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T-$p(x)$-SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WITH AN L^1-DUAL DATUM

Abstract. We establish the existence of a T-$p(x)$-solution for the $p(x)$-elliptic problem

$$-\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f - \operatorname{div} F \quad \text{in } \Omega,$$

where Ω is a bounded open domain of $\mathbb{R}^N$, $N \geq 2$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the natural growth condition and the coercivity condition, but with only a weak monotonicity condition. The right hand side f lies in $L^1(\Omega)$ and F belongs to $\prod_{i=1}^N L^{p'(\cdot)}(\Omega)$.

1. Introduction. In this work we are concerned with the problem of existence of a T-$p(x)$-solution for a class of nonlinear elliptic equations of the type

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f - \operatorname{div} F & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a *Carathéodory* function (that is, $a(\cdot, s, \xi)$ is measurable on Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, and $a(x,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) and $g(x,u)$ is a nonlinear term which satisfies some suitable conditions (see (3.1) and (3.2) below). The right hand side f is in $L^1(\Omega)$ and F lies in $\prod_{i=1}^N L^{p'(\cdot)}(\Omega)$ where $p(\cdot) : \Omega \to \mathbb{R}$ is a measurable function satisfying some hypotheses (see Section 2). The vector function $a(\cdot)$ is supposed to satisfy

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the following assumptions:

- For almost every $x \in \Omega$, and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, with $\gamma(\cdot)$ a continuous function and $k(\cdot) \in L^{p'(\cdot)}(\Omega)$,

$$|a(x,s,\xi)| \le k(x) + |s|^{p(x)-1} + [\gamma(s)|\xi|]^{p(x)-1}. \tag{1.2}$$

- For almost every $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, and for some constant $\alpha > 0$,

$$a(x,s,\xi)\cdot\xi \ge \alpha|\xi|^{p(x)}. \tag{1.3}$$

- For almost every $x \in \Omega$ and all $(s,\xi,\overline{\xi}) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ with $\xi \neq \overline{\xi}$,

$$(a(x,s,\xi) - a(x,s,\overline{\xi}))\cdot(\xi - \overline{\xi}) > 0. \tag{1.4}$$

Our objective in this paper is to study the existence of a possible solution of (1.1) in the framework of Sobolev spaces with variable exponent under only some weak monotonicity condition.

Hypotheses (1.3) and (1.4) are natural extensions of the classical assumptions in the study of nonlinear monotone operators of divergence form for constant $p(\cdot) \equiv p$ (see [19]). However, the growth condition (1.2) is not a natural hypothesis. This is due to the function $\gamma(\cdot)$ introduced in (1.2) (this makes the term $a(x,u,\nabla u)$ not necessarily bounded in $\prod_{i=1}^N L^{p'(\cdot)}(\Omega)$), so, proving the existence of a solution seems to be an arduous task. To overcome this difficulty we use the framework of T-$p(x)$-solutions (this is the first aim of this paper). The formula of this solution is written in the form of an equality (see Definition 3.1 below). However, the formula for the entropy solution (see [6] for instance) is an inequality. So we can say that the T-$p(x)$-solution is an entropy solution with equality.

One of our motivations for studying (1.1) comes from applications of electro-rheological fluids, an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an extremal electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Růžička (we refer to [26], [28] for more details).

Another important application is related to image processing [11] where the diffusion operator is used to underline the borders of the distorted image and to eliminate the noise. We also mention that our space appears in elasticity [26] and in the calculus of variations with variable exponents [2]–[22].

Before starting, we list some remarks about solvability of (1.1). Firstly, in the case where $p = p(x)$ we can cite several studies such as: [5], [16], [21], [13]. Secondly, it should be noted that in all recent works, the strict monotonicity condition (1.4) is assumed. When trying to relax this condition on $a(\cdot)$, the classical monotone operator methods developed by Višik [28], Minty [24], Browder [10], Brézis [9], Lions [19] and others are not applicable.

The second aim of our paper is to treat the problem (1.1) when (1.4) is replaced by the weak monotonicity condition

$$(a(x,s,\xi)-a(x,s,\overline{\xi}))\cdot(\xi-\overline{\xi})\geq 0. \tag{1.5}$$

Here we cannot use the classical method of almost everywhere convergence of the gradient for the approximation of solutions because there is no guarantee that $\nabla u_n \to \nabla u$ a.e. in Ω. To overcome this difficulty we use some new techniques based on the L^1-version of Minty's lemma. When $p(\cdot)=p=$ constant, the problem (1.1) is studied under the weak monotonicity assumption (1.5) in [7] and in [4] (in the last work the degenerate or singular operator is treated). Finally, our third aim in this paper is to generalize [4] and [7] to the case where $p=p(x)$. Note also that this article can be seen as a generalization of [5], [21], [23] and as a continuation of [4]. Recently, in the case $p=p(x)$, Wittbold and Zimmermann [29] have proved the existence and uniqueness of a renormalized solution to nonlinear elliptic equations of the form

$$\begin{cases} -\operatorname{div}(a(x,\nabla u))+g(u)=f-\operatorname{div}F & \text{in } \Omega,\\ u\equiv 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The notion of renormalized solutions has been introduced, for the first time, by Lions and DiPerna [14] in their study of the Boltzmann equations. See also P.-L. Lions [20] for a few applications to fluid mechanics models. The equivalence between entropy and renormalized solutions was developed by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [12] for the study of nonlinear elliptic problems. Moreover, this equivalence was generalized to parabolic equations with smooth measure data by J. Droniou and A. Prignet [15].

In the case of the Dirichlet problem in divergence form with variable growth, modeled on the $p(x)$-Laplace equation, M. Sanchón and J. M. Urbano [27] proved the existence and uniqueness of an entropy solution for L^1 data.

Note that, in our work, if $f\in L^{p'(\cdot)}(\Omega)$, then (1.1) admits no weak solution because the term $a(x,s,\xi)$ is not necessarily in $L^{p'(\cdot)}(\Omega)$ due to the introduction of the function $\gamma(\cdot)$ in (H_1) (see Remark 5.2 below). However, in other works [27], [29], $a(\cdot,s,\xi)\in L^{p'(\cdot)}(\Omega)$ and consequently (1.1) has a weak solution.

This paper is organized as follows: In the second section, we introduce some basic properties of the generalized Lebesgue and Sobolev–Lebesgue spaces. In the third section, we prove some technical lemmas after giving the basic assumptions. In the fourth section, we begin by studying an approximate problem $(\mathcal{P}_n)$ for our main problem $(\mathcal{P})$, which will be useful in proving the main result in the last section. The latter proof is divided into three steps.

2. Mathematical preliminaries. This section is devoted to introducing some definitions and properties of generalized Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and Lebesgue–Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, where Ω is a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, that will be needed throughout the paper (for further details about these notions and results, we refer the reader to [17], [18] and [30] for instance).

We set

$$\mathcal{C}_+(\overline{\Omega}) = \{p \in \mathcal{C}(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For every $p \in \mathcal{C}_+(\overline{\Omega})$ we define

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined as

$$L^{p(\cdot)}(\Omega) = \Big\{u : u \text{ is a measurable real-valued function,} \\ \exists \lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}\,dx < \infty\Big\},$$

normed by the so-called *Luxemburg* norm,

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}\,dx \leq 1\Big\}.$$

The $L^{p(\cdot)}(\Omega)$ spaces have some properties similar to those of the classical Lebesgue spaces. They are Banach spaces ([18, Theorem 2.5]). They are reflexive if and only if $1 < p^- \leq p^+ < \infty$ ([18, Corollary 2.7]) and the continuous functions are dense if $p^+ < \infty$ ([18, Theorem 2.11]). The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. And for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder inequality

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p^+}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

holds.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the modular of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)}(u) : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

If $u \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$ then the following relations hold:

- If $\|u\|_{p(\cdot)} > 1$, then $\|u\|_{p(\cdot)}^{p_-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$.
- If $\|u\|_{p(\cdot)} < 1$, then $\|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p_-}$.

We also have

$$\|u\|_{p(\cdot)} \to 0 \quad \text{if and only if} \quad \rho_{p(\cdot)}(u) \to 0.$$

Next, we define the *generalized Lebesgue–Sobolev space* $W^{1,p(\cdot)}(\Omega)$ as

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is endowed with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

We define

$$W_0^{1,p(\cdot)}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

$W^{-1,p'(\cdot)}(\Omega)$ is the dual space of $W_0^{1,p(\cdot)}(\Omega)$.

We end this section by recalling the following important properties of these spaces which will be needed throughout the following.

Proposition 2.1 ([18]).

(1) *$W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are Banach spaces, which are separable if $p \in L^\infty(\Omega)$ and reflexive if $1 < p_- < p^+ < \infty$.*

(2) *If $q \in \mathcal{C}_+(\overline{\Omega})$ with $q(x) < p^*(x)$ then we have the compact embedding*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega),$$

where $p^*(x) = Np(x)/(N - p(x))$ for all $p(x) < N$. Since $p(x) < p^*(x)$ for all $x \in \Omega$, in particular

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega). \tag{2.1}$$

(3) *There exists a constant $c > 0$ with $\|u\|_{p(\cdot)} \leq c\|\nabla u\|_{p(\cdot)}$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$, hence $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.*

3. Basic assumptions and technical lemmas

3.1. Basic assumptions. First, we suppose that the Carathéodory function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following assumptions:

$$|a_i(x, s, \xi)| \leq k(x) + |s|^{p(x)-1} + (\gamma(s)|\xi|)^{p(x)-1}; \tag{H_1}$$

$$(a(x, s, \xi) - a(x, s, \overline{\xi})) \cdot (\xi - \overline{\xi}) \geq 0; \tag{H_2}$$

$$\sum_{i=1}^N a_i(x, s, \xi) \cdot \xi_i \geq \alpha \sum_{i=1}^N |\xi_i|^{p(x)}, \tag{H_3}$$

for almost every $x \in \Omega$ all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and all $i \in \{1,\dots,N\}$, where $k(\cdot)$ is a positive function in $L^{p'(\cdot)}(\Omega)$, $\gamma(\cdot)$ is a continuous function and α is a positive constant.

Next, we consider the following $p(x)$-*Dirichlet problem*:

$$(\mathcal{P}) \quad \begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$ and F lies in the dual space $\prod_{i=1}^N L^{p'(\cdot)}(\Omega)$. Moreover, $g(x,s)$ is a Carathéodory function satisfying

$$g(x,s)s \geq 0, \tag{3.1}$$

$$\sup_{|s|\leq n} |g(x,s)| = h_n(x) \in L^1(\Omega). \tag{3.2}$$

For all $k > 1$ and s in $\mathbb{R}$, the truncation T_k is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ ks/|s| & \text{if } |s| > k. \end{cases}$$

Definition 3.1. Let u be a measurable function such that $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$. Then u is called a T-$p(x)$-*solution* of the problem $(\mathcal{P})$ if

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\varphi)\,dx = \int_\Omega fT_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Definition 3.2. Let X be a Banach reflexive space and let X^* be its dual space. We say that the operator $L : X \to X^*$ is *pseudo-monotone* if

$$\left.\begin{matrix} u_n \rightharpoonup u \text{ weakly in } X \\ \limsup_{n\to\infty} \langle Lu_n, u_n - u\rangle \leq 0 \end{matrix}\right\} \Rightarrow \begin{cases} Lu_n \rightharpoonup Lu \text{ weakly in } X^*, \\ \langle Lu_n, u_n\rangle \to \langle Lu, u\rangle. \end{cases}$$

The symbol $\rightharpoonup$ denotes weak convergence.

3.2. Some technical lemmas

Lemma 3.3. *Let $q \in \mathcal{C}_+(\overline{\Omega})$, $g \in L^{q(\cdot)}(\Omega)$ and $(g_n)_n \in L^{q(\cdot)}(\Omega)$ with $\|g_n\|_{q(\cdot)} \leq C$, where C is a positive constant. If $g_n(x) \to g(x)$ almost everywhere in Ω, then $g_n \rightharpoonup g$ in $L^{q(\cdot)}(\Omega)$.*

Proof. We set

$$E(N) = \{x \in \Omega : |g_n(x) - g(x)| \leq 1,\ \forall n \geq N\}.$$

Then

$$\operatorname{meas}(E(N)) \to \operatorname{meas}(\Omega) \quad \text{as } N \to \infty.$$

Let

$$\mathcal{F} = \{\varphi \in L^{q'(\cdot)}(\Omega) : \varphi \equiv 0 \text{ almost everywhere in } \Omega \setminus E(N) \text{ for some } N\}.$$

We shall show that $\mathcal{F}$ is dense in $L^{q'(\cdot)}(\Omega)$. Let $f \in L^{q'(\cdot)}(\Omega)$, and put

$$f_N(x) = \begin{cases} f(x) & \text{if } x \in E(N), \\ 0 & \text{if } x \in \Omega \setminus E(N). \end{cases}$$

Then

$$\begin{aligned} \rho_{q'(\cdot)}(f_N - f) &= \int_\Omega |f_N(x) - f(x)|^{q'(x)}\,dx \\ &= \int_{E(N)} |f_N(x) - f(x)|^{q'(x)}\,dx + \int_{\Omega\setminus E(N)} |f_N(x) - f(x)|^{q'(x)}\,dx \\ &= \int_{\Omega\setminus E(N)} |f(x)|^{q'(x)}\,dx = \int_\Omega |f(x)|^{q'(x)}\chi_{\Omega\setminus E(N)}\,dx. \end{aligned}$$

Taking $\psi_N(x) = |f(x)|^{q'(x)}\chi_{\Omega\setminus E(N)}(x)$ for almost every x in Ω, we obtain

$$\psi_N \to 0 \text{ almost everywhere in } \Omega \quad \text{and} \quad |\psi_N| \le |f|^{q'(x)}.$$

Thus by the dominated convergence theorem, we conclude that

$$\rho_{q'(\cdot)}(f_N - f) \to 0 \quad \text{as } N \to \infty.$$

Therefore, $f_N \to f$ in $L^{q'(\cdot)}(\Omega)$. Consequently, $\mathcal{F}$ is dense in $L^{q'(\cdot)}(\Omega)$.

Now, we will show that

$$\lim_{n\to\infty} \int_\Omega \varphi(x)(g_n(x) - g(x))\,dx = 0 \quad \text{for all } \varphi \in \mathcal{F}.$$

Suppose $\varphi \equiv 0$ in $\Omega \setminus E(N)$. We put $\phi_n = \varphi(g_n - g)$. Since $|\varphi(x)|\,|g_n(x) - g(x)| \le |\varphi(x)|$ almost everywhere in $E(N)$ and since $\phi_n \to 0$ almost everywhere in Ω, thanks (again) to the dominated convergence theorem we obtain $\phi_n \to 0$ in $L^1(\Omega)$ as desired.

Finally, by the density of $\mathcal{F}$ in $L^{q'(\cdot)}(\Omega)$, we conclude that

$$\lim_{n\to\infty} \int_\Omega \varphi g_n\,dx = \int_\Omega \varphi g\,dx \quad \text{for all } \varphi \in L^{q'(\cdot)}(\Omega),$$

which proves that $g_n \rightharpoonup g$ in $L^{q(\cdot)}(\Omega)$.

Lemma 3.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function with $F(0) = 0$, and $p \in \mathcal{C}_+(\overline{\Omega})$. If $u \in W_0^{1,p(\cdot)}(\Omega)$, then $F(u) \in W_0^{1,p(\cdot)}(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Proof. First, we consider the case where

$$F \in \mathcal{C}^1(\Omega) \quad \text{and} \quad F' \in L^\infty(\Omega).$$

Let u in $W_0^{1,p(\cdot)}(\Omega)$. Since $\overline{\mathcal{C}_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)} = W_0^{1,p(\cdot)}(\Omega)$, there exists a sequence $(u_n)_n \subset \mathcal{C}_0^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{1,p(\cdot)}(\Omega)$, hence $u_n \to u$ almost everywhere on Ω and $\nabla u_n \to \nabla u$ almost everywhere on Ω.

Therefore,

$$|F(u_n)| = |F(u_n) - F(0)| \leq \|F'\|_\infty \|u_n\|,$$

implying

$$|F(u_n)|^{p(x)} \leq \|F'\|_\infty^{p^+} \|u_n\|^{p(x)} \quad \text{and} \quad \left|\frac{\partial F}{\partial x_i}(u_n)\right|^{p(x)} = \left|F'(u_n)\frac{\partial u_n}{\partial x_i}\right|^{p(x)}.$$

So $F(u_n) \in W_0^{1,p(\cdot)}(\Omega)$ and $F(u_n)$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, implying $F(u_n) \rightharpoonup v$ in $W_0^{1,p(\cdot)}(\Omega)$. Thus, $F(u_n) \to v$ in $L^{p(\cdot)}(\Omega)$ (strongly) by (2.1). So, $F(u_n) \to v$ almost everywhere in Ω, hence $v = F(u) \in W_0^{1,p(\cdot)}(\Omega)$.

Now let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz. Then

$$F_n = F * \rho_n \to F$$

uniformly on every compact set, where ρ_n is the regularizing function. We have $F_n \in \mathcal{C}^1(\mathbb{R})$ and $F_n' \in L^\infty(\mathbb{R})$, therefore by the foregoing, we have $F_n(u) \in W_0^{1,p(\cdot)}(\Omega)$, $F_n(u) \to F(u)$ for almost everywhere on Ω, and also $(F_n(u))_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$ and $F_n(u) \rightharpoonup \overline{v}$ in $W_0^{1,p(\cdot)}(\Omega)$ (weakly). So, by using (2.1) we obtain

$$F_n(u) \to \overline{v} \quad \text{in } L^{p(\cdot)}(\Omega).$$

Finally, $F_n(u) \to \overline{v}$ for almost everywhere in Ω, and consequently $\overline{v} = F(u) \in W_0^{1,p(\cdot)}(\Omega)$.

LEMMA 3.5. *Let $u \in W_0^{1,p(\cdot)}(\Omega)$. Then $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ with $k > 0$. Moreover, $T_k(u) \to u$ in $W_0^{1,p(\cdot)}(\Omega)$ as $k \to \infty$.*

Proof. For $k > 0$, let

$$T_k : \mathbb{R} \to \mathbb{R}^+, \qquad s \mapsto T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ ks/|s| & \text{if } |s| > k. \end{cases}$$

Since T_k is uniformly Lipschitz and $T_k(0) = 0$, so using Lemma 3.4 we have $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$. Moreover,

$$\begin{aligned}\int_\Omega |T_k(u)-u|^{p(x)}\,dx &+ \int_\Omega |\nabla T_k(u)-\nabla u|^{p(x)}\,dx\\ &= \int_{|u|\le k} |T_k(u)-u|^{p(x)}\,dx + \int_{|u|>k} |T_k(u)-u|^{p(x)}\,dx\\ &\quad + \int_{|u|\le k} |\nabla T_k(u)-\nabla u|^{p(x)}\,dx + \int_{|u|>k} |\nabla T_k(u)-\nabla u|^{p(x)}\,dx\\ &= \int_{|u|>k} |T_k(u)-u|^{p(x)}\,dx + \int_{|u|>k} |\nabla u|^{p(x)}\,dx.\end{aligned}$$

Since $T_k(u)\to u$ as $k\to\infty$, and by using the dominated convergence theorem, we have

$$\int_{|u|>k} |T_k(u)-u|^{p(x)}\,dx + \int_{|u|>k} |\nabla u|^{p(x)}\,dx \to 0 \qquad \text{as } k\to\infty.$$

Finally, $\|T_k(u)-u\|_{W_0^{1,p(\cdot)}(\Omega)}\to 0$ as $k\to\infty$.

LEMMA 3.6. *Let $(u_n)_n\subset W_0^{1,p(\cdot)}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$. Then $T_k(u_n)\rightharpoonup T_k(u)$ in $W_0^{1,p(\cdot)}(\Omega)$.*

Proof. We have $u_n\rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$. So, by the compact embedding (2.1) we have $u_n\to u$ in $L^{p(\cdot)}(\Omega)$, and hence $u_n\to u$ almost everywhere on Ω. On the other hand,

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p(x)} dx = \sum_{i=1}^N \int_\Omega \left|T_k'(u_n)\frac{\partial u_n}{\partial x_i}\right|^{p(x)} dx \le \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p(x)} dx < \infty.$$

Thus, $(T_k(u_n))_n$ is bounded on $W_0^{1,p(\cdot)}(\Omega)$, so there exists $v_k\in W_0^{1,p(\cdot)}(\Omega)$ such that

$$T_k(u_n)\rightharpoonup v_k \quad \text{in } W_0^{1,p(\cdot)}(\Omega) \quad \text{as } n\to\infty.$$

Therefore, by the compact embedding (2.1) again, we have

$$T_k(u_n)\to v_k \quad \text{almost everywhere in } \Omega.$$

And since $T_k(u_n)\to T_k(u)$ almost everywhere in Ω, we deduce that

$$v_k = T_k(u) \quad\text{and}\quad T_k(u_n)\rightharpoonup T_k(u) \quad \text{in } W_0^{1,p(\cdot)}(\Omega).$$

4. The approximate problem. Let $(f_n)_n$ be a sequence of functions in $L^\infty(\Omega)$ which is strongly convergent to f in $L^1(\Omega)$ such that $\|f_n\|_{L^1}\le \|f\|_{L^1}$, and consider the following approximate problem:

$$(\mathcal{P}_n)\qquad \begin{cases} u_n\in W_0^{1,p(\cdot)}(\Omega),\\ -\operatorname{div}(a(x,T_n(u_n),\nabla u_n)) + g_n(x,u_n) = f_n - \operatorname{div}(F) \quad \text{in } \Omega,\end{cases}$$

where

$$g_n(x,s) = \frac{g(x,s)}{1+\frac{1}{n}|g(x,s)|}.$$

In this section we will prove the existence of a solution to $(\mathcal{P}_n)$ under certain conditions. This is contained in the following theorem;

THEOREM 4.1. *Let B_k be the operator defined by*

$$B_k : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega),$$
$$u \mapsto B_k u = -\operatorname{div}(a(x,T_k(u),\nabla u) + g_k(x,u).$$

The operator B_k is bounded, hemi-continuous, coercive and pseudo-monotone.

By using [19] and Theorem 4.1, we obtain

THEOREM 4.2. *Problem $(\mathcal{P}_n)$ admits a solution u_n in $W_0^{1,p(\cdot)}(\Omega)$.*

Proof of Theorem 4.1

• *B_k is bounded*: For $u,v \in W_0^{1,p(\cdot)}(\Omega)$,

$$|\langle B_k u, v\rangle| = \Big|\int_\Omega a(x,T_k(u),\nabla u)\nabla v\,dx + \int_\Omega g_k(x,u)v\,dx\Big|$$
$$\le \Big(\frac{1}{p'^-}+\frac{1}{p^-}\Big)\|a(x,T_k(u),\nabla u)\|_{p'(\cdot)}\cdot\|\nabla v\|_{p(\cdot)} + \int_\Omega |kv(x)|\,dx$$
$$\le C_1\Big(1+\int_\Omega \big(k(x)+|T_k(u)|^{p(x)-1}+(\gamma(T_k(u))|\nabla u|)^{p(x)-1}\big)^{p'(x)}\,dx\Big)^{1/p'_s}\|v\|_{1,p(\cdot)}$$
$$\le C_1\Big(1+\int_\Omega C_2\big(k^{p'(x)}+|T_k(u)|^{p(x)}+(\gamma(T_k(u)))^{p(x)}|\nabla u|^{p(x)}\big)\,dx\Big)^{1/p'_s}\|v\|_{1,p(\cdot)}$$
$$\le C_3\|v\|_{1,p(\cdot)},$$

because $\gamma(\cdot)$ is a continuous function, thus $\operatorname{supp}(T_k(u)) \subset [-k,k]$, which implies that $\gamma(T_k(u))$ is bounded in $W^{1,p(\cdot)}(\Omega)$; here C_1, C_2 and C_3 are positive constants and

$$p'_s = \begin{cases} p'^- & \text{if } \|a(x,T_k(u),\nabla u)\|_{p'(\cdot)} > 1,\\ p'^+ & \text{if } \|a(x,T_k(u),\nabla u)\|_{p'(\cdot)} \le 1.\end{cases}$$

• *B_k is hemi-continuous*: Let t be a real variable tending to t_0. We have

$$a_i(x,T_k(u+tv),\nabla(u+tv)) \to a_i(x,T_k(u+t_0v),\nabla(u+t_0v))$$

almost everywhere in Ω and for $i \in \{1,\dots,N\}$. As moreover $(a_i(u,T_k(u+tv), \nabla(u+tv)))_t$ is bounded in $L^{p'(\cdot)}(\Omega)$, by Lemma 3.3, $a(x,T_k(u+tv),\nabla(u+tv)) \rightharpoonup a(x,T_k(u+t_0v),\nabla(u+t_0v))$ in $(L^{p'(\cdot)}(\Omega))^N$ as $t \to t_0$.

Let $w \in W_0^{1,p(\cdot)}(\Omega)$. Since $\nabla w \in (L^{p(\cdot)}(\Omega))^N$ and

$$\frac{g(x,u+tv)}{1+\frac{1}{k}|g(x,u+tv)|} \to \frac{g(x,u+t_0v)}{1+\frac{1}{k}|g(x,u+t_0v)|} \quad \text{in } L^{p'(\cdot)}(\Omega) \quad \text{as } t \to t_0,$$

we have

$$\langle B_k(u+tv), w\rangle \to \langle B_k(u+t_0v), w\rangle \quad \text{as } t \to t_0.$$

• *B_k is coercive*: For $u \in W_0^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned}\frac{\langle B_k u, u\rangle}{\|u\|_{1,p(\cdot)}} &= \frac{\int_\Omega a(x,T_k(u),\nabla u)\cdot\nabla u\,dx}{\|u\|_{1,p(\cdot)}} + \frac{\int_\Omega g_k(x,u)u\,dx}{\|u\|_{1,p(\cdot)}} \\ &\geq \frac{\alpha\int_\Omega |\nabla u|^{p(x)}\,dx}{\|u\|_{1,p(\cdot)}} \geq \alpha\frac{\|u\|_{1,p(\cdot)}^{p_s}}{\|u\|_{1,p(\cdot)}} \\ &\geq \alpha\|u\|_{1,p(\cdot)}^{p_s-1} \to +\infty \text{ as } \|u\|_{1,p(\cdot)} \to \infty,\end{aligned}$$

where

$$p_s = \begin{cases} p^- & \text{if } \|u\|_{p(\cdot)} \leq 1, \\ p^+ & \text{if } \|u\|_{p(\cdot)} > 1. \end{cases}$$

• *B_k is pseudo-monotone*: Let $(u_j)_{j\in\mathbb{N}} \subset W_0^{1,p(\cdot)}(\Omega)$ be such that

$$u_j \rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega) \quad \text{and} \quad \limsup\,\langle B_k u_j, u_j - u\rangle \leq 0. \tag{4.1}$$

We decompose the operator B_k as $B_k = A_k + G_k$, where

$$\langle A_k u, v\rangle = \int_\Omega a(x,T_k(u),\nabla u)\nabla v\,dx \quad \text{and} \quad \langle G_k u, v\rangle = \int_\Omega g_k(x,u)v\,dx,$$

for all u, v in $W_0^{1,p(\cdot)}(\Omega)$.

STEP 1: $B_k u_j \rightharpoonup B_k u$. First, we show that

$$\lim_{j\to\infty}\langle G_k u_j, u_j - u\rangle = 0.$$

Indeed,

$$\begin{aligned}|\langle G_k u_j, u_j - u\rangle| &= \left|\int_\Omega \frac{g(x,u_j)}{1+\frac{1}{k}|g(x,u_j)|}(u_j - u)\,dx\right| \\ &\leq \int_\Omega \left|\frac{g(x,u_j)}{1+\frac{1}{k}|g(x,u_j)|}\right||u_j - u|\,dx \\ &\leq \int_\Omega k|u_j - u|\,dx \leq C\|u_j - u\|_{p(\cdot)} \to 0 \quad \text{as } j \to \infty,\end{aligned}$$

thanks to (2.1). By (4.1), we have

$$\begin{aligned}&\limsup\langle A_k u_j + G_k u_j, u_j - u\rangle \\ &\qquad = \limsup\langle A_k u_j, u_j - u\rangle + \limsup\,\langle G_k u_j, u_j - u\rangle \leq 0,\end{aligned}$$

which implies that

$$\limsup \langle A_k u_j, u_j - u\rangle \le 0 \tag{4.2}$$

Now since $u_j \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$, we have $\partial u_j/\partial x_i \rightharpoonup \partial u/\partial x_i$ in $L^{p(\cdot)}(\Omega)$ for all $i = 1,\dots,N$. And since $(A_k(u_j))_j$ is bounded in $W^{-1,p'(\cdot)}(\Omega)$, there exist h_k and h_{ki} such that

$$\begin{aligned} &A_k u_j \rightharpoonup h_k \text{ in } W^{-1,p'(\cdot)}(\Omega),\\ &a_i(\cdot, T_k(u_j), \nabla u_j) \rightharpoonup h_{ki} \text{ in } L^{p'(\cdot)}(\Omega) \quad \forall i = 1,\dots,N. \end{aligned} \tag{4.3}$$

So, by (4.2) and (4.3) we obtain

$$\limsup \langle A_k u_j, u_j\rangle \le \langle h_k, u\rangle. \tag{4.4}$$

Moreover, by using (H_2), we can write

$$\sum_{i=1}^{N} \int_{\Omega} \big(a_i(x, T_k(u_j), \nabla v) - a_i(x, T_k(u_j), \nabla u_j)\big)\left(\frac{\partial v}{\partial x_i} - \frac{\partial u_j}{\partial x_i}\right) dx \ge 0$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$, hence

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial u_j}{\partial x_i}\, dx \ge{}& \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial v}{\partial x_i}\, dx\\ &- \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla v)\frac{\partial v}{\partial x_i}\, dx\\ &+ \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla v)\frac{\partial u_j}{\partial x_i}\, dx. \end{aligned} \tag{4.5}$$

Since, $u_j \to u$ in $L^{p(\cdot)}(\Omega)$ thanks to (2.1), we get $u_j \to u$ almost everywhere in Ω. Then, by (H_1) and the dominated convergence theorem, we obtain

$$a_i(x, T_k(u_j), \nabla v) \to a_i(x, T_k(u), \nabla v) \text{ in } L^{p'(\cdot)}(\Omega) \quad \text{for all } i = 1,\dots,N. \tag{4.6}$$

Thus,

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla v)\frac{\partial v}{\partial x_i}\, dx \to \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u), \nabla v)\frac{\partial v}{\partial x_i}\, dx, \tag{4.7}$$

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla v)\frac{\partial u_j}{\partial x_i}\, dx \to \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u), \nabla v)\frac{\partial u}{\partial x_i}\, dx. \tag{4.8}$$

By using (4.3), we get

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial v}{\partial x_i}\, dx \to \sum_{i=1}^{N} \int_{\Omega} h_{ki}\frac{\partial v}{\partial x_i}\, dx.$$

Letting $j \to \infty$ in (4.5) and using (4.6)–(4.8), we deduce that

$$\begin{aligned}\lim_{j\to\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_j), \nabla u_j) \frac{\partial u_j}{\partial x_i}\,dx \geq \sum_{i=1}^{N} \int_{\Omega} h_{ki} \frac{\partial v}{\partial x_i}\,dx \\ + \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u), \nabla v) \frac{\partial u}{\partial x_i}\,dx \\ - \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u), \nabla v) \frac{\partial v}{\partial x_i}\,dx.\end{aligned}$$

By invoking (4.4), we get

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} h_{ki} \frac{\partial u}{\partial x_i}\,dx \geq \sum_{i=1}^{N} \int_{\Omega} h_{ki} \frac{\partial v}{\partial x_i}\,dx + \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u), \nabla v) \frac{\partial u}{\partial x_i}\,dx \\ - \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u), \nabla v) \frac{\partial v}{\partial x_i}\,dx.\end{aligned}$$

So

$$\sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u), \nabla v) - h_{ki}) \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \geq 0 \quad \text{for all } v \in W_0^{1,p(\cdot)}(\Omega).$$

Taking $v = u + tw$ with first $t = 1$ and then $t = -1$, and using the technique of Minty, we have

$$\int_{\Omega} \big(a(x, T_k(u), \nabla(u + tw)) - h_k\big) \nabla w\,dx = 0 \quad \text{for all } w \in W_0^{1,p(\cdot)}(\Omega).$$

Consequently,

$$A_k u = h_k \tag{4.9}$$

is an element of $W^{-1,p'(\cdot)}(\Omega)$, so we deduce that

$$A_k u_j \rightharpoonup A_k u \quad \text{in } W^{-1,p'(\cdot)}(\Omega).$$

Now, since $g_k(x, u_j) \to g_k(x, u)$ almost everywhere in Ω as $j \to \infty$, and $|g_k(x, u_j)| \leq k$, the dominated convergence theorem yields

$$g_k(x, u_j) \rightharpoonup g_k(x, u) \quad \text{in } L^{p'(\cdot)}(\Omega). \tag{4.10}$$

Finally,

$$(A_k + G_k)(u_j) \rightharpoonup (A_k + G_k)(u) \quad \text{in } W^{-1,p'(\cdot)}(\Omega).$$

STEP 2: $\langle B_k u_j, u_j\rangle \to \langle B_k u, u\rangle$. According to (4.4) and (4.9), we have

$$\limsup \langle A_k u_j, u_j\rangle \leq \langle A_k u, u\rangle = \langle h_k, u\rangle,$$

and by (4.10), $\langle G_k u_j, u_j\rangle \to \langle G_k u, u\rangle$, hence

$$\limsup(\langle A_k u_j, u_j\rangle + \langle G_k u_j, u_j\rangle) \leq \langle A_k u, u\rangle + \langle G_k u, u\rangle,$$

thus,

$$\limsup \langle B_k u_j, u_j\rangle \le \langle B_k u, u\rangle.$$

So it suffices to prove that

$$\liminf \langle B_k u_j, u_j\rangle \ge \langle B_k u, u\rangle.$$

We have

$$\begin{aligned}
\langle B_k u_j, u_j\rangle &= \sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial u_j}{\partial x_i}\,dx + \int_{\Omega} g_k(x,u_j)u_j\,dx\\
&= \sum_{i=1}^{N}\int_{\Omega} \big(a_i(x, T_k(u_j), \nabla u_j) - a_i(x, T_k(u_j), \nabla u)\big)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)dx\\
&\quad + \sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)dx\\
&\quad + \sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial u}{\partial x_i}\,dx + \int_{\Omega} g_k(x,u_j)u_j\,dx.
\end{aligned}$$

Now, since

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial u_j}{\partial x_i}\,dx + \int_{\Omega} g_k(x,u_j)u_j\,dx \ge 0,$$

we have

$$\begin{aligned}
\langle B_k u_j, u_j\rangle \ge{}& \sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)dx\\
&+ \sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial u}{\partial x_i}\,dx + \int_{\Omega} g_k(x,u_j)u_j\,dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\liminf\langle B_k u_j, u_j\rangle \ge{}& \liminf\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u)\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)dx\\
&+ \liminf\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_k(u_j), \nabla u_j)\frac{\partial u}{\partial x_i}\,dx + \liminf\int_{\Omega} g_k(x,u_j)u_j\,dx.
\end{aligned}$$

So, we deduce that

$$\liminf \langle B_k u_j, u_j\rangle \ge \sum_{i=1}^{N}\int_{\Omega} h_i\frac{\partial u}{\partial x_i}\,dx + \int_{\Omega} g_k(x,u)u\,dx \ge \langle B_k u, u\rangle.$$

Consequently, B_k is pseudo-monotone.

5. Main result. Now, we are in a position to rephrase our main result under convenient hypotheses. Precisely, we may state the following.

THEOREM 5.1. *Under the assumptions* (H_1)–(H_3), *problem* $(\mathcal{P})$ *admits at least one* T-$p(x)$-*solution.*

REMARKS 5.2. In the formulation of problem $(\mathcal{P})$, we have $a(x,u,\nabla u)$ instead of $a(x,T_n(u_n),\nabla u_n)$ and the term $a(x,u,\nabla u)$ is not necessarily in $L^{p'(\cdot)}(\Omega)$, nor in $L^1(\Omega)$, therefore $(\mathcal{P})$ need not have a weak solution. For example, if

$$a(x,u,\nabla u)=\exp[(p(x)-1)\|u\|_{p(\cdot)}]\cdot\|\nabla u\|_{p(\cdot)}^{p(x)-2}\nabla u,$$

with $\gamma(s)=\exp[(p(x)-1)s]$ and $g(x,u)=\alpha(x)u|u|^q$, with α a positive function in $L^1(\Omega)$ and q a positive constant, then the problem

$$\begin{cases} T_k(u)\in W_0^{1,p(\cdot)}(\Omega), F\in\prod_{i=1}^N L^{p'(\cdot)}(\Omega),\\ -\operatorname{div}\big(\exp[(p(x)-1)\|u\|]\|\nabla u\|^{p(x)-2}\nabla u\big)+\alpha(x)u|u|^q=f-\operatorname{div}F \quad \text{in } \Omega,\\ u=0 \quad \text{on } \partial\Omega, \end{cases}$$

has a T-$p(x)$-solution but no weak solution.

We recall that in the following calculations, the symbol C is a constant with changing value.

Proof of Theorem 5.1

STEP I: *The approximate problem and a priori estimate.* We recall that $(f_n)_n$ is a sequence of $L^\infty(\Omega)$ functions which is strongly convergent to f in $L^1(\Omega)$ such that

$$\|f_n\|_{L^1}\le\|f\|_{L^1}\quad \text{for all } n\in\mathbb{N}.$$

Let $u_n\in W_0^{1,p(\cdot)}(\Omega)$ be a solution of the approximate problem $(\mathcal{P}_n)$, whose existence is guaranteed by Theorem 4.2. Choosing $T_k(u_n)$ as a test function in $(\mathcal{P}_n)$, we have

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\nabla T_k(u_n)\,dx+\int_\Omega g_n(x,u_n)T_k(u_n)\,dx = \int_\Omega f_nT_k(u_n)\,dx+\int_\Omega F\nabla T_k(u_n)\,dx.$$

Using $\nabla T_k(u_n)=\nabla u_n\chi_{\{|u_n|\le k\}}$ and thanks to the coercivity condition (H_3), we obtain

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\nabla T_k(u_n)\,dx\ge\alpha\sum_{i=1}^N\int_\Omega\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p(x)}dx.$$

Since $g_n(x,u_n)T_k(u_n) \geq 0$, one has

$$\alpha \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx \leq k\|f\|_{L^1} + \sum_{i=1}^{N} \int_{\Omega} |F_i| \left| \frac{\partial T_k(u_n)}{\partial x_i} \right| dx$$
$$\leq k\|f\|_{L^1} + \sum_{i=1}^{N} \int_{\Omega} |F_i| \left(\frac{\alpha}{2}\right)^{-1/p(x)} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right| \left(\frac{\alpha}{2}\right)^{1/p(x)} dx.$$

Now, by Young's inequality, we obtain

$$\alpha \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx \leq k\|f\|_{L^1} + \sum_{i=1}^{N} \int_{\Omega} \frac{C(\alpha)}{p'(x)} |F_i|^{p'(x)}\, dx$$
$$+ \sum_{i=1}^{N} \int_{\Omega} \frac{\alpha}{2p(x)} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx.$$

So,

$$\alpha \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx \leq k\|f\|_{L^1} + \sum_{i=1}^{N} \int_{\Omega} C(\alpha, p'^{-}) |F_i|^{p'(x)}\, dx$$
$$+ \sum_{i=1}^{N} \int_{\Omega} \frac{\alpha}{2p^{-}} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx.$$

Then

$$\left(1 - \frac{1}{2p^{-}}\right) \alpha \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx \leq k\left(\|f\|_{L^1} + \frac{C(\alpha, p'^{-})}{k} \sum_{i=1}^{N} \int_{\Omega} |F_i|^{p'(x)}\, dx \right)$$

for $k \geq 1$, which implies that

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx \leq Ck \quad \text{for all } k > 1. \tag{5.1}$$

STEP II: *Local convergence in measure of* u_n. We prove that $(u_n)_n$ converges to some function u locally in measure (and therefore we can always assume that the convergence is a.e. after passing to a subsequence). We shall show that $(u_n)_n$ is a Cauchy sequence in measure in any ball B_R.

For $k > 0$ large enough, we have

$$k \operatorname{meas}(\{|u_n| > k\} \cap B_R) = \int_{\{|u_n|>k\} \cap B_R} |T_k(u_n)|\, dx \leq \int_{B_R} |T_k(u_n)|\, dx$$
$$\leq C\|\nabla T_k(u_n)\|_{p(\cdot)} \leq C\left(\int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^{p(x)} dx \right)^{1/p_s}$$
$$\leq Ck^{1/p_s}$$

with

$$p_s = \begin{cases} p^- & \text{if } \|\nabla T_k(u_n)\|_{p(\cdot)} \le 1, \\ p^+ & \text{if } \|\nabla T_k(u_n)\|_{p(\cdot)} > 1, \end{cases}$$

which implies

$$\text{meas}(\{|u_n| > k\} \cap B_R) \le \frac{C}{k^{1-1/p_s}} \quad \text{for all } k > 1. \tag{5.2}$$

We have, for every $\delta > 0$,

$$\begin{aligned} \text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) \le \ & \text{meas}(\{|u_n| > k\} \cap B_R) \\ & + \text{meas}(\{|u_m| > k\} \cap B_R) \\ & + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \delta\}). \end{aligned} \tag{5.3}$$

Since $(T_k(u_n))_n$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, there exists v_k in $W_0^{1,p(\cdot)}(\Omega)$ such that

$$\begin{aligned} T_k(u_n) &\rightharpoonup v_k \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega), \\ T_k(u_n) &\to v_k \quad \text{strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega \text{ (by (2.1))}. \end{aligned}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$. Then by (5.2) and (5.3), there exists some $k(\varepsilon) > 0$ such that

$$\text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon \quad \text{for all } n, m \ge n_0(k(\varepsilon), \delta, R).$$

This proves that $(u_n)_n$ is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega), \\ T_k(u_n) &\to T_k(u) \quad \text{strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega \text{ (by (2.1))}. \end{aligned} \tag{5.4}$$

STEP III: *Equi-integrability of the nonlinearities.* We need to prove that

$$g_n(x, u_n) \to g(x, u) \quad \text{strongly in } L^1(\Omega). \tag{5.5}$$

It is enough to prove the equi-integrability of $g_n(x, u_n)$. We take $T_{l+1}(u_n) - T_l(u_n)$ as a test function in $(\mathcal{P}_n)$ to obtain

$$\begin{aligned} \int_\Omega a(x, T_n(u_n), \nabla u_n) \nabla (T_{l+1}(u_n) - T_l(u_n))\, dx & \\ + \int_\Omega g_n(x, u_n)(T_{l+1}(u_n) - T_l(u_n))\, dx & \\ = \int_\Omega f_n (T_{l+1}(u_n) - T_l(u_n))\, dx + \sum_{i=1}^N \int_\Omega F_i \nabla (T_{l+1}(u_n) - T_l(u_n))\, dx&, \end{aligned}$$

which implies that

$$\int_{\{l\le|u_n|\le l+1\}} a(x,T_n(u_n),\nabla u_n)\nabla u_n\,dx+\int_{\{|u_n|\ge l+1\}}|g_n(x,u_n)|\,dx$$
$$\le C\int_{\{|u_n|\ge l\}}|f_n|\,dx+\sum_{i=1}^{N}\int_{\{l\le|u_n|\le l+1\}}|F_i|\Big(\frac{\alpha}{2}\Big)^{-1/p(x)}|\nabla u_n|\Big(\frac{\alpha}{2}\Big)^{1/p(x)}\,dx.$$

By Young's inequality,

$$\int_{\{l\le|u_n|\le l+1\}} a(x,T_n(u_n),\nabla u_n)\nabla u_n\,dx+\int_{\{|u_n|\ge l+1\}}|g_n(x,u_n)|\,dx$$
$$\le C\int_{\{|u_n|\ge l\}}|f_n|\,dx+C(\alpha,p'^{-})\sum_{i=1}^{N}\int_{\{|u_n|\ge l\}}|F_i|^{p'(x)}\,dx$$
$$+\frac{\alpha}{2p^{-}}\sum_{i=1}^{N}\int_{l\le\{|u_n|\le l+1\}}\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p(x)}\,dx.$$

Thus, by the coercivity condition (H_3),

$$\int_{\{|u_n|\ge l+1\}}|g_n(x,u_n)|\,dx\le C\int_{\{|u_n|\ge l\}}|f_n|\,dx+C(\alpha,p'^{-})\sum_{i=1}^{N}\int_{\{|u_n|\ge l\}}|F_i|^{p'(x)}\,dx.$$

Let $\varepsilon>0$. Then there exists $l(\varepsilon)\ge 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}}|g_n(x,u_n)|\,dx\le\frac{\varepsilon}{2}. \tag{5.6}$$

For any measurable subset $E\subset\Omega$, we have

$$\int_E|g_n(x,u_n)|\,dx\le\int_{E\cap\{|u_n|\le l(\varepsilon)\}}|g_n(x,u_n)|\,dx+\int_{E\cap\{|u_n|>(\varepsilon)\}}|g_n(x,u_n)|\,dx$$
$$\le\int_E|h_{l(\varepsilon)}(x)|\,dx+\int_{E\cap\{|u_n|>(\varepsilon)\}}|g_n(x,u_n)|\,dx.$$

In view of (3.2) there exists $\eta(\varepsilon)>0$ such that

$$\int_E|h_{l(\varepsilon)}(x)|\,dx\le\frac{\varepsilon}{2} \tag{5.7}$$

for all E such that $\operatorname{meas}(E)<\eta(\varepsilon)$. Finally, by combining (5.6) and (5.7) one easily sees that

$$\int_E|g_n(x,u_n)|\,dx\le\varepsilon\quad\text{for all } E \text{ such that } \operatorname{meas}(E)<\eta(\varepsilon).$$

STEP IV: *The intermediate inequality.* In this step, we shall prove that for all $\varphi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_{\Omega} a(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)\,dx + \int_{\Omega} g_n(x,u_n)T_k(u_n-\varphi)\,dx \le \int_{\Omega} f_n T_k(u_n-\varphi)\,dx + \int_{\Omega} F\nabla T_k(u_n-\varphi)\,dx. \tag{5.8}$$

We now choose $T_k(u_n-\varphi)$ as a test function in $(\mathcal{P}_n)$, with φ in $W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and n large enough $(n\ge k+\|\varphi\|_\infty)$, to obtain

$$\int_{\Omega} a(x,T_n(u_n),\nabla u_n)\nabla T_k(u_n-\varphi)\,dx + \int_{\Omega} g_n(x,u_n)T_k(u_n-\varphi)\,dx = \int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx + \int_{\Omega} g_n(x,u_n)T_k(u_n-\varphi)\,dx,$$

which implies that

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx + \int_{\Omega} g_n(x,u_n)T_k(u_n-\varphi)\,dx = \int_{\Omega} f_n T_k(u_n-\varphi)\,dx + \int_{\Omega} F\nabla T_k(u_n-\varphi)\,dx.$$

Note that since $n\ge k+\|\varphi\|_\infty$, we have $T_n(u_n)=u_n$.

Adding and subtracting the term $\int_\Omega a(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)\,dx$ yields

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx + \int_{\Omega} a(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)\,dx - \int_{\Omega} a(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)\,dx + \int_{\Omega} g_n(x,u_n)T_k(u_n-\varphi)\,dx = \int_{\Omega} f_n T_k(u_n-\varphi)\;\,dx + \int_{\Omega} F\nabla T_k(u_n-\varphi)\,dx. \tag{5.9}$$

Thanks to the weak monotonicity condition (H_2) and the definition of the truncation function, we have

$$\int_{\Omega} (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\varphi))\nabla T_k(u_n-\varphi)\,dx \ge 0. \tag{5.10}$$

Combining (5.9) and (5.10), we obtain (5.8).

Step V: *Passing to the limit.* We shall prove that for $\varphi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_{\Omega} a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\,dx + \int_{\Omega} g(x,u)T_k(u-\varphi)\,dx \le \int_{\Omega} fT_k(u-\varphi)\,dx + \int_{\Omega} F\nabla T_k(u-\varphi)\,dx.$$

First, we claim that

$$\int_\Omega a(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)\,dx \to \int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\,dx \quad \text{as } n\to\infty.$$

Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^{1,p(\cdot)}(\Omega)$, with $M = k+\|\varphi\|_\infty$, we have

$$T_k(u_n-\varphi) \rightharpoonup T_k(u-\varphi) \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega), \tag{5.11}$$

which gives

$$\frac{\partial T_k}{\partial x_i}(u_n-\varphi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(u-\varphi) \quad \text{weakly in } L^{p(\cdot)}(\Omega) \text{ for all } i=1,\ldots,N. \tag{5.12}$$

Now, thanks to (H_1),

$$|a_i(x,T_M(u_n),\nabla\varphi)|^{p'(x)} \le \big(k(x)+|T_M(u_n)|^{p(x)-1}+(\gamma_0|\nabla\varphi|)^{p(x)-1}\big)^{p'(x)},$$

thus

$$|a_i(x,T_M(u_n),\nabla\varphi)|^{p'(x)} \le \beta\big(k(x)^{p'(x)}+|T_M(u_n)|^{p(x)}+\gamma_0^{p(x)}|\nabla\varphi|^{p(x)}\big), \tag{5.13}$$

with $\gamma_0=\sup\{|\gamma(s)| : |s|\le k+\|\varphi\|_\infty\}$, and β a positive constant.

Now, since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^{1,p(\cdot)}(\Omega)$ and by (2.1), we have

$$T_M(u_n)\to T_M(u) \quad \text{strongly in } L^{p(\cdot)}(\Omega).$$

Thus,

$$|a_i(x,T_M(u_n),\nabla\varphi)|^{p'(x)} \to |a_i(x,T_M(u),\nabla\varphi)|^{p'(x)} \quad \text{a.e. in } \Omega,$$

and

$$\beta\big(k(x)^{p'(x)}+|T_M(u_n)|^{p(x)}+\gamma_0^{p(x)}|\nabla\varphi|^{p(x)}\big) \to \beta\big(k(x)^{p'(x)}+|T_M(u)|^{p(x)}+\gamma_0^{p(x)}|\nabla\varphi|^{p(x)}\big),$$

a.e. in Ω. According to Vitali's theorem, we deduce that

$$a_i(x,T_M(u_n),\nabla\varphi)\to a_i(x,T_M(u),\nabla\varphi) \text{ strongly } L^{p'(\cdot)}(\Omega) \text{ as } n\to\infty. \tag{5.14}$$

Combining (5.11), (5.14) and the fact that $T_M(u_n)=u_n$ (since $M=k+\|\varphi\|_\infty$), one has

$$\int_\Omega a(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)\,dx \to \int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\,dx \quad \text{as } n\to\infty. \tag{5.15}$$

Secondly, we show that

$$\int_\Omega f_nT_k(u_n-\varphi)\,dx \to \int_\Omega fT_k(u-\varphi)\,dx. \tag{5.16}$$

We have $f_nT_k(u_n-\varphi)\to fT_k(u-\varphi)$ a.e. in Ω and $|f_nT_k(u_n-\varphi)|\le k|f_n|$ and $k|f_n|\to k|f|$ in $L^1(\Omega)$. By using Vitali's theorem a second time, we obtain (5.16).

Similarly thanks to (5.5), we can show that

$$\int_\Omega g_n(x,u_n)T_k(u_n-\varphi)\,dx \to \int_\Omega g(x,u)T_k(u-\varphi)\,dx \quad \text{as } n\to\infty. \tag{5.17}$$

In view of (5.12) and since $F\in\prod_{i=1}^N L^{p'(\cdot)}(\Omega)$, we obtain

$$\int_\Omega F\nabla T_k(u_n-\varphi)\,dx \to \int_\Omega F\nabla T_k(u-\varphi)\,dx \quad \text{as } n\to\infty. \tag{5.18}$$

Thanks to (5.15), (5.16) and (5.18) we can pass to the limit in (5.8), so that for all $\varphi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we deduce

$$\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\varphi)\,dx \le \int_\Omega fT_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx. \tag{5.19}$$

Now we introduce an L^1-version of the Minty lemma.

Lemma 5.3. *Let u be a measurable function such that $T_k(u)$ belongs to $W_0^{1,p(\cdot)}(\Omega)$ for every $k>0$. The following assertions are equivalent:*

(i) $$\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\varphi)\,dx \le \int_\Omega fT_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx,$$

(ii) $$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\varphi)\,dx = \int_\Omega fT_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx,$$

for all $\varphi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and for all $k>0$.

In view of Lemma 5.3, the proof of Theorem 5.1 is finished.

Proof of Lemma 5.3. (ii)⇒(i). This is easily proved by adding and subtracting

$$\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)\,dx$$

and then using the weak monotonicity condition (H_2).

(i)⇒(ii). Let h and k be positive real numbers, let $\lambda\in\,]-1,1[$ and $\psi\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$. Choosing

$$\varphi = T_h(u-\lambda T_k(u-\psi))\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$$

as a test function in (i), we have

$$I_{hk} \le J_{hk} \tag{5.20}$$

with

$$I_{hk} = \int_\Omega a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx$$
$$+\int_\Omega g(x,u)T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx = I'_{hk}+I''_{hk},$$

and

$$J_{hk} = \int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx+\int_\Omega F\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx.$$

We set

$$A_{hk} = \{x\in\Omega : |u-T_h(u-\lambda T_k(u-\psi))|\le k\},$$
$$B_{hk} = \{x\in\Omega : |u-\lambda T_k(u-\psi)|\le h\}.$$

Then we obtain

$$I'_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx$$
$$+\int_{A_{kh}\cap B^C_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx$$
$$+\int_{A^C_{kh}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx.$$

Since $\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))$ is different from zero only on A_{kh}, we have

$$\int_{A^C_{kh}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx = 0. \tag{5.21}$$

Moreover, if $x\in B^C_{hk}$, we have $\nabla T_h(u-\lambda T_k(u-\psi)) = 0$ and using the coercivity condition (H_3), we deduce that

$$\int_{A_{kh}\cap B^C_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx$$
$$= \int_{A_{kh}\cap B^C_{hk}} a(x,u,0)\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx = 0. \tag{5.22}$$

From (5.21) and (5.22), we obtain

$$I'_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx.$$

Letting $h \to \infty$, and $|\lambda| \le 1$, we have

$$A_{kh} \to \{x : |\lambda|\,|T_k(u-\psi)| \le k\} = \Omega,$$
$$B_{hk} \to \Omega, \quad \text{which implies} \quad A_{kh} \cap B_{hk} \to \Omega.$$

By using Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{h\to\infty} \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx$$
$$= \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\psi))\nabla T_k(u-\psi)\,dx,$$

which implies that

$$\lim_{h\to\infty} I'_{hk} = \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\psi))\nabla T_k(u-\psi)\,dx.$$

Moreover, it is easy to see that

$$\lim_{h\to\infty} \int_\Omega g(x,u)T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx = \lambda \int_\Omega g(x,u)T_k(u-\psi)\,dx,$$

which implies that

$$\lim_{h\to+\infty} I_{hk} = \lambda \int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\psi))\nabla T_k(u-\psi)\,dx + \lambda \int_\Omega g(x,u)T_k(u-\psi)\,dx. \tag{5.23}$$

On the other hand,

$$J_{hk} = \int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx + \int_\Omega F\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx.$$

Thus,

$$\lim_{h\to\infty} \int_\Omega fT_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx + \int_\Omega F\nabla T_k(u-T_h(u-\lambda T_k(u-\psi)))\,dx$$
$$= \lambda \int_\Omega fT_k(u-\psi)\ \ dx + \lambda \int_\Omega F\nabla T_k(u-\psi)\,dx,$$

which implies that

$$\lim_{h\to\infty} J_{hk} = \lambda \int_\Omega fT_k(u-\psi)\ \ dx + \lambda \int_\Omega F\nabla T_k(u-\psi)\,dx. \tag{5.24}$$

Using (5.23), (5.24) and passing to the limit in (5.20), we obtain

$$\lambda\Big(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\psi)\nabla T_k(u-\psi)\,dx + \int_\Omega g(x,u)T_k(u-\psi)\,dx\Big)$$
$$\le \lambda\Big(\int_\Omega fT_k(u-\psi)\,dx + \int_\Omega F\nabla T_k(u-\psi)\,dx\Big)$$

for every $\psi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and for $k > 0$. Choosing $\lambda > 0$, dividing both sides by λ, and then letting λ tend to zero, we obtain

$$(5.25) \quad \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\psi)\,dx \leq \int_\Omega f T_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx.$$

Doing the same for $\lambda < 0$, we obtain

$$(5.26) \quad \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\psi)\,dx \geq \int_\Omega f T_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx.$$

Combining (5.25) and (5.26), we conclude that

$$(5.27) \quad \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u)T_k(u-\psi)\,dx = \int_\Omega f T_k(u-\varphi)\,dx + \int_\Omega F\nabla T_k(u-\varphi)\,dx.$$

This completes the proof of Lemma 5.3.

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