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ENTROPY SOLUTIONS FOR NONLINEAR UNILATERAL PARABOLIC INEQUALITIES IN ORLICZ–SOBOLEV SPACES

Abstract. We discuss the existence of entropy solution for the strongly nonlinear unilateral parabolic inequalities associated to the nonlinear parabolic equations $\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u) + \Phi(u)) + g(u)M(|\nabla u|) = \mu$ in Q, in the framework of Orlicz–Sobolev spaces without any restriction on the N-function of the Orlicz spaces, where $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator and $\Phi \in C^0(\mathbb{R},\mathbb{R}^N)$. The function $g(u)M(|\nabla u|)$ is a nonlinear lower order term with natural growth with respect to $|\nabla u|$, without satisfying the sign condition, and the datum μ belongs to $L^1(Q)$ or $L^1(Q)+W^{-1,x}E_{\overline{M}}(Q)$.

1. Introduction. Let Q be the cylinder $\Omega \times (0,T)$, where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^N$ with the segment property, and let M and P be two N-functions such that $P \ll M$. In the present paper, we consider the following boundary value problem:

$$(1.1)\qquad \begin{cases} u \geq \psi \quad \text{a.e. in } Q, \\ \dfrac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(u)\big) + g(u)M(|\nabla u|) = f \quad \text{in } Q, \\ u = 0 \qquad\qquad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (t,s,ξ) in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) such that for all ξ and ξ^* in $\mathbb{R}^N$, $\xi \neq \xi^*$,

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$$(1.2)\qquad a(x,t,s,\xi)\xi \geq \alpha M(|\xi|),$$

$$(1.3)\qquad [a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0,$$

$$(1.4)\qquad |a(x,t,s,\xi)| \leq c(x,t) + k_1\overline{P}^{-1}M(k_2|s|) + k_3\overline{M}^{-1}M(k_4|\xi|),$$

where $c(\cdot,\cdot)$ belongs to $E_{\overline{M}}(Q)$, $c \geq 0$, $k_i \geq 0$ $(i = 1,2,3,4)$ and $\alpha > 0$. Moreover,

(1.5) $\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous function,

(1.6) $f \in L^1(Q)$,

(1.7) $u_0 \in L^1(\Omega)$, $\quad u_0 \geq \psi(x)$ a.e. in Ω,
$\psi \in L^\infty(\Omega) \cap W_0^{1,x}E_M(\Omega)$, $\quad \psi(x) \leq 0$ a.e. in Ω,

(1.8) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is an integrable continuous function.

Remark 1.1. We remark that if $u_0 \in L^1(\Omega)$, $u_0 \geq 0$, then the convex subset $\{v \in L^1(\Omega) : v \geq \psi\}$ is nonempty.

In [P] where $\Phi \equiv 0$, the author has shown the existence of a renormalized solution for the corresponding equation. In [P], the function $a(x,t,u,\nabla u)$ was assumed to satisfy a polynomial growth condition with respect to u and ∇u. When trying to relax this restriction on the function $a(x,t,u,\nabla u)$, we are led to replace the space $L^p(0,T;W_0^{1,p}(\Omega))$ by an inhomogeneous Orlicz–Sobolev space $W^{1,x}L_M(Q)$ built from an Orlicz space L_M instead of L^p, where the N-function M which defines L_M is related to the actual growth of the Carathéodory function. Recently M. Kbiri Alaoui et al. [KMS] proved the existence result for the obstacle problem associated to (1.1) in the setting of Orlicz–Sobolev spaces where $\Phi \equiv 0$. The above problem does not admit, in general, a weak solution since the fields $a(x,t,u,\nabla u)$ and $\Phi(u)$ do not belong in $(L^1_{\mathrm{loc}}(Q))^N$ in general. For analogous elliptic or parabolic problems in the setting of Sobolev spaces or Orlicz–Sobolev spaces, we refer the reader to [ABT, ABM15, ABM, ABMR, AB, BE, D, E, EM1, EM3, GM, L, R, YBM1, YBM2].

This paper is motivated by recent advances in mathematical modeling of non-Newtonian fluids and elastic mechanics, in particular, electrorheological fluids (smart fluids). This important class of fluids is characterized by the change of viscosity which depends on the electric field. These fluids, also known as ER fluids, have many applications in elastic mechanics, fluid dynamics etc.

The scope of the present paper is to solve the obstacle problem associated to (1.1) in the case where $f \in L^1(Q) + W^{-1,x}E_{\overline{M}}(Q)$ and without assuming any growth restriction on M, $\Phi(u) \not\equiv 0$, while the function $g(u)M(|\nabla u|)$ does not satisfy the sign condition. The existence of solutions is proved via a sequence of penalized problems.

2. Existence results. This section is devoted to establishing the following existence theorem.

THEOREM 2.1. *Assume that* (1.2)–(1.8) *hold. Then there exists a solution of problem* (1.1) *in the following sense:*

$$\begin{cases} u \geq \psi \quad \text{a.e. in } Q, \quad T_k(u) \in W_0^{1,x} L_M(Q), \quad S_k(u(\cdot,t)) \in L^1(\Omega), \\ \int_\Omega S_k(u(T)-v(T))\,dx + \left\langle \frac{\partial v}{\partial t}, T_k(u-v) \right\rangle + \int_Q a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt \\ \qquad + \int_Q \Phi(u)\nabla T_k(u-v)\,dx\,dt \leq \int_Q g(u)M(|\nabla u|)T_k(u-v)\,dx\,dt \\ \qquad\qquad + \int_Q fT_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0-v(0))\,dx, \end{cases}$$

for all $k > 0$ *and* $v \in W_0^{1,x} L_M(Q) \cap L^\infty(Q)$ *such that* $\partial v/\partial t \in W^{-1,x} L_{\overline{M}}(Q) + L^1(Q)$ *and* $v \geq \psi$. *Here* S_k *is the truncation defined by* $S_k(\tau) = \int_0^\tau T_k(s)\,ds$ *where the standard truncation function* T_k, $k > 0$, *is defined for all* $s \in \mathbb{R}$ *by* $T_k(s) = \max\{-k, \min\{k, s\}\}$.

The proof is divided into four steps.

STEP 1: *Approximate problems and a priori estimate.* Consider the following approximate problem:

$$\text{(2.1)} \qquad \begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}\big(a(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\big) - nT_n((u_n-\psi)^-) \\ \qquad = g(u_n)M(|\nabla u_n|) + f_n \quad \text{in } Q, \\ u_n(x,0) = u_{0n}(x) \quad \text{in } \Omega, \end{cases}$$

where we have set $\Phi_n(s) = \Phi(T_n(s))$. For fixed $n > 0$, since Φ is continuous, it is obvious that $|\Phi_n(t)| \leq \max_{|t|\leq n} |\Phi(t)| = C_n$. Moreover, the sequence $(f_n) \subset \mathcal{D}(Q)$ is such that $f_n \to f$ strongly in $L^1(Q)$ and $(u_{0n}) \subset \mathcal{D}(\Omega)$ is such that $u_{0n} \to u_0$ strongly in $L^1(\Omega)$. By Lemma 3.1 of [KMS], there exists a weak solution $u_n \geq 0$ in $W_0^{1,x} L_M(Q)$ of problem (2.1). Let $h > 0$ and consider the test function $v = T_h(u_n)\exp(\int_0^{u_n} g(s)\,ds)$ in (2.1). We have

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big) \right\rangle \\ &\qquad + \int_{\{u_n\leq h\}} a(\cdot,u_n,\nabla u_n)\nabla u_n \exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \\ &\qquad + \int_Q a(\cdot,u_n,\nabla u_n)\nabla u_n T_h(u_n) g(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \end{aligned}$$

$$+\int_Q \Phi_n(u_n)\nabla\Big(T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big)\,dx\,dt$$
$$-\int_Q nT_n((u_n-\psi)^-)T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt$$
$$=\int_Q g(u_n)M(|\nabla u_n|)T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt$$
$$+\int_Q f_nT_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt.$$

The Lipschitz character of Φ_n and the Stokes formula together with the boundary condition $u_n=0$ on $(0,T)\times\partial\Omega$ make it possible to obtain

$$\int_Q \Phi_n(u_n)\nabla\Big(T_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big)\,dx\,dt=0. \tag{2.2}$$

Using (2.2) and (1.2), and since $u_n\ge 0$ gives $T_h(u_n)\ge 0$, we have

$$\Big\langle\frac{\partial u_n}{\partial t},T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big\rangle$$
$$+\int_{\{u_n\le h\}} M(|\nabla u_n|)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)$$
$$-n\int_Q T_n((u_n-\psi)^-)T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt$$
$$\le\int_Q f_nT_h(u_n-T_k(u_n))\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt.$$

We have

$$\Big\langle\frac{\partial u_n}{\partial t},T_h(u_n)\exp\Big(\int_0^{u_n} g(s)\,ds\Big)\Big\rangle$$
$$=\int_\Omega\int_0^{u_n(x,T)} T_h(s)\exp\Big(\int_0^{s} g(s)\,ds\Big)-\int_\Omega\int_0^{u_0} T_h(s)\exp\Big(\int_0^{s} g(s)\,ds\Big)\ge -Ch,$$

where

$$C=\|u_0\|_{L^1(\Omega)}\int_0^{u_0} T_h(s)\exp\Big(\int_0^{u_0} g(s)\,ds\Big).$$

So, we obtain

$$-n\int_Q T_n((u_n-\psi)^-)T_h(u_n)\le Ch,$$

and since $\psi(x) \le 0$ a.e. in Ω,

$$T_h(u_n) \le T_h((u_n - \psi)$$

and

$$-\int_Q nT_n((u_n-\psi)^-)T_h(u_n-\psi) \le -\int_Q nT_n((u_n-\psi)^-)T_h(u_n) \le Ch.$$

Then

$$\int_Q nT_n((u_n-\psi)^-)\frac{T_h((u_n-\psi)^-)}{h} \le C.$$

Letting h to tend to zero, one has

$$0 \le \int_Q nT_n((u_n-\psi)^-) \le C. \tag{2.3}$$

If we use $v = T_k(u_n)\exp(\int_0^{u_n} g(s)\,ds)$ as a test function in (2.1), then as above we obtain

$$\int_Q M(|\nabla T_k(u_n)|)\exp\Big(\int_0^{u_n} g(s)\,ds\Big) \le C_1 k. \tag{2.4}$$

Thus $(T_k(u_n))_n$ is bounded in $W_0^{1,x}L_M(Q)$, and so there exist some $w_k \in W_0^{1,x}L_M(Q)$ such that

$$\begin{aligned} T_k(u_n) &\rightharpoonup w_k \quad \text{weakly in } W_0^{1,x}L_M(Q) \text{ for } \sigma\big(\textstyle\prod L_M, \prod E_{\overline{M}}\big),\\ T_k(u_n) &\to w_k \quad \text{strongly in } E_M(Q) \text{ and a.e. in } Q, \end{aligned}$$

where $\prod L_M$ is the product of $N+1$ copies of L_M and $\prod E_M$ is the product of $N+1$ copies of E_M.

Let η_k be the nondecreasing $C^2(\mathbb{R})$ function with

$$\eta_k(s) = \begin{cases} s, & |s| \le k/2,\\ k\operatorname{sign}(s), & |s| \ge k. \end{cases}$$

Multiplying the approximating equation by $\eta_k'(u_n)$, we get

$$\begin{aligned} \frac{\partial \eta_k(u_n)}{\partial t} - \operatorname{div}\big(a(x,t,u_n,\nabla u_n)\eta_k'(u_n)\big) + a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \eta_k''(u_n)\\ - \operatorname{div}\big(\Phi_n(u_n)\eta_k'(u_n)\big) + \Phi_n(u_n)\eta_k''(u_n)\nabla u_n\\ = g(u_n)M(|\nabla u_n|)\eta_k'(u_n) + f_n\eta_k'(u_n) + nT_n((u_n-\psi)^-)\eta_k'(u_n) \end{aligned}$$

in the distribution sense. We deduce that $\eta_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$ and $\partial\eta_k(u_n)/\partial t$ in $W^{-1,x}L_{\overline{M}}(Q)+L^1(Q)$. By the Corollary of [EM1], $\eta_k(u_n)$ is compact in $L^1(Q)$.

In the same way as in [P] we obtain, for every $k > 0$,

$$
\begin{aligned}
T_k(u_n) &\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,x} L_M(Q) \text{ for } \sigma\big(\textstyle\prod L_M, \prod E_{\overline{M}}\big), \\
T_k(u_n) &\to T_k(u) \quad \text{strongly in } L^1(Q) \text{ and a.e. in } Q.
\end{aligned} \tag{2.5}
$$

Now using the estimate (2.3) and Fatou's Lemma, we obtain

$$(u-\psi)^- = 0,$$

and so $u \geq \psi$.

STEP 2: *Almost everywhere convergence of the gradients*

LEMMA 2.2. *Let u_n be a solution of the approximate problem* (2.1). *Then there exists a subsequence also denoted by u_n such that*

$$\nabla u_n \to \nabla u \qquad \textit{a.e. in } Q.$$

The proof is similar to Step 2 in [KMS]. Now, there exists a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } Q.$$

We deduce that

$$a(\cdot, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(\cdot, T_k(u), \nabla T_k(u)) \quad \text{in } (L_{\overline{M}}(Q))^N \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

STEP 3: *Modular convergence of the truncations.* We use the same technique as in [GM] in the parabolic case. The functions v_j, χ_j^s, $\varepsilon(n,j,\mu,i,s,m)$, χ^s and $\varepsilon(n,j,s)$ below are as in Step 2 in [KMS]. By using the same argument as in Step 2 in [KMS], we prove that

$$\int_Q \big(a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k^*(u_n) - \nabla T_k(v_j)\chi_j^s\big) \times \exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt \leq \varepsilon(n,j,\mu,i,s,m),$$

where

$$T_k^*(s) = \Big(\int_0^{T_k(s)} \exp\Big(\int_0^t g(s)\,ds\Big)\,dt\Big)\Big(\exp\Big(-\int_0^\infty g(s)\,ds\Big)\Big).$$

We can also deduce that

$$\int_Q \big(a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k^*(u)\chi^s)\big)\big(\nabla T_k^*(u_n) - \nabla T_k^*(u)\chi^s\big) \times \exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt$$

$$= \int_Q \big(a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s)\big)\big(\nabla T_k^*(u_n) - \nabla T_k(v_j)\chi_j^s\big) \times \exp\Big(\int_0^{u_n} g(s)\,ds\Big)\,dx\,dt + \varepsilon(n,j,s).$$

Then

$$\int_Q a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt \le \int_Q a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u)\chi^s\,dx\,dt + \int_Q a(\cdot, T_k(u_n), \nabla T_k^*(u)\chi^s)(\nabla T_k^*(u_n) - T_k(u)\chi^s)\,dx\,dt + \varepsilon(n,j,\mu,i,s,m).$$

We deduce that

$$\limsup_n \int_Q a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt \le \int_Q a(\cdot, T_k(u), \nabla T_k(u))\nabla T_k^*(u)\chi^s\,dx\,dt + \lim_n \varepsilon(n,j,\mu,i,s,m),$$

so that

$$\limsup_n \int_Q a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt \le \int_Q a(\cdot, T_k(u), \nabla T_k(u))\nabla T_k^*(u)\chi^s\,dx\,dt \le \liminf_n \int_Q a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n)\,dx\,dt,$$

as $n \to \infty$. Hence

$$a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k^*(u_n) \to a(\cdot, T_k(u), \nabla T_k(u))\nabla T_k^*(u) \quad \text{in } L^1(Q).$$

Using the same argument as above, we obtain

$$a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) \to a(\cdot, T_k(u), \nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(Q),$$

and Vitali's theorem and (1.2) give

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{for the modular convergence in } (L_M(Q))^N.$$

STEP 4: *Passing to the limit.* Using the approximate function of Lemma 3.2 of [KMS], the passing to the limit is easy, as in [EM2, EM3].

REMARK 2.3. A similar result can be proved when dealing with the right-hand side in $L^1(Q) + W^{-1,x}E_{\overline{M}}(Q)$.

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