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ENTROPY SOLUTIONS FOR NONHOMOGENEOUS ANISOTROPIC $\Delta_{\vec{p}(\cdot)}$ PROBLEMS

Abstract. We study a class of anisotropic nonlinear elliptic equations with variable exponent $\vec{p}(\cdot)$ growth. We obtain the existence of entropy solutions by using the truncation technique and some a priori estimates.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$. Our aim is to prove the existence of entropy solutions for the anisotropic nonlinear elliptic problem

$$(1.1)\qquad \begin{cases} Au + |u|^{p_0(x)-2}u = f - \operatorname{div}\phi(u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $Au = -\sum_{i=1}^{N} D^i(|D^i u|^{p_i(x)-2} D^i u)$, $f \in L^1(\Omega)$ and $\phi(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$.

The function $\phi(u)$ does not belong in $(L^1_{\mathrm{loc}}(\Omega))^N$ because the function $\phi(\cdot)$ is just assumed to be continuous on $\mathbb{R}$, so that proving existence of a weak solution seems to be an arduous task. To overcome this difficulty, we use some techniques in the framework of entropy solutions.

In the particular case when $p_i(\cdot) = p(\cdot)$ for any $i \in \{0, 1, \ldots, N\}$, the operator involved in (1.1) is the $p(\cdot)$-Laplace operator, i.e. $\Delta_{p(\cdot)}(u) := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. This differential operator is a natural generalization of the isotropic p-Laplace operator $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(\cdot)$-Laplace operator possesses a more complicated nonlinearity than the p-Laplace operator, due to the fact that $\Delta_{p(\cdot)}$ is not homogeneous.

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Anisotropic problems like (1.1) have strong physical background. They emerge, for instance, from the mathematical description of the dynamics of fluids with different conductivities in different directions. We refer to the extensive books by Antontsev, Díaz and Shmarev [1] and Bear [4] for discussions in this direction. They also appear in biology, as a model describing the spread of an epidemic disease through a heterogeneous habitat.

In this paper, we will extend the results of [2, 3, 6] to the anisotropic variable exponent case, and our main ideas and methods come from [3, 7, 10]. The outline of this paper is as follows. In Section 2, we give some mathematical preliminaries and in Section 3 we prove the existence of entropy solutions.

2. A brief overview on variable exponent spaces. This section is related to anisotropic Lebesgue and Sobolev spaces with variable exponent (cf. [10]) that will enable us to study the problem (1.1) with sufficient accuracy. For a deeper treatment of Lebesgue and Sobolev spaces with variable exponent we refer to [2, 8, 11] and references therein.

Let

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{measurable function } p(\cdot) : \overline{\Omega} \to \mathbb{R} \mid 1 < p^- \leq p^+ < \infty\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) \mid x \in \overline{\Omega}\}, \qquad p^+ = \operatorname{ess\,sup}\{p(x) \mid x \in \overline{\Omega}\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite, and the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The Banach space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is separable and uniformly convex, hence reflexive. Its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Finally, we have the *generalized Hölder inequality*

$$\Big|\int_\Omega uv\, dx\Big| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{2.1}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

Let

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space.

For exponents $\vec{p}(\cdot) : \overline{\Omega} \to \mathbb{R}^{N+1}$, $\vec{p}(\cdot) = (p_0(\cdot), \dots, p_N(\cdot))$, we assume that $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ for every $i \in \{0, \dots, N\}$ and

$$\underline{p} = \min\{p_i^- \mid i = 0, \dots, N\} > 1. \tag{2.2}$$

We denote

$$D^0 u = u, \qquad D^i u = \partial u/\partial x_i \quad \text{for } i = 1, \dots, N.$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_0(\cdot)}(\Omega) \mid D^i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}. \tag{2.3}$$

We also define $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.3). The space $(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)})$ is a reflexive Banach space (cf. [10]).

Let us introduce the following notations:

$$p_-^+ = \max\{p_1^-, \dots, p_N^-\}, \qquad p_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}, \qquad p_{-,\infty} = \max\{p_-^+, p_-^*\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{2.4}$$

We have the following result (cf. [10]):

Theorem 2.1. *Assume $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. Assume that (2.4) is fulfilled. For any $q(\cdot) \in C(\overline{\Omega})$ satisfying $1 < q(x) < p_{-,\infty}$ for all $x \in \overline{\Omega}$, the embedding*

$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \text{ is continuous and compact.}$$

Proposition 2.2. *Denote the dual of the Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ by $W^{-1,\vec{p}'(\cdot)}(\Omega)$ (cf. [6] for the constant exponent case). For each $F \in W^{-1,\vec{p}'(\cdot)}(\Omega)$ there exist $f_i \subset L^{p_i'(\cdot)}(\Omega)$, $i = 0, \dots, N$, such that $F = f_0 - \sum_{i=1}^N D^i f_i$.*

Moreover, for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have $\langle F, u\rangle = \sum_{i=0}^N \int_\Omega f_i D^i u \, dx$.

We define a norm on the dual space by $\|F\|_{-1,\vec{p}'(\cdot)} = \sum_{i=0}^N \|f_i\|_{p_i'(\cdot)}$.

3. Existence of entropy solutions. First of all, we introduce a space in which we will prove the existence of entropy solutions. We define $\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$

to be the set of all measurable functions $u : \Omega \to \mathbb{R}$ which satisfy $T_k(u) := \max(-k, \min(k, u)) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$.

PROPOSITION 3.1. *Let $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. Then there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that $D^i T_k(u) = v_i \chi_{\{|u|<k\}}$ for a.e. $x \in \Omega$ and all $k > 0$, where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional gradient of u, that is, $v_i = D^i u$.*

LEMMA 3.2. *Let $(u_n)_n$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and*

(3.1)
$$S_n = \sum_{i=0}^{N} \int_{\Omega} [|D^i u_n|^{p_i(x)-2} D^i u_n - |D^i u_n|^{p_i(x)-2} D^i u_n](D^i u_n - D^i u)\, dx \to 0.$$

Then $u_n \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Proof. Recall the well-known inequality

$$\big(|a|^{p-2}a - |b|^{p-2}b\big)(a-b) \geq \begin{cases} 2^{2-p}|a-b|^p & \text{if } p \geq 2, \\ (p-1)\frac{|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } 1 < p < 2, \end{cases} \qquad \forall a, b \in \mathbb{R}.$$

Take $\Omega_i = \{x \in \Omega \mid 1 < p_i(x) < 2\}$. Then we have

$$\sum_{i=0}^{N} \int_{\Omega} |D^i u_n - D^i u|^{p_i(x)}\, dx = \sum_{i=0}^{N} \int_{\Omega\setminus\Omega_i} |D^i u_n - D^i u|^{p_i(x)}\, dx + \sum_{i=0}^{N} \int_{\Omega_i} |D^i u_n - D^i u|^{p_i(x)}\, dx. \tag{3.2}$$

On the one hand, it is clear that

$$\sum_{i=0}^{N} 2^{2-p_i^+} \int_{\Omega\setminus\Omega_i} |D^i u_n - D^i u|^{p_i(x)}\, dx \leq \sum_{i=0}^{N} \int_{\Omega\setminus\Omega_i} 2^{2-p_i(x)} |D^i u_n - D^i u|^{p_i(x)}\, dx \leq S_n. \tag{3.3}$$

On the other hand, using the generalized Hölder inequality we get

$$\sum_{i=0}^{N} \int_{\Omega_i} |D^i u_n - D^i u|^{p_i(x)}\, dx \tag{3.4}$$
$$= \sum_{i=0}^{N} \int_{\Omega_i} \frac{|D^i u_n - D^i u|^{p_i(x)}}{(|D^i u_n| + |D^i u|)^{p_i(x)(2-p_i(x))/2}} (|D^i u_n| + |D^i u|)^{p_i(x)(2-p_i(x))/2}\, dx$$

$$
\begin{aligned}
&\le 2\sum_{i=0}^{N}\bigg\|\frac{|D^iu_n-D^iu|^{p_i(\cdot)}}{(|D^iu_n|+|D^iu|)^{p_i(\cdot)(2-p_i(\cdot))/2}}\bigg\|_{L^{2/p_i(\cdot)}(\Omega_i)}\\
&\quad\times\|(|D^iu_n|+|D^iu|)^{p_i(\cdot)(2-p_i(\cdot))/2}\|_{L^{2/2-p_i(\cdot)}(\Omega_i)}\\
&\le 2\sum_{i=0}^{N}\max\bigg\{\bigg(\int_{\Omega_i}\frac{|D^iu_n-D^iu|^2}{(|D^iu_n|+|D^iu|)^{2-p_i(x)}}\,dx\bigg)^{1/2},\int_{\Omega_i}\frac{|D^iu_n-D^iu|^2}{(|D^iu_n|+|D^iu|)^{2-p_i(x)}}\,dx\bigg\}\\
&\quad\times\Big(\int_{\Omega_i}(|D^iu_n|+|D^iu|)^{p_i(x)}\,dx+1\Big)^{(2-p_i^-)/2}\\
&\le 2\sum_{i=0}^{N}\max\{(p_i^--1)^{-1/2}S_n^{1/2},(p_i^--1)^{-1}S_n\}\\
&\quad\times\Big(\int_{\Omega_i}(|D^iu_n|+|D^iu|)^{p_i(x)}\,dx+1\Big)^{(2-p_i^-)/2}.
\end{aligned}
$$

By combining (3.1) and (3.2)–(3.4), we deduce that $u_n\to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Definition 3.3. A measurable function u is called an *entropy solution* of the nonlinear anisotropic elliptic problem (1.1) if $u\in\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$ and

$$
\begin{aligned}
\sum_{i=1}^{N}\int_{\Omega}|D^iu|^{p_i(x)-2}D^iuD^iT_k(u-v)\,dx+\int_{\Omega}|u|^{p_0(x)-2}uT_k(u-v)\,dx\\
\le\int_{\Omega}fT_k(u-v)\,dx+\sum_{i=1}^{N}\int_{\Omega}\phi_i(u)D^iT_k(u-v)\,dx
\end{aligned}
$$

for every $v\in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$.

Theorem 3.4. *If $f\in L^1(\Omega)$ and $\phi(\cdot)\in\mathcal{C}^0(\mathbb{R},\mathbb{R}^N)$, then the problem* (1.1) *has an entropy solution.*

3.1. Proof of Theorem 3.4

Step 1: *Approximate problems.* Let $(f_n)_n$ be a sequence in $W^{-1,\vec{p}'(\cdot)}(\Omega)\cap L^1(\Omega)$ such that $f_n\to f$ in $L^1(\Omega)$ and $|f_n|\le|f|$. We consider the approximate problem

$$
\begin{cases}Au_n+|u_n|^{p_0(x)-2}u_n=f_n-\operatorname{div}\phi_n(u_n),\\ u_n\in W_0^{1,\vec{p}(\cdot)}(\Omega).\end{cases}\tag{3.5}
$$

We define the operator $R_n: W_0^{1,\vec{p}(\cdot)}(\Omega)\to W^{-1,\vec{p}'(\cdot)}(\Omega)$ by

$$
\langle R_n(u),v\rangle=\int_{\Omega}|u|^{p_0(x)-2}uv\,dx-\int_{\Omega}\phi_n(u)\nabla v\,dx\qquad\forall u,v\in W_0^{1,\vec{p}(\cdot)}(\Omega),
$$

with $\phi_n(u) = (\phi_{1,n}(u), \dots, \phi_{N,n}(u))$. The generalized Hölder inequality gives

$$
\begin{aligned}
(3.6)\quad \Big|\int_\Omega \phi_n(u)\nabla v\,dx\Big| &\le \sum_{i=1}^N \int_\Omega |\phi_{i,n}(u)|\,|D^i v|\,dx \\
&\le \sum_{i=1}^N \bigg(\frac{1}{p_i^-} + \frac{1}{(p_i')^-}\bigg) \|\phi_i(T_n(u))\|_{p_i'(\cdot)} \|D^i v\|_{p_i(\cdot)} \\
&\le 2\sum_{i=1}^N \Big[\sup_{|s|\le n} (|\phi_i(s)|+1)^{(p_i')^+}|\Omega| + 1\Big]^{1/(p_i')^-} \|v\|_{1,\vec{p}(\cdot)} \\
&\le C_1\|v\|_{1,\vec{p}(\cdot)}.
\end{aligned}
$$

Lemma 3.5. *The operator $B_n = A + R_n$ from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$
(3.7)\qquad \frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \to \infty \quad \textit{if} \quad \|v\|_{1,\vec{p}(\cdot)} \to \infty,\ \forall v \in W_0^{1,\vec{p}(\cdot)}(\Omega).
$$

Proof. Using Hölder's inequality and (3.6), it is clear that the operator B_n is bounded. For the coercivity, we have for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$
\begin{aligned}
\langle B_n u, u\rangle &\ge \sum_{i=0}^N \int_\Omega |D^i u|^{p_i(x)}\,dx - \sum_{i=1}^N \int_\Omega |\phi_{i,n}(u)|\,|D^i u|\,dx \\
&\ge \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - (N+1) - C_1\|u\|_{1,\vec{p}(\cdot)}.
\end{aligned}
$$

Hence B_n is coercive in the sense of (3.7).

It remains to show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$
(3.8)\qquad \begin{cases} u_k \rightharpoonup u \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n \quad \text{in } W^{-1,\vec{p}'(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \le \langle \chi_n, u\rangle. \end{cases}
$$

We will prove that

$$
\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \quad \text{as } k \to \infty.
$$

Firstly, since $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, it follows that $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence denoted again $(u_k)_k$. We have

$$
(3.9)\quad |D^i u_k|^{p_i(x)-2} D^i u_k \rightharpoonup |D^i u|^{p_i(x)-2} D^i u \quad \text{in } L^{p_i'(\cdot)}(\Omega) \text{ for } i = 1, \dots, N
$$

and

$$
(3.10)\qquad |u_k|^{p_0(x)-2} u_k \rightharpoonup |u|^{p_0(x)-2} u \quad \text{in } L^{p_0'(\cdot)}(\Omega).
$$

Also, since $\phi_n = \phi \circ T_n$ is a bounded continuous function and $u_k \to u$ a.e. in Ω, by using the Lebesgue dominated convergence theorem, we deduce that

$$\phi_{i,n}(u_k) \to \phi_{i,n}(u) \quad \text{in } L^{p_i'(\cdot)}(\Omega) \text{ for } i = 1, \dots, N, \tag{3.11}$$

which implies that $\chi_n = B_n u$.

On the one hand, it is clear that for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned} \langle \chi_n, v \rangle &= \lim_{k\to\infty} \sum_{i=0}^{N} \int_{\Omega} |D^i u_k|^{p_i(x)-2} D^i u_k D^i v \, dx \\ &\quad - \lim_{k\to\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_{i,n}(u_k) D^i v \, dx \\ &= \sum_{i=0}^{N} \int_{\Omega} |D^i u|^{p_i(x)-2} D^i u D^i v \, dx - \sum_{i=1}^{N} \int_{\Omega} \phi_{i,n}(u) D^i v \, dx. \end{aligned} \tag{3.12}$$

From (3.8) and (3.12), we obtain

$$\limsup_{k\to\infty} \langle B_n u_k, u_k \rangle \le \sum_{i=0}^{N} \int_{\Omega} |D^i u|^{p_i(x)} \, dx - \sum_{i=1}^{N} \int_{\Omega} \phi_{i,n}(u) D^i u \, dx.$$

Thanks to (3.11), we have

$$\sum_{i=1}^{N} \int_{\Omega} \phi_{i,n}(u_k) D^i u_k \, dx \to \sum_{i=1}^{N} \int_{\Omega} \phi_{i,n}(u) D^i u \, dx. \tag{3.13}$$

Therefore

$$\limsup_{k\to\infty} \sum_{i=0}^{N} \int_{\Omega} |D^i u_k|^{p_i(x)} \, dx \le \sum_{i=0}^{N} \int_{\Omega} |D^i u|^{p_i(x)} \, dx. \tag{3.14}$$

On the other hand, we have

$$\sum_{i=0}^{N} \int_{\Omega} (|D^i u_k|^{p_i(x)-2} D^i u_k - |D^i u|^{p_i(x)-2} D^i u)(D^i u_k - D^i u) \, dx \ge 0. \tag{3.15}$$

Hence

$$\begin{aligned} \sum_{i=0}^{N} \int_{\Omega} |D^i u_k|^{p_i(x)} \, dx &\ge \sum_{i=0}^{N} \int_{\Omega} |D^i u_k|^{p_i(x)-2} D^i u_k D^i u \, dx \\ &\quad + \sum_{i=0}^{N} \int_{\Omega} |D^i u|^{p_i(x)-2} D^i u (D^i u_k - D^i u) \, dx, \end{aligned}$$

using (3.9) and (3.10), we get

$$\liminf_{k\to\infty} \sum_{i=0}^{N} \int_{\Omega} |D^i u_k|^{p_i(x)}\, dx \geq \sum_{i=0}^{N} \int_{\Omega} |D^i u|^{p_i(x)}\, dx.$$

This implies by using (3.14) that

$$\lim_{k\to\infty} \sum_{i=0}^{N} \int_{\Omega} |D^i u_k|^{p_i(x)}\, dx = \sum_{i=0}^{N} \int_{\Omega} |D^i u|^{p_i(x)}\, dx. \tag{3.16}$$

According to (3.12), (3.13) and (3.16), we obtain

$$\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \quad \text{as } k\to\infty.$$

Finally, by using the classical theorem of Lions [9] and as a conclusion of this step, there exists at least one weak solution $u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the problem (3.5).

STEP 2: *A priori estimates.* Taking $T_k(u_n)$ as a test function in (3.5), we get

$$\begin{aligned} \sum_{i=0}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)}\, dx &\leq \sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)}\, dx \\ &\quad + \int_{\Omega} |u_n|^{p_0(x)-2} u_n T_k(u_n)\, dx \\ &\leq k\|f\|_1 + \sum_{i=1}^{N} \int_{\Omega} \phi_{i,n}(T_k(u_n)) D^i T_k(u_n)\, dx. \end{aligned} \tag{3.17}$$

Take $\Phi_{i,n}(t) = \int_0^t \phi_{i,n}(\tau)\, d\tau$. Then $\Phi_{i,n}(0) = 0$ and $\Phi_{i,n} \in \mathcal{C}^1(\mathbb{R})$. In view of the Green formula, we have

$$\begin{aligned} \int_{\Omega} \phi_{i,n}(T_k(u_n)) D^i T_k(u_n)\, dx &= \int_{\Omega} D^i \Phi_{i,n}(T_k(u_n))\, dx \\ &= \int_{\partial\Omega} \Phi_{i,n}(T_k(u_n)) n_i\, d\sigma = 0, \end{aligned} \tag{3.18}$$

since $u_n = 0$ on $\partial\Omega$, with $\Phi_n = (\Phi_{1,n}, \dots, \Phi_{N,n})$ and $\vec{n} = (n_1, \dots, n_N)$ the normal vector on $\partial\Omega$. It follows from (3.17) that

$$\sum_{i=0}^{N} \|D^i T_k(u_n)\|_{p_i(x)}^{\underline{p}} \leq \sum_{i=0}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)}\, dx + N + 1 \leq k\|f\|_1 + N + 1.$$

Consequently,

$$\|T_k(u_n)\|_{1,\vec{p}(\cdot)} \leq C_2 k^{1/\underline{p}} \quad \text{for all } k \geq 1. \tag{3.19}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure. Indeed, by using the generalized Hölder inequality and (3.19), we have

$$
\begin{aligned}
k \operatorname{meas}\{|u_n| > k\} &\leq \int_\Omega |T_k(u_n)|\,dx \\
&\leq \left(\frac{1}{p_0^-} + \frac{1}{(p_0')^-}\right)\|1\|_{p_0'(\cdot)}\|T_k(u_n)\|_{p_0(x)} \\
&\leq 2(|\Omega|+1)^{1/(p_0')^-}\|T_k(u_n)\|_{1,\vec{p}(\cdot)} \leq C_3 k^{1/\underline{p}},
\end{aligned} \tag{3.20}
$$

which yields

$$
\operatorname{meas}\{|u_n| > k\} \leq C_3 \frac{1}{k^{1-1/\underline{p}}} \to 0 \quad \text{as } k \to \infty. \tag{3.21}
$$

We can now apply the same procedure as in [3] and [7] to prove that $(u_n)_n$ is a Cauchy sequence in measure and so converges almost everywhere, for a subsequence, to some measurable function u. Therefore,

$$
T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \tag{3.22}
$$

and in view of the Lebesgue dominated convergence theorem,

$$
T_k(u_n) \to T_k(u) \quad \text{in } L^{p_0(\cdot)}(\Omega). \tag{3.23}
$$

To prove the equi-integrability of $|u_n|^{p_0(x)-1}$, taking $T_1(u_n - T_h(u_n))$ as a test function in (3.5), we obtain

$$
\begin{aligned}
&\sum_{i=1}^N \int_{\{h\leq|u_n|\leq h+1\}} |D^i u_n|^{p_i(x)}\,dx + \int_{\{h\leq|u_n|\}} |u_n|^{p_0(x)-2}u_n T_1(u_n - T_h(u_n))\,dx \\
&\quad= \int_{\{h\leq|u_n|\}} f_n T_1(u_n - T_h(u_n))\,dx + \sum_{i=1}^N \int_{\{h\leq|u_n|\leq h+1\}} \phi_{i,n}(u_n) D^i u_n\,dx.
\end{aligned} \tag{3.24}
$$

By the Green formula, it is clear that

$$
\begin{aligned}
&\int_{\{h\leq|u_n|\leq h+1\}} \phi_{i,n}(u_n) D^i u_n\,dx \\
&\quad= \int_\Omega \phi_i(T_{h+1}(u_n)) D^i T_{h+1}(u_n)\,dx - \int_\Omega \phi_i(T_h(u_n)) D^i T_h(u_n)\,dx \\
&\quad= \int_\Omega D^i \Phi_i(T_{h+1}(u_n))\,dx - \int_\Omega D^i \Phi_i(T_h(u_n))\,dx \\
&\quad= \int_{\partial\Omega} \Phi_i(T_{h+1}(u_n)) \cdot n_i\,d\sigma - \int_{\partial\Omega} \Phi_i(T_h(u_n)) \cdot n_i\,d\sigma = 0.
\end{aligned} \tag{3.25}
$$

Then

$$\int_{\{h+1\le|u_n|\}} |u_n|^{p_0(x)-1}\,dx \le \int_{\{h\le|u_n|\}} |u_n|^{p_0(x)-2}u_nT_1(u_n-T_h(u_n))\,dx$$
$$\le \int_{\{h\le|u_n|\}} |f|\,dx \to 0 \quad \text{as } h\to\infty.$$

Let $\eta>0$ be fixed. Then there exists $h(\eta)>0$ such that

$$\int_{\{h(\eta)\le|u_n|\}} |u_n|^{p_0(x)-1}\,dx \le \frac{\eta}{2}. \tag{3.26}$$

On the other hand, there exists $\lambda(\eta)>0$ such that for all $E\subseteq\Omega$ with $|E|<\lambda(\eta)$, we have

$$\int_E |T_{h(\eta)}(u_n)|^{p_0(x)-1}\,dx \le \frac{\eta}{2}. \tag{3.27}$$

By combining (3.26) and (3.27), one easily has

$$\int_E |u_n|^{p_0(x)-1}\,dx \le \int_E |T_{h(\eta)}(u_n)|^{p_0(x)-1}\,dx + \int_{\{h(\eta)\le|u_n|\}} |u_n|^{p_0(x)-1}\,dx \le \eta \tag{3.28}$$

for any $E\subset\Omega$ with $|E|<\lambda(\eta)$. Thus, we have proved that $(|u_n|^{p_0(x)-2}u_n)_n$ is uniformly equi-integrable. In view of Vitali's Theorem we conclude that

$$|u_n|^{p_0(x)-2}u_n \to |u|^{p_0(x)-2}u \quad \text{strongly in } L^1(\Omega). \tag{3.29}$$

STEP 3: *Convergence of the gradient.* Denote by $\varepsilon_1(n),\varepsilon_2(n),\ldots$ various functions of real numbers which converge to 0 as n tends to ∞.

Let $h>k>0$ and $M=4k+h$. Choosing $\omega_n=T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))$ as a test function in (3.5), we get

$$\sum_{i=1}^N \int_\Omega |D^iu_n|^{p_i(x)-2}D^iu_nD^i\omega_n\,dx + \int_\Omega |u_n|^{p_0(x)-2}u_n\omega_n\,dx$$
$$= \int_\Omega f_n\omega_n\,dx + \sum_{i=1}^N \int_\Omega \phi_{i,n}(u_n)D^i\omega_n\,dx.$$

Since ω_n and u_n have the same sign on $\{|u_n|>k\}$, and $\omega_n=T_k(u_n)-T_k(u)$ on $\{|u_n|\le k\}$ and $D^i\omega_n=0$ on $\{|u_n|>M\}$ for $i=1,\ldots,N$, it follows that

$$\begin{aligned}\sum_{i=1}^N \int_\Omega &|D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^i\omega_n\,dx \\ &+ \int_{\{|u_n|\le k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\,dx \\ &\le \int_\Omega f_n\omega_n\,dx + \sum_{i=1}^N \int_\Omega \phi_{i,n}(T_M(u_n))D^i\omega_n\,dx.\end{aligned} \tag{3.30}$$

On the one hand, taking $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$, we get

$$\begin{aligned}(3.31)\quad &\sum_{i=1}^{N}\int_{\Omega} |D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^i\omega_n\,dx \\ &= \sum_{i=1}^{N}\int_{\Omega} |D^iT_k(u_n)|^{p_i(x)-2}D^iT_k(u_n)(D^iT_k(u_n) - D^iT_k(u))\,dx \\ &\quad + \sum_{i=1}^{N}\int_{\{|u_n|>k\}\cap\{|z_n|\le 2k\}} |D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^iz_n\,dx \\ &\ge \sum_{i=1}^{N}\int_{\Omega} [|D^iT_k(u_n)|^{p_i(x)-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i(x)-2}D^iT_k(u)] \\ &\qquad\qquad \times [D^iT_k(u_n) - D^iT_k(u)]\,dx \\ &\quad + \sum_{i=1}^{N}\int_{\Omega} |D^iT_k(u)|^{p_i(x)-2}D^iT_k(u)(D^iT_k(u_n) - D^iT_k(u))\,dx \\ &\quad - \sum_{i=1}^{N}\int_{\{|u_n|>k\}} |D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^iT_k(u)\,dx.\end{aligned}$$

For the second term on the right-hand side of (3.31), since $|D^iT_k(u)|^{p_i(x)-2}D^iT_k(u) \in L^{p_i'(\cdot)}(\Omega)$ and $D^iT_k(u_n) \rightharpoonup D^iT_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, we have

$$(3.32)\quad \varepsilon_1(n) = \sum_{i=1}^{N}\int_{\Omega} |D^iT_k(u)|^{p_i(x)-2}D^iT_k(u)(D^iT_k(u_n) - D^iT_k(u))\,dx \to 0.$$

Concerning the last term on the right-hand side of (3.31), we have

$$|D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n) \rightharpoonup |D^iT_M(u)|^{p_i(x)-2}D^iT_M(u) \quad \text{in } L^{p_i'(\cdot)}(\Omega).$$

Thus

$$\begin{aligned}(3.33)\quad \varepsilon_2(n) &= \int_{\{|u_n|>k\}} |D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^iT_k(u)\,dx \\ &\to \int_{\{|u|>k\}} |D^iT_M(u)|^{p_i(x)-2}D^iT_M(u)D^iT_k(u)\,dx = 0.\end{aligned}$$

By combining (3.31)–(3.33), we deduce that

$$\begin{aligned}(3.34)\quad &\sum_{i=1}^{N}\int_{\Omega} \big[|D^iT_k(u_n)|^{p_i(x)-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i(x)-2}D^iT_k(u)\big] \\ &\qquad\qquad \times\ [D^iT_k(u_n) - D^iT_k(u)]\,dx \\ &\qquad \le \sum_{i=1}^{N}\int_{\Omega} |D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^i\omega_n\,dx + \varepsilon_3(n).\end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned}&\int_{\{|u_n|\le k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\,dx\\&\ge \int_\Omega \big(|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u)\big)(T_k(u_n)-T_k(u))\,dx\\&\quad-\int_\Omega |T_k(u)|^{p_0(x)-1}|T_k(u_n)-T_k(u)|\,dx-\int_{\{|u_n|>k\}} k^{p_0(x)-1}|T_k(u_n)-T_k(u)|\,dx.\end{aligned}$$

In view of the Lebesgue dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ in $L^{p_0(\cdot)}(\Omega)$. Thus the second and the last terms on the right-hand side of the previous inequality converge to 0 as n goes to ∞, and we get

$$\begin{aligned}&\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n)-|T_k(u)|^{p_0(x)-2}T_k(u))\big(T_k(u_n)-T_k(u)\big)\,dx\\&\qquad\le \int_{\{|u_n|\le k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\,dx+\varepsilon_4(n),\end{aligned}\tag{3.35}$$

Thanks to (3.30) and (3.34)–(3.35), we obtain

$$\begin{aligned}&\sum_{i=0}^N\int_\Omega \big[|D^iT_k(u_n)|^{p_i(x)-2}D^iT_k(u_n)-|D^iT_k(u)|^{p_i(x)-2}D^iT_k(u)\big]\\&\qquad\qquad\times[D^iT_k(u_n)-D^iT_k(u)]\,dx\\&\quad\le\sum_{i=1}^N\int_\Omega |D^iT_M(u_n)|^{p_i(x)-2}D^iT_M(u_n)D^i\omega_n\,dx\\&\qquad+\int_{\{|u_n|\le k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n)-T_k(u))\,dx+\varepsilon_5(n)\\&\quad\le\int_\Omega f_n\omega_n\,dx+\sum_{i=1}^N\int_\Omega \phi_{i,n}(T_M(u_n))D^i\omega_n\,dx+\varepsilon_5(n).\end{aligned}\tag{3.36}$$

We have

$$\int_\Omega f_n\omega_n\,dx=\int_\Omega fT_{2k}(u-T_h(u))\,dx+\varepsilon_6(n),\tag{3.37}$$

and for n large enough,

$$\phi_{i,n}(T_M(u_n))=\phi_i(T_M(u_n)).$$

It follows that

$$\int_\Omega \phi_{i,n}(T_M(u_n))D^i\omega_n\,dx=\int_\Omega \phi_i(T_M(u))D^iT_{2k}(u-T_h(u))\,dx+\varepsilon_7(n).\tag{3.38}$$

Similarly to (3.25), we can prove that

$$\int_\Omega \phi_i(T_M(u))D^iT_{2k}(u-T_h(u))\,dx = \int_\Omega \phi_i(T_{2k+h}(u))D^iT_{2k+h}(u)\,dx - \int_\Omega \phi_i(T_h(u))D^iT_h(u)\,dx = 0. \tag{3.39}$$

Therefore, by letting n then h go to ∞ in (3.36), and thanks to (3.37)–(3.39), we deduce that

$$\sum_{i=0}^{N}\int_\Omega \big(|D^iT_k(u_n)|^{p_i(x)-2}D^iT_k(u_n) - |D^iT_k(u)|^{p_i(x)-2}D^iT_k(u)\big) \times (D^iT_k(u_n) - D^iT_k(u))\,dx \to 0, \tag{3.40}$$

and in view of Lemma 3.2, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega) \quad \text{and} \quad D^iu_n \to D^iu \quad \text{a.e. in } \Omega. \tag{3.41}$$

STEP 4: *Passing to the limit.* By using $T_k(u_n-\varphi)$ as a test function in (3.5), with $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we get

(3.42)
$$\sum_{i=1}^{N}\int_\Omega |D^iu_n|^{p_i(x)-2}D^iu_nD^iT_k(u_n-\varphi)\,dx + \int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\varphi)\,dx = \int_\Omega f_nT_k(u_n-\varphi)\,dx + \sum_{i=1}^{N}\int_\Omega \phi_{i,n}(u_n)D^iT_k(u_n-\varphi)\,dx.$$

Put $M = k + \|\varphi\|_\infty$ and let n be large enough ($n \ge M$). It is clear that if $|u_n| > M$ then

$$|u_n-\varphi| \ge |u_n| - \|\varphi\|_\infty > k,$$

therefore

$$\{|u_n-\varphi| \le k\} \subseteq \{|u_n| \le M\}.$$

For the first term on the right-hand side, according to Fatou's Lemma, we have

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega |D^iu_n|^{p_i(x)-2}D^iu_nD^iT_k(u_n-\varphi)\,dx &\ge \int_\Omega \big[|D^iT_M(u)|^{p_i(x)-2}D^iT_M(u) - |D^i\varphi|^{p_i(x)-2}D^i\varphi\big] \\ &\qquad\times[D^iT_M(u)-D^i\varphi]\chi_{\{|u-\varphi|\le k\}}\,dx \\ &\quad + \int_\Omega |D^i\varphi|^{p_i(x)-2}D^i\varphi(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx,\end{aligned} \tag{3.43}$$

and we get

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}|D^iu_n|^{p_i(x)-2}D^iu_nD^iT_k(u_n-\varphi)\,dx\\
&\quad\geq\sum_{i=1}^{N}\int_{\Omega}|D^iT_M(u)|^{p_i(x)-2}D^iT_M(u)(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&\quad=\sum_{i=1}^{N}\int_{\Omega}|D^iu|^{p_i(x)-2}D^iuD^iT_k(u-\varphi)\,dx.
\end{aligned}$$

On the other hand, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak-$\star$ in $L^\infty(\Omega)$ and thanks to (3.29) we obtain

$$\int_{\Omega}|u_n|^{p_0(x)-2}u_nT_k(u_n-\varphi)\,dx\to\int_{\Omega}|u|^{p_0(x)-2}uT_k(u-\varphi)\,dx \tag{3.44}$$

and

$$\int_{\Omega}f_nT_k(u_n-\varphi)\,dx\to\int_{\Omega}fT_k(u-\varphi)\,dx. \tag{3.45}$$

Again, since $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $\phi_{i,n}(u_n)=\phi_i(T_M(u_n))$ in $\{|u_n-\varphi|\leq k\}$ for $n\geq M$, we have

$$\int_{\Omega}\phi_{i,n}(u_n)D^iT_k(u_n-\varphi)\,dx\to\int_{\Omega}\phi_i(u)D^iT_k(u-\varphi)\,dx. \tag{3.46}$$

This completes the proof of Theorem 3.4.

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