|^2 v\,dx + \int_{\mathbb{R}^3} \operatorname{curl} v\,|\omega|^2\omega\,dx \\ &= II_1 + II_2 + II_3.\end{aligned} \tag{3.25}$$

Integrating by parts and applying Hölder's and Young's inequalities for $II_2$ and $II_3$ , it follows that

$$II_2 + II_3 \le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2|v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \omega|^2|\omega|^2\,dx + C(\|v\|_4^4 + \|\omega\|_4^4). \tag{3.26}$$

Applying the operator $\nabla\operatorname{div}$ to the first equation of (1.1), one formally has

$$\nabla P = \sum_{i,j=1}^{3} R_i R_j(\nabla(v_i v_j)),$$

where $R_j$ denotes the $j$th Riesz operator. By the Calderón–Zygmund inequality, we have

$$\|\nabla P\|_{L^2} \le C\|v\cdot\nabla v\|_{L^2}. \tag{3.27}$$

Concerning the term $II_1$, by the Cauchy inequality and (3.27), we have

$$\begin{aligned} II_1 &\le \|\nabla P\|_{L^4}\|v\|_{L^4}^3 \le C\|\nabla P\|_{L^2}^{1/2}\|\nabla P\|_{\mathrm{BMO}}^{1/2}\|v\|_{L^4}^3 \\ &\le C\|v\cdot\nabla v\|_{L^2}^{1/2}\|\nabla P\|_{\mathrm{BMO}}^{1/2}\|v\|_{L^4}^3 \\ &\le \tfrac{1}{2}\|v\cdot\nabla v\|_{L^2}^2 + C\|\nabla P\|_{\mathrm{BMO}}^{2/3}\|v\|_{L^4}^4,\end{aligned} \tag{3.28}$$

where we used the interpolation inequality (see [29])

$$\|f\|_{L^{2r}}^2 \le C\|f\|_{\mathrm{BMO}}\|f\|_{L^r}^2.$$

Inserting (3.26) and (3.28) into (3.25), it follows that

$$\begin{aligned}
(3.29)\qquad & \frac{d}{dt}(\|v\|_4^4+\|\omega\|_4^4) \le C\|\nabla P\|_{\mathrm{BMO}}^{2/3}\|v\|_{L^4}^4 + C(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\|\nabla P\|_{\mathrm{BMO}}^{2/3}(\|v\|_4^4+\|\omega\|_4^4) + C(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C(1+\|\nabla P\|_{\dot B^0_{\infty,\infty}}^{2/3}\ln^{1/3}(1+\|\nabla P\|_{H^2}))(\|v\|_4^4+\|\omega\|_4^4) + C(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C(1+\|\nabla P\|_{\dot B^0_{\infty,\infty}}^{2/3}\ln^{1/3}(1+\|\Lambda^3 v\|_{L^2}))(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(1+\frac{\|\nabla P\|_{\dot B^0_{\infty,\infty}}^{2/3}}{(1+\ln(1+\|\nabla P\|_{\dot B^0_{\infty,\infty}}))^{2/3}}\ln(1+\|\Lambda^3 v\|_{L^2})\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(1+\frac{\|\nabla P\|_{\dot B^0_{\infty,\infty}}^{2/3}}{(1+\ln(1+\|\nabla P\|_{\dot B^0_{\infty,\infty}}))^{2/3}}\ln(1+\|\Lambda^3 v\|_{L^2}+\|\Lambda^3 \omega\|_{L^2})\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(1+\frac{\|\nabla P\|_{\dot B^0_{\infty,\infty}}^{2/3}}{(1+\ln(1+\|\nabla P\|_{\dot B^0_{\infty,\infty}}))^{2/3}}\ln(1+y(t))\bigg)(\|v\|_4^4+\|\omega\|_4^4)
\end{aligned}$$

where we used the inequalities (see [17, 18])

$$\begin{aligned}
&\|P\|_{H^{s-1}} \le C + C\|\Lambda^s v\|_{L^2}^2,\\
&\|f\|_{\mathrm{BMO}} \le C(1+\|f\|_{\dot B^0_{\infty,\infty}}\ln^{1/2}(1+\|f\|_{H^{s-1}})), \qquad s > n/2+1,
\end{aligned}$$

and $y(t)$ is defined by (3.6).

Applying Gronwall's lemma to inequality (3.29) on the interval $[T_0, t]$, one has

$$(3.30)\qquad \sup_{T_0\le s\le t}(\|v\|_4^4+\|\omega\|_4^4) \le C_0(1+y(t))^{C\varepsilon}$$

provided that

$$\int_{T_0}^{t} \frac{\|\nabla P\|_{\dot B^0_{\infty,\infty}}^{2/3}}{(1+\ln(1+\|\nabla P\|_{\dot B^0_{\infty,\infty}}))^{2/3}}\,ds < \varepsilon,$$

and where $C_0$ is a positive constant depending on $T_0$.

Multiplying both sides of the first equation of (1.1) by $v$ and the second by $\omega$, and integrating by parts over $\mathbb{R}^3$, we get

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2+\int_{\mathbb{R}^3}|\nabla v|^2\,dx\le\int_{\mathbb{R}^3}v\operatorname{curl}\omega\,dx\le\int_{\mathbb{R}^3}|\nabla\omega|\,|v|\,dx \le\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\omega|^2\,dx+\int_{\mathbb{R}^3}|v|^2\,dx, \tag{3.31}$$

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2+\int_{\mathbb{R}^3}|\nabla\omega|^2\,dx+\int_{\mathbb{R}^3}|\operatorname{div}\omega|^2\,dx+2\int_{\mathbb{R}^3}|\omega|^2\,dx \le\int_{\mathbb{R}^3}\omega\operatorname{curl}v\,dx\le\int_{\mathbb{R}^3}\omega|\nabla v|^2\,dx\le\frac{1}{4}\int_{\mathbb{R}^3}|\nabla v|^2\,dx+\int_{\mathbb{R}^3}|\omega|^2\,dx. \tag{3.32}$$

Combining (3.31) and (3.32), using Gronwall's inequality, we infer that

$$\|v\|_{L^\infty(0,T;L^2)}+\|v\|_{L^2(0,T;H^1)}\le C, \tag{3.33}$$

$$\|\omega\|_{L^\infty(0,T;L^2)}+\|\omega\|_{L^2(0,T;H^1)}\le C. \tag{3.34}$$

Note that one has to estimate the $L^2$-norm of $\nabla v$ and $\nabla\omega$. We multiply both sides of the first equation of (1.1) by $-\Delta v$ and the second by $-\Delta\omega$, then by integrating by parts over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2+\|\Delta v\|_{L^2}^2&=\int_{\mathbb{R}^3}(v\cdot\nabla)v\cdot\Delta v\,dx-\int_{\mathbb{R}^3}\operatorname{curl}\omega\,\Delta v\,dx\\ &\le\|v\|_{L^4}\|\nabla v\|_{L^4}\|\Delta v\|_{L^2}+\|\nabla\omega\|_{L^2}\|\Delta v\|_{L^2}\\ &\le\|v\|_{L^4}\|v\|_{L^4}^{1/5}\|\Delta v\|_{L^2}^{4/5}\|\Delta v\|_{L^2}+\tfrac{1}{16}\|\Delta v\|_{L^2}^2+C\|\nabla\omega\|_{L^2}^2\\ &\le\tfrac{1}{8}\|\Delta v\|_{L^2}^2+C\|v\|_{L^4}^{12}+C\|\omega\|_{L^2}\|\Delta\omega\|_{L^2}\\ &\le\tfrac{1}{8}\|\Delta v\|_{L^2}^2+\tfrac{1}{8}\|\Delta\omega\|_{L^2}^2+C\|v\|_{L^4}^{12}+C\|\omega\|_{L^2}^2,\end{aligned} \tag{3.35}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+\|\nabla\operatorname{div}\omega\|_{L^2}^2+2\|\nabla\omega\|_{L^2}^2\\ &\quad=\int(v\cdot\nabla)\omega\cdot\Delta\omega\,dx-\int\operatorname{curl}v\,\Delta\omega\,dx\\ &\quad\le\|v\|_{L^4}\|\nabla\omega\|_{L^4}\|\Delta\omega\|_{L^2}+\|\nabla v\|_{L^2}\|\Delta\omega\|_{L^2}\\ &\quad\le\|v\|_{L^4}\|\omega\|_{L^4}^{1/5}\|\Delta\omega\|_{L^2}^{4/5}\|\Delta\omega\|_{L^2}+C\|v\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{1/2}\|\Delta\omega\|_{L^2}\\ &\quad\le\tfrac{1}{8}\|\Delta v\|_{L^2}^2+\tfrac{1}{8}\|\Delta\omega\|_{L^2}^2+C\|v\|_{L^4}^{10}\|\omega\|_{L^4}^2+C\|v\|_{L^2}^2,\end{aligned} \tag{3.36}$$

where we have used the Gagliardo–Nirenberg inequality:

$$\|\nabla f\|_{L^4}\le C\|f\|_{L^4}^{1/5}\|\Delta f\|_{L^2}^{4/5},\qquad\|\nabla f\|_{L^2}\le C\|f\|_{L^2}^{1/2}\|\Delta f\|_{L^2}^{1/2}.$$

By (3.33)–(3.36), we deduce that

$$\|\nabla v\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2\le C(1+y(t))^{3C\varepsilon}(t-T_0)+\|\nabla v(\cdot,T_0)\|_{L^2}^2+\|\nabla\omega(\cdot,T_0)\|_{L^2}^2. \tag{3.37}$$

From (3.37), the estimate of the $H^3$-norms of $v$ and $\omega$ is as in the proof of Theorem 1.1.

Finally we show that Theorem 1.2 holds under condition (1.4). We start from (3.28). Inserting (3.26) and (3.28) into (3.25) it follows that

$$
\begin{aligned}
(3.38)\qquad \frac{d}{dt}(\|v\|_4^4+\|\omega\|_4^4) &\le C\|\nabla P\|_{\mathrm{BMO}}^{2/3}\|v\|_{L^4}^4 + C(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\|\nabla P\|_{\mathrm{BMO}}^{2/3}(\|v\|_4^4+\|\omega\|_4^4) + C(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(\frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}(1+\ln(1+\|\nabla P\|_{\mathrm{BMO}}))\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(\frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}(1+\ln(1+\|\Delta P\|_{L^3}))\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(\frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}(1+\ln(1+\|\nabla v\|_{L^6}^2))\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(\frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}(1+\ln(1+\|\Delta v\|_{L^2}^2))\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(\frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}(1+\ln(1+\|\Lambda^3 v\|_{L^2}))\bigg)(\|v\|_4^4+\|\omega\|_4^4)\\
&\le C\bigg(\frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}(1+\ln(1+y(t))\bigg)(\|v\|_4^4+\|\omega\|_4^4),
\end{aligned}
$$

where we used the relation $-\Delta p = \partial_i\partial_j(v_iv_j)$ and

$$\|\Delta v\|_{L^2}^2 \le C\|v\|_{L^2}^{2/3}\|\Lambda^3 v\|_{L^2}^{4/3},$$

and $y(t)$ is defined by (3.6). Applying Gronwall's inequality to (3.38) for the interval $[T_0,t]$, one has

$$\sup_{T_0\le s\le t}(\|v\|_4^4+\|\omega\|_4^4) \le C_0(1+y(t))^{C\varepsilon}$$

provided that

$$\int_{T_0}^{t} \frac{\|\nabla P\|_{\mathrm{BMO}}^{2/3}}{1+\ln(1+\|\nabla P\|_{\mathrm{BMO}})}\,ds < \varepsilon,$$

and where $C_0$ is a positive constant depending on $T_0$. The remaining estimate is analogous to (3.31)-(3.37). Thus the proof of Theorem 1.2 is complete.

**Acknowledgements.** The author would like to thank Professor Changxing Miao for his useful suggestions, and the referee for his (her) comments and suggestions.

This research was supported by the National Natural Science Foundation of China (Nos. 11026048 and 11171026), Beijing Natural Science Foundation (No. 2112023) and PCSIRT of China.

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*Received on 2.11.2011;*
*revised version on 6.12.2011* (2113)