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IMPROVED BALL CONVERGENCE OF NEWTON'S METHOD UNDER GENERAL CONDITIONS

Abstract. We present ball convergence results for Newton's method in order to approximate a locally unique solution of a nonlinear operator equation in a Banach space setting. Our hypotheses involve very general majorants on the Fréchet derivatives of the operators involved. In the special case of convex majorants our results, compared with earlier ones, have at least as large radius of convergence, no less tight error bounds on the distances involved, and no less precise information on the uniqueness of the solution.

1. Introduction. Let X and Y be Banach spaces and let D be a nonempty convex subset of X. In the present paper $F : D \subseteq X \to Y$ is Fréchet-differentiable.

Many problems in computational sciences can be brought in the form of the nonlinear equation

$$F(x) = 0 \tag{1.1}$$

using mathematical modeling.

The solutions $x^\star$ of equation (1.1) can rarely be found in closed form. That is why most solution methods for these equations are iterative.

The most popular iterative procedure for generating a sequence approximating $x^\star$ is undoubtedly Newton's method:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad n \geq 0,\, x_0 \in \mathcal{D}. \tag{1.2}$$

Here $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, denotes the Fréchet derivative of the operator F [4], [10].

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Newton's method converges quadratically to $x^\star$ (under certain conditions [3], [4], [10]), and requires one function evaluation and one inverse at each step. Therefore the efficiency index $E = p^{1/q}$ (p is the order of convergence, and q the number of function evaluations per iterative step) is $E = \sqrt{2}$ [4], [14].

The convergence for Newton-type methods under very general conditions has been studied by several authors. A survey of recent results can be found in [4], [10] (and the references there; see also [3], [13], [14]).

In this study we suppose that the nonlinear operator equation (1.1) has a solution $x^\star$. An interesting problem is to estimate the radius of the convergence ball of Newton's method (1.2). An open ball $U(x^\star, r) \subseteq \mathcal{D}$ with center $x^\star$ and radius r is called a *convergence ball* of an iterative method if the sequence generated by this iterative method starting from any initial value in it converges. The convergence ball of an iterative method is very important in computational mathematics, because it shows the extent of difficulty of choosing initial points. Ball convergence theorems can be immediately obtained by specializing results on Newton-type methods in the case of Newton's method [1]–[7], [9], [12]. However, a more direct approach is expected to generate more exact results.

In particular, we are motivated by optimization considerations and the work of Ferreira [8], where it is claimed that the best possible radius of convergence can be obtained for Newton's method using the information (F, F') under convex majorants.

In this paper, we introduce even more general majorant conditions that do not necessarily imply the convexity of the functions involved. We provide a ball convergence result for Newton's method (1.2). If we specialize our theorem in the case of convex majorants we obtain a result which, in comparison with earlier ones [8], [12]–[15] (under the same hypotheses and computational cost), has

(a) at least as large radius of convergence;
(b) no less tight error estimates on the distance $\|x_n - x^\star\|$ $(n \geq 0)$;
(c) no less precise information on the uniqueness of the solution $x^\star$.

Advantages (a)–(c) are important in computational mathematics, since they allow a wider choice of initial guesses x_0, and the computation of fewer iterative steps to obtain the same desired error tolerance. Numerical examples further validating the theoretical results are also provided in this study.

2. Ball convergence of Newton's method. We shall show the main local convergence result for Newton's method (1.2).

THEOREM 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{D} \subseteq \mathcal{X}$ an open set, and $F : \mathcal{D} \to \mathcal{Y}$ a continuously Fréchet-differentiable operator. Let $x^\star \in \mathcal{D}$, $R > 0$, and $\alpha = \sup\{t \in [0, R) : U(x^\star, t\} \subset \mathcal{D}\}$. Suppose that $F(x^\star) = 0$, $F'(x^\star)$ is invertible, and there exist functions $f, f_0 : [0, R) \to \mathbb{R}$ continuously differentiable such that for all $\theta \in [0, 1]$, $x \in U(x^\star, \alpha)$:*

(H1) $\|F'(x^\star)^{-1} \ (F'(x) - F'(x^\star + \theta(x - x^\star)))\| \le f'(\|x - x^\star\|) - f'(\theta\|x - x^\star\|)$;

(H2) $\|F'(x^\star)^{-1} \ (F'(x) - F'(x^\star))\| \le f_0'(\|x - x^\star\|) - f_0'(0)$;

(H3) *let* $\beta = \sup\{t \in [0, R) : f_0'(t) < 0\}$, $\gamma = \sup\{t \in (0, \beta) : r_{f_0,f} \le 1\}$, $\delta = \min\{\alpha, \gamma\}$, *where* $r_{f_0,f} : [0, \delta)^3 \to \mathbb{R}$ *is given for* $0 \le t < v < \delta$, $x \in U(x^\star, \delta)$ *by*

$$r_{f_0,f} = r_{f_0,f}(t, v, \|x - x^\star\|) = -\frac{e(t, v)}{f_0'(\|x - x^\star\|)},$$
$$e(t, v) = f(v) - f(t) - f'(v)(v - t);$$

(H4) *the functions* $t \mapsto f'(t) - f'(\theta t)$, $t \mapsto f_0'(t) - f_0'(0)$ *are increasing for all* $t \in [0, \infty)$, $\theta \in [0, 1]$;

(H5) $f_0(t) \le f(t)$ *and* $f_0'(t) \le f'(t)$ *for* $t \in [0, R)$;

(H6) $f_0'(0) \ge -1$.

Then the sequence $\{x_n\}$ generated by Newton's method (1.2) *for $x_0 \in U(x^\star, \delta) - \{x^\star\}$ is well defined, remains in $U(x^\star, \delta)$ for all $n \ge 0$, and converges to $x^\star$, which is the unique solution of $F(x) = 0$ in $U(x^\star, \alpha)$.*

From now on we assume that the hypotheses of Theorem 2.1 hold. We shall prove Theorem 2.1 through a series of lemmas:

LEMMA 2.2. *If $x \in \overline{U}(x^\star, t)$ and $t \in [0, \beta_0)$, where $\beta_0 = \min\{\alpha, \beta\}$, then $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$, and*

$$\|F'(x)^{-1}F'(x^\star)\| \le -\frac{1}{f_0'(\|x - x^\star\|)} \le -\frac{1}{f_0'(t)} \le -\frac{1}{f'(t)}. \tag{2.1}$$

Proof. Using hypotheses (H2)–(H6), we obtain

$$\begin{aligned} \|F'(x^\star)^{-1} \ (F'(x) - F'(x^\star))\| &\le f_0'(\|x - x^\star\|) - f_0'(0) \\ &\le f_0'(\|x - x^\star\|) + 1 \le f_0'(t) + 1 < 1. \end{aligned} \tag{2.2}$$

It follows from (2.2), and the Banach lemma on invertible operators [4], [10] that $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$, so that (2.1) holds true. ∎

LEMMA 2.3. *Let $x \in U(x^\star, t)$, and $0 \le t < v < \beta_0$. Set*

$$\begin{aligned} E(x^\star, x) &= F(x) - F(x^\star) - F'(x)(x - x^\star) \\ &= \int_0^1 [F'(x^\star + \theta(x - x^\star)) - F'(x)](x - x^\star)\, d\theta. \end{aligned}$$

Then

$$\|F'(x^\star)^{-1}E(x^\star,x)\| \le e(t,v)\|x^\star - x\|, \tag{2.3}$$

$$r_{f_0,f} \le 1, \tag{2.4}$$

where $e(t,v)$ *and* $r_{f_0,f}$ *are given in* (H3).

Proof. Using (H1), (H4), and the definition of E we obtain

$$\begin{aligned} \|F'(x^\star)^{-1}E(x^\star,x)\| &\le \int_0^1 \|F'(x^\star)^{-1}(F'(x^\star+\theta\,(x-x^\star)) - F'(x))\|\,\|x-x^\star\|\,d\theta \\ &\le \int_0^1 (f'(\|x-x^\star\|) - f'(\theta\|x-x^\star\|))\|x-x^\star\|\,d\theta, \end{aligned} \tag{2.5}$$

which implies estimate (2.3).

In view of (H5), we get

$$\frac{\int_0^1(f'(t)-f'(\theta t))\,d\theta}{-f_0'(t)} \le 1, \tag{2.6}$$

which together with (H3) implies (2.4). ∎

Proof of Theorem 2.1. According to Lemmas 2.2 and 2.3, it remains to show $x_n \in U(x^\star,\delta)$ $(n \ge 1)$, $\lim_{n\to\infty} x_n = x^\star$, and the uniqueness part.

By hypothesis $x_0 \in U(x^\star,\delta)$. Let us assume $x_k \in U(x^\star,\delta)$ for all $k \le n$. We shall show $x_{k+1} \in U(x^\star,\delta)$. Using (1.2), and Lemma 2.3 for $x = x_n$, we get

$$\|x_{k+1} - x^\star\| < \|x_k - x^\star\| < \delta, \tag{2.7}$$

which shows $x_{k+1} \in U(x^\star,\delta)$, and $\lim_{k\to\infty} x_k = x^\star$.

Finally, to show uniqueness in $U(x^\star,\alpha)$, let $y^\star$ be a solution of $F(x)=0$ in $U(x^\star,\alpha)$. Define a linear operator $\mathcal{M}$ by

$$\mathcal{M} = \int_0^1 F'(x^\star + \theta(y^\star - x^\star))\,d\theta.$$

Using (2.1) with $x^\star + \theta\,(y^\star - x^\star) \in U(x^\star,\alpha)$ replacing x, we conclude $\mathcal{M}^{-1}$ exists. It then follows from the identity

$$F(y^\star) - F(x^\star) = \mathcal{M}\,(y^\star - x^\star) \tag{2.8}$$

that $x^\star = y^\star$. ∎

If f_0, f satisfy certain convexity conditions, then Proposition 2.4 below specializes to Theorem 2.1 in [8, p. 748]. The proofs are omitted, since they follow from the corresponding ones in [8], where the estimate

$$\|F'(x)^{-1}F'(x^\star)\| \le -\frac{1}{f'(\|x-x^\star\|)} \tag{2.9}$$

was used instead of the at least as tight

$$\|F'(x)^{-1}F'(x^\star)\| \le -\frac{1}{f_0'(\|x-x^\star\|)} \tag{2.10}$$

(see Lemma 2.2).

Proposition 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{D} \subseteq \mathcal{X}$ an open set, and $F : \mathcal{D} \to \mathcal{Y}$ a continuously Fréchet-differentiable operator. Let $x^\star \in \mathcal{D}$, $R > 0$, and $\alpha = \sup\{t \in [0,R) : U(x^\star,t) \subset \mathcal{D}\}$. Suppose that $F(x^\star) = 0$, $F'(x^\star)$ is invertible, and there exist functions $f, f_0 : [0,R) \to \mathbb{R}$ twice continuously differentiable such that for all $x \in U(x^\star,\alpha)$, $\theta \in [0,1]$:*

- (H1) $\|F'(x^\star)^{-1}(F'(x) - F'(x^\star + \theta(x - x^\star)))\| \le f'(\|x-x^\star\|) - f'(\theta\|x-x^\star\|)$;
- (H2) $\|F'(x^\star)^{-1}(F'(x) - F'(x^\star))\| \le f_0'(\|x-x^\star\|) - f_0'(0)$;
- ($\overline{\text{H}}$3) *let* $\beta = \sup\{t \in [0,R) : f_0'(t) < 0\}$, $\gamma = \sup\{t \in (0,\beta) : \overline{r}_{f_0,f} \le 1\}$, $\delta = \min\{\alpha,\gamma\}$, *where* $\overline{r}_{f_0,f} : [0,\delta)^3 \to \mathbb{R}$ *is given for* $0 \le t < v < \delta$, $x \in U(x^\star,\delta)$ *by*
$$\overline{r}_{f_0,f} = \overline{r}_{f_0,f}(t,v,\|x-x^\star\|) = -\frac{e(t,v)\|x-x^\star\|}{(v-t)^2 f_0'(\|x-x^\star\|)},$$
$$e(t,v) = f(v) - f(t) - f'(v)(v-t);$$
- ($\overline{\text{H}}$4) f_0', f' *are convex and strictly increasing functions on* $[0,R)$;
- (H5) $f_0(t) \le f(t)$ *and* $f_0'(t) \le f'(t)$ *for* $t \in [0,R)$;
- ($\overline{\text{H}}$6) $f_0(0) = f(0) = 0$ *and* $f_0'(0) = f'(0) = -1$.

Then the sequence $\{x_n\}$ generated by Newton's method (1.2) *is well defined, remains in $U(x^\star,\alpha)$ for all $n \ge 0$, and converges to a unique solution $x^\star$ in $U(x^\star,\lambda)$ of $F(x) = 0$, where*

$$\lambda = \sup\{0 < t < \alpha : f_0(t) < 0\}. \tag{2.11}$$

Moreover, the scalar sequence $\{t_n\}$ given for $t_0 = \|x^\star - x_0\|$ by

$$t_{n+1} = \left|t_n - \frac{f(t_n)}{f'(t_n)}\right| \left|\frac{f'(t_n)}{f_0'(t_n)}\right| \quad (n \ge 0) \tag{2.12}$$

is well defined, remains in $(0,\alpha)$, is strictly decreasing, and converges to zero.

Furthermore, $\{t_{n+1}/t_n^2\}$ is strictly decreasing,

$$\|x^\star - x_{n+1}\| \le \frac{t_{n+1}}{t_n^2}\|x^\star - x_n\|^2 \quad (n \ge 0), \tag{2.13}$$

$$\|x^\star - x_n\| \le t_0\left(\frac{t_1}{t_0}\right)^{2^n-1} \quad (n \ge 0), \tag{2.14}$$

and

$$\frac{t_{n+1}}{t_n^2} \le \frac{f''(t_0)}{2|f_0'(t_0)|} \qquad (n \ge 0). \tag{2.15}$$

If, additionally $\overline{r}_{f_0,f}(\gamma,\gamma,\gamma) = 1$, *and* $\gamma < \alpha$, *then* $\alpha = \gamma$ *is the best possible convergence radius.*

Proof. As noted above, the proof follows as in Theorem 2.1 in [8] (simply replace (2.9) by (2.10) in the computation of the upper bounds of the norm $\|F'(x)^{-1}F'(x^\star)\|$) with the exception of the uniqueness part. We have

$$y^\star - x^\star = -\int_0^1 F'(x^\star)^{-1}[F'(x^\star + \theta(y^\star - x^\star)) - F'(x^\star)](y^\star - x^\star)\,d\theta. \tag{2.16}$$

Using (H2), ($\overline{\mathrm{H}}$4), ($\overline{\mathrm{H}}$6) and (2.16) with $x = x^\star + \theta(y^\star - x^\star)$, and $\theta = 0$, we get

$$\begin{aligned} \|y^\star - x^\star\| &\le \int_0^1 [f_0'(\theta\|y^\star - x^\star\|) - f_0'(0)]\|y^\star - x^\star\|\,d\theta \\ &= f_0(\|y^\star - x^\star\|) - f_0(0) - f_0'(0)\|y^\star - x^\star\|, \end{aligned} \tag{2.17}$$

which implies

$$f_0(\|y^\star - x^\star\|) \ge 0. \tag{2.18}$$

The function f_0 is strictly convex and $f(t) < 0$ in $[0,\alpha)$. That is, 0 is the unique solution of $f_0(t) = 0$ in $[0,\alpha)$. Hence, estimate (2.18) implies $\|y^\star - x^\star\| = 0$. Thus, $x^\star = y^\star$. ■

Remark 2.5. If $f_0' = f'$ on $[0,R)$, then Proposition 2.4 reduces to Theorem 2.1 in [8]. Otherwise, i.e. if

$$f_0'(t) < f'(t), \qquad t \in [0,R), \tag{2.19}$$

then our results are finer with advantages as stated in the abstract.

Note that (H2) is not an addition to (H1), since f_0 always exists in this case. Hence, through Theorem 2.1 we have studied the local convergence of Newton's method under very general majorants (not necessarily convex).

Moreover, using Proposition 2.4 we expanded the applicability of Theorem 2.1 using the same hypotheses on (F, F'), and convex majorants.

In the next section, we provide numerical examples where (2.19) holds.

3. Applications

Example 3.1. Assume there exists $L > 0$ such that

$$\|F'(x^\star)^{-1}(F'(y) - F'(x))\| \le L\|x - y\| \quad \text{for all } x, y \in \overline{U}(x_0, R) \subseteq \mathcal{D}. \tag{3.1}$$

Define a scalar majorant function $f : [0, R] \to (-\infty, \infty)$ by

$$f(t) = \frac{L}{2}t^2 - t \tag{3.2}$$

and set

$$f_0(t) = f(t), \quad t \in [0, R]. \tag{3.3}$$

It then follows from Proposition 2.4 (or Theorem 2.1 in [8]) that we can set

$$r_{RTW} = \frac{2}{3L}, \tag{3.4}$$

which is the radius of convergence obtained by Rheinboldt [12], [4], and Traub [14].

It follows from (3.1) that there exists $L_0 > 0$ such that

$$\|F'(x^\star)^{-1}(F'(x) - F'(x^\star))\| \le L_0\|x - x^\star\| \quad \text{for all } x \in \overline{U}(x_0, R). \tag{3.5}$$

Clearly

$$L_0 \le L \tag{3.6}$$

and L/L_0 can be arbitrarily large [2]–[4].

Define

$$f_0(t) = \frac{L_0}{2}t^2 - t. \tag{3.7}$$

It then follows from Proposition 2.4 that we can set

$$r_{AH} = \frac{2}{2L_0 + L}. \tag{3.8}$$

By comparing (3.4) with (3.8) we conclude that

$$r_{RTW} \le r_{AH}. \tag{3.9}$$

If strict inequality holds in (3.6), then so is the case in (3.9). Note also that

$$\frac{r_{RTW}}{r_{AH}} = \frac{2\frac{L_0}{L} + 1}{3} \to \frac{1}{3} \quad \text{as } \frac{L_0}{L} \to 0.$$

Hence, our approach triples (at most) the radius of convergence given by (3.4) ([3], [4], [8], [10], [12]–[15]).

Example 3.2. Let $f : [0, R) \to (-\infty, \infty)$ be a twice continuously differentiable function with f' convex. Then F satisfies (2.2) if and only if

$$\|F'(x^\star)^{-1}F''(x)\| \le f''(\|x - x^\star\|) \quad \text{for all } x \in \mathcal{D} \cap U(x^\star, R) \tag{3.10}$$

(see Lemma 14 in [8] or [15]).

Define a function f on $[0, R)$ by

$$f(t) = \frac{\gamma t^2}{1 - \gamma t} - t, \tag{3.11}$$

where $R < 1/\gamma$ for some $\gamma > 0$.

If for example F is an analytic operator, then (3.10) is satisfied for

$$\gamma^\star = \sup_{k\geq 2} \left\| \frac{F'(x^\star)^{-1}F^{(k)}(x^\star)}{k!} \right\|^{1/(k-1)}. \tag{3.12}$$

Smale [13] and Wang [15] have used (3.11) to provide a convergence analysis for Newton's method (1.2). In particular Wang [15] showed convergence for F being only twice Fréchet continuously differentiable for γ satisfying

$$\gamma^\star \leq \gamma. \tag{3.13}$$

We have also used (3.11) to provide a convergence analysis for the secant method [5] (see also [4]).

Define

$$f_0(t) = f(t), \qquad t \in [0, R). \tag{3.14}$$

For analytic operators F we obtain Smale's radius of convergence [13]:

$$t_S^\star = \frac{5-\sqrt{13}}{6\gamma^\star}, \tag{3.15}$$

and for twice Fréchet continuously differentiable operator F we obtain Wang's radius [15]:

$$t_W^\star = \frac{5-\sqrt{13}}{6\gamma}. \tag{3.16}$$

In what follows we shall show that we can enlarge the radii given by (3.15) and (3.16).

We can see that for f given by (3.11), condition (3.10) or equivalently (2.2) implies that there exists $\gamma_0 > 0$ satisfying

$$\gamma_0 \leq \gamma, \tag{3.17}$$

so that $f_0 : [0, 1/\gamma_0) \to (-\infty, \infty)$ satisfies condition (2.1) for $R \in [0, 1/\gamma_0)$. Note also that γ/γ_0 can be arbitrarily large [2]–[4]. It follows by (3.17) that there exists $a \in [0, 1]$ such that

$$\gamma_0 = a\gamma. \tag{3.18}$$

Set

$$b = 1 - a, \tag{3.19}$$

and define a scalar polynomial P_a by

$$P_a(t) = 3a^2t^3 + a(6b-a)t^2 + (3b^2 - 2ab - 1)t - b^2. \tag{3.20}$$

Then for fixed a, we get

$$P_a(0) = -b^2 \leq 0 \quad \text{and} \quad P_a(1) = 1. \tag{3.21}$$

Using (3.21) and the intermediate value theorem we conclude that there exists $t_a \in [0, 1)$ such that $P_a(t_a) = 0$. Denote by t_a the minimal number in

$[0,1)$ satisfying $P_a(t_a) = 0$. Define

$$t_a^\star = \frac{1 - t_a}{\gamma}. \tag{3.22}$$

In particular for $a = 1$, $t_1 = (1 + \sqrt{13})/6$, and consequently

$$t_a^\star = \frac{5 - \sqrt{13}}{6\gamma} = t_W^\star. \tag{3.23}$$

It is simple algebra to show that for all $a \in [0,1]$, $P_a(t_1) \geq 0$, which implies

$$t_a \leq t_1 \tag{3.24}$$

and

$$t_1^\star \leq t_a^\star. \tag{3.25}$$

We also note that strict inequality holds in (3.24) and (3.25) for $a \neq 1$.

As an example, let $a = 1/2$. Then we obtain

$$\begin{aligned} t_{1/2} &= 0.65185 < t_1 = \frac{1 + \sqrt{13}}{6} = 0.76759, \\ t_1^\star &= \frac{0.23241}{\gamma} < \frac{0.34815}{\gamma} = t_{1/2}^\star. \end{aligned} \tag{3.26}$$

Finally note that clearly if strict inequality holds in (2.4), i.e., in (3.6) or (3.17), then our estimates on $\|x_{n+1} - x^\star\|$ $(n \geq 0)$ are finer (more precise) than the corresponding ones in [1], [8], [10], [12]–[15] (see e.g. (2.10)).

These results are also obtained under the same computational cost since in practice the evaluation of L (or γ) requires that of L_0 (or γ_0).

REMARK 3.3. As noted in [2]–[6], [11], [14], [16] the local results obtained here can be used for projection methods such us Arnold's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite projection methods, and in connection with the mesh independence principle to develop the cheapest and most efficient mesh refinement strategies.

REMARK 3.4. The local results obtained can also be used to solve equations of the form $F(x) = 0$, where F' satisfies the autonomous differential equation ([4], [10])

$$F'(x) = T(F(x)), \tag{3.27}$$

where $T : \mathcal{Y} \to \mathcal{X}$ is a known continuous operator. Since $F'(x^\star) = T(F(x^\star))$ $= T(0)$, we can apply our results without actually knowing the solution $x^\star$ of $F(x) = 0$.

EXAMPLE 3.5. Let $\mathcal{X} = \mathcal{Y} = (-\infty, \infty)$, $\mathcal{D} = \overline{U}(0,1)$, and define a function F on $\mathcal{D}$ by

$$F(x) = e^x - 1. \tag{3.28}$$

Then, for $x^\star = 0$, we can set $T(x) = x + 1$ in (3.27). Using (3.1), (3.7) and (3.28) we obtain $L_0 = e - 1 < L = e$. In view of (3.4) and (3.8), we get

$$r_{RTW} = 0.2571658439 < r_{AH} = 0.3249472314.$$

Hence, (2.19) holds and our radius of convergence is larger than the one provided in [8], [10], [12]–[15].

Example 3.6. Let $\mathcal{X} = \mathcal{Y} = C[0,1]$, the space of continuous functions defined on $[0,1]$, and equipped with the max-norm. Let $\mathcal{D} = U(0,1)$. Define a function F on $\mathcal{D}$ by

$$F(h)(x) = h(x) - 5\int_0^1 x\theta h^3(\theta)\,d\theta.$$

Then

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\theta h^2(\theta)u(\theta)\,d\theta \qquad \text{for all } u \in \mathcal{D}.$$

Then, as in Example 3.5, we get for $h^\star(x) = 0$, $L_0 = 7.5 < L = 15$,

$$r_{RTW} = 0.0\overline{4} < 0.0\overline{6} = r_{AH}.$$

Note again that (2.19) holds, and our convergence radius is larger than before ([8], [10], [12]–[15]).

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