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ON THE p-BIHARMONIC OPERATOR WITH CRITICAL SOBOLEV EXPONENT

Abstract. We study the existence of solutions for a p-biharmonic problem with a critical Sobolev exponent and Navier boundary conditions, using variational arguments. We establish the existence of a precise interval of parameters for which our problem admits a nontrivial solution.

1. Introduction. Let Ω be a regular bounded domain in $\mathbb{R}^N$. Consider the fourth order nonlinear eigenvalue problem

$$\Delta_p^2 u = \lambda |u|^{p-2}u + |u|^{p^*-2}u \quad \text{in } \Omega, \qquad u \in W_0^{2,p}(\Omega), \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and p^* is the critical Sobolev exponent defined by

$$p^* = Np/(N-2p) \quad \text{with} \quad 2 < 2p < N.$$

$\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the fourth order operator called the p-*biharmonic operator*. For $p = 2$, the linear operator $\Delta_2^2 = \Delta^2 = \Delta\Delta$ is the iterated Laplacian that up to a multiplicative positive constant often appears in the Navier–Stokes equations as being the viscosity term. Its inverse operator denoted by $(\Delta^2)^{-1}$ is the celebrated Green operator [Lio1].

Not that the biharmonic equation $\Delta^2 u = 0$ is a linear partial differential equation of fourth order which appears in quantum mechanics and in the theory of linear elasticity modeling Stokes flows.

It is well known that fourth order elliptic problems arise in many applications, such as micro-electromechanical systems, thin film theory, nonlinear surface diffusion in solids, interface dynamics, flows in Hele-Shaw cells, phase

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field models of multi-phase systems and the deformation of a nonlinear elastic beam (see for example [FW] and [M] and references therein).

This paper is motivated by the work of Gazzola et al. [GGS], who studied (1.1) for the case $p = 2$, $p^* = 2N/(N-4)$, with $N \geq 5$. We use a variational technique to prove the existence of a sequence of positive eigenvalues of problem (1.1). For $\lambda = 0$ El Khalil et al. [EMT] proved that the nonlinear boundary eigenvalue problem

$$\begin{cases} \Delta_p^2 u = |u|^{p^*-2}u & \text{in } \Omega, \\ |\Delta u|^{p-2}\Delta u = \mu \left|\dfrac{\partial u}{\partial n}\right|^{p-2} \dfrac{\partial u}{\partial n} & \text{on } \partial\Omega, \\ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \end{cases}$$

where μ is a real parameter which plays the role of an eigenvalue, has at least one increasing sequence of positive eigenvalues.

To present our result concerning (1.1), consider the auxiliary problem

$$\Delta_p^2 u = \lambda |u|^{p-2}u \quad \text{in } \Omega, \qquad u \in W_0^{2,p}(\Omega). \tag{1.2}$$

2. Preliminaries

Definition 2.1. We say that a function $u \in W_0^{2,p}(\Omega)$ is a *weak solution* of (1.2) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v \, dx = \lambda \int_\Omega |u|^{p-2}uv \, dx \quad \text{for all } v \in W_0^{2,p}(\Omega).$$

If u is not identically zero, then we say that λ is an *eigenvalue* of (1.2) corresponding to the eigenfunction u.

The main objective of this work is to show that problem (1.1) has a nontrivial solution for λ in a precise interval. This interval is obtained by using a variational technique based on Ljusternik–Schnirelmann theory on C^1-manifolds [Sz] applied to the auxiliary eigenvalue problem (1.2). We prove that (1.2) has at least one increasing sequence $(\lambda_k)_{k\geq 1}$ of positive eigenvalues. In fact, we give a direct characterization of λ_k involving a mini-max argument over sets of genus greater than k. We set

$$\lambda_1 = \inf\Big\{ \|\Delta v\|_p^p : v \in W_0^{2,p}(\Omega), \int_\Omega |v|^p \, dx = 1 \Big\}, \tag{2.1}$$

where $\|\Delta v\|_p = (\int_\Omega |\Delta v|^p)^{1/p}$ denotes the norm of v in $W_0^{2,p}(\Omega)$. Let us notice that $W_0^{2,p}(\Omega)$ equipped with this norm is a uniformly convex Banach space for $1 < p < \infty$. The norm $\|\Delta \cdot\|_p$ is uniformly equivalent on $W_0^{2,p}(\Omega)$ to the usual norm of $W_0^{2,p}(\Omega)$ [GT].

We see that the value defined in (2.1) can be written as

$$\lambda_1 = \inf_{u\in W_0^{2,p}(\Omega),\, u\not\equiv 0} \frac{\int_\Omega |\Delta u|^p\,dx}{\int_\Omega |u|^p\,dx}. \tag{2.2}$$

Finally, let us point out that problem (1.2) is naturally well defined taking into account the compact embedding $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$.

Definition 2.2. Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle\cdot,\cdot\rangle$. We shall deal with mappings T acting from X into X^*. Strong convergence in X (and in X^*) is denoted by $\to$, and weak convergence by $\rightharpoonup$. A mapping T is said to belong to the class (S^+) if for any sequence u_n in X converging weakly to $u \in X$ with $\limsup_{n\to\infty}\langle \Delta_p^2 u_n, u_n - u\rangle \le 0$, it follows that u_n converges strongly to u in X. We then write $T \in (S^+)$.

Consider now the following two functionals defined on $W_0^{2,p}(\Omega)$:

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p\,dx \quad \text{and} \quad \varphi(u) = \frac{1}{p}\int_\Omega |u|^p\,ds,$$

and set $\mathcal{M} = \{u \in W_0^{2,p}(\Omega) : p\varphi(u) = 1\}$.

Lemma 2.3.

(i) *Φ and φ are even, and of class C^1 on $W_0^{2,p}(\Omega)$.*
(ii) *$\mathcal{M}$ is a closed C^1-manifold.*

Proof. It is clear that φ and Φ are even and of class C^1 on $W_0^{2,p}(\Omega)$ and $\mathcal{M} = \varphi^{-1}\{1/p\}$. Therefore $\mathcal{M}$ is closed. The derivative operator φ' satisfies $\varphi'(u) \ne 0$ for $u \in \mathcal{M}$, i.e., $\varphi'(u)$ is onto for all $u \in \mathcal{M}$. Hence φ is a submersion, which proves that $\mathcal{M}$ is a C^1-manifold. ∎

Remark 2.4 ([EKT, Remark 3.2]). The functional $J : W_0^{2,p}(\Omega) \to W_0^{-2,p'}(\Omega)$ defined by

$$J(u) = \|\Delta u\|_p^{2-p}\Delta_p^2 u \quad \text{if } u \ne 0 \quad \text{and} \quad J(0) = 0,$$

is the duality mapping of $(W_0^{2,p}(\Omega), \|\Delta\cdot\|_p)$ associated with the gauge function $t \mapsto |t|^{p-2}t$.

The following lemma is the key to showing existence.

Lemma 2.5.

(i) *φ' is completely continuous.*
(ii) *The functional Φ satisfies the Palais–Smale condition on $\mathcal{M}$, i.e., for $\{u_k\} \subset \mathcal{M}$, if $\{\Phi(u_k)\}_k$ is bounded and*

$$\alpha_k := \Phi'(u_k) - \beta_k\varphi'(u_k) \to 0 \quad \text{as } k \to \infty, \tag{2.3}$$

where $\beta_k = \langle \Phi'(u_k), u_k\rangle / \langle \varphi'(u_k), u_k\rangle$, then $\{u_k\}_{k\ge1}$ has a subsequence convergent in $W_0^{2,p}(\Omega)$.

Proof. (i) Let $(u_n)_n \subset W_0^{2,p}(\Omega)$ and $u_n \rightharpoonup u$ (weakly) in $W_0^{2,p}(\Omega)$. By Sobolev embedding we deduce that $(u_n)_n$ converges strongly to u in $L^p(\Omega)$, and there exists $w \in L_+^p(\Omega)$ such that

$$|u| \leq w \quad \text{a.e. in } \Omega.$$

Since $w \in L^{p-1}(\Omega)$, it follows from the Lebesgue Dominated Convergence Theorem that

$$|u_n|^{p-2}u_n \to |u|^{p-2}u \quad \text{in } L^{p'}(\Omega).$$

That is,

$$\varphi'(u_n) \to \varphi'(u) \quad \text{in } L^{p'}(\Omega).$$

Recall that the embeddings

$$W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{and} \quad L^{p'}(\Omega) \hookrightarrow W^{-2,p'}(\Omega)$$

are compact. Thus

$$\varphi'(u_n) \to \varphi'(u) \quad \text{in } W^{-2,p'}(\Omega).$$

This proves (i).

For the proof of (ii), we refer the reader to [EKT]. The duality mapping J of Remark 2.4 satisfies the condition (S^+). Therefore, $u_n \to u$ in $W_0^{2,p}(\Omega)$. ∎

3. Main results. Set

$$\Gamma_j = \{K \subset \mathcal{M} : K \text{ symmetric, compact and } \gamma(K) \geq j\},$$

where $\gamma(K) = j$ is the genus of K, i.e., the smallest integer j such that there exists an odd continuous map from K to $\mathbb{R}^j \setminus \{0\}$.

Let us now state our first main result using Ljusternik–Schnirelmann theory.

MAIN THEOREM 3.1. *For any integer* $j \geq 1$,

$$\lambda_j := \inf_{K \in \Gamma_j} \max_{u \in K} p\Phi(u)$$

is a critical value of Φ *restricted to* $\mathcal{M}$. *More precisely, there exists* $u_j \in K$ *such that*

$$\lambda_j = p\Phi(u_j) = \sup_{u \in K} p\Phi(u)$$

and u_j *is a solution of* (1.2) *associated to the positive eigenvalue* λ_j. *Moreover,*

$$\lambda_j \to \infty \quad \text{as } j \to \infty.$$

Proof. We only need to prove that for any $j \geq 1$, $\Gamma_j \neq \emptyset$ and the last assertion. The proof is similar to the proof in [EKT] with appropriate modifications. ∎

COROLLARY 3.2 (see [EKT]).

(i) $\lambda_1 = \inf\{\|\Delta v\|_p^p : v \in W_0^{2,p}(\Omega),\ \int_\Omega |u|^p\,dx = 1\}$,
(ii) $0 < \lambda_1 \leq \cdots \leq \lambda_n \to \infty$.

We now turn to the fourth order eigenvalue problem (1.1).

DEFINITION 3.3. We say that a function $u \in W_0^{2,p}(\Omega)$ is a *weak solution* of (1.1) if

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta v \, dx - \lambda \int_\Omega |u|^{p-2} uv \, ds = \int_\Omega |u|^{p^*-2} uv \, dx \tag{3.1}$$

for all $v \in W_0^{2,p}(\Omega)$. If $u \in W_0^{2,p}(\Omega) \setminus \{0\}$, then we say u is an *eigenfunction* of problem (1.1).

LEMMA 3.4 ([A]). *Assume that Ω is a bounded C^2 domain in $\mathbb{R}^N$. Then $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $p \le q \le Np/(N-2p)$.*

We formulate our second main result of this paper as follows.

MAIN THEOREM 3.5. *There exists $\lambda_1^* < \lambda_1$ such that if $\lambda \in (\lambda_1^*, \lambda_1)$, then* (1.1) *admits a least-energy solution u_λ; these solutions satisfy*

$$u_\lambda \to 0 \text{ in } W_0^{2,p}(\Omega) \quad \text{and} \quad \frac{u_\lambda}{\|\Delta u_\lambda\|_p} \to u^* \text{ in } W_0^{2,p}(\Omega) \quad \text{as } \lambda \to \lambda_1, \tag{3.2}$$

where u^* is an eigenfunction of (1.2) *such that $\|\Delta u^*\|_p = 1$, associated to the principal eigenvalue λ_1 defined by* (2.2).

Proof. Consider the minimization problem

$$\Lambda_\lambda = \inf_{u \in W_0^{2,p}(\Omega) \setminus \{0\}} \frac{\|\Delta u\|_p^p - \lambda \|u\|_p^p}{\|u\|_{p^*}^p}. \tag{3.3}$$

The existence of a least energy solution follows from the proposition below. ∎

PROPOSITION 3.6. *Assume that $0 < \lambda < \lambda_1$. If $\Lambda_\lambda < S$, then the minimum in* (3.3) *is achieved; here S is the best Sobolev constant for the embedding $W^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.*

Proof. Let $(u_m)_{m \ge 0}$ be a minimizing sequence for Λ_λ such that

$$\|u_m\|_{p^*}^p = 1. \tag{3.4}$$

Then

$$\|\Delta u_m\|_p^p - \lambda \|u_m\|_p^p = \Lambda_\lambda + o(1) \quad (\text{as } m \to \infty). \tag{3.5}$$

Moreover, from (2.2), we have

$$\|\Delta u_m\|_p^p = \Lambda_\lambda + \lambda \|u_m\|_p^p + o(1) \le \Lambda_\lambda + \frac{\lambda}{\lambda_1} \|\Delta u_m\|_p^p + o(1), \tag{3.6}$$

which implies that (u_m) is bounded in $W_0^{2,p}(\Omega)$.

Exploiting the compactness of the embedding $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$, we deduce that there exists $u \in W_0^{2,p}(\Omega)$ such that

$$u_m \rightharpoonup u \text{ in } W_0^{2,p}(\Omega) \quad \text{and} \quad u_m \to u \text{ in } L^p(\Omega), \tag{3.7}$$

up to a subsequence. That is, if we set $v_m := u_m - u$, then

$$v_m \rightharpoonup 0 \text{ in } W_0^{2,p}(\Omega) \quad \text{and} \quad v_m \to 0 \text{ in } L^p(\Omega). \tag{3.8}$$

On the other hand, in view of (3.4), we have $\|\Delta u_m\|_p^p \geq S$, so that from (3.5), we obtain

$$\lambda \|u_m\|_p^p = \|\Delta u_m\|_p^p - \Lambda_\lambda + o(1) \geq S - \Lambda_\lambda + o(1),$$

which remains bounded away from 0 since $\Lambda_\lambda < S$. From this, we deduce that $u \neq 0$. Now, thanks to (3.7) and (3.8), we may rewrite (3.5) as

$$\|\Delta u\|_p^p + \|\Delta v_m\|_p^p - \lambda \|u\|_p^p = \Lambda_\lambda + o(1). \tag{3.9}$$

Moreover, by (3.4) and the Brezis–Lieb Lemma [BL], we have

$$\begin{aligned} 1 = \|u + v_m\|_{p^*}^{p^*} = \|u\|_{p^*}^{p^*} + \|v_m\|_{p^*}^{p^*} + o(1) &\leq \|u\|_{p^*}^p + \|v_m\|_{p^*}^p + o(1) \\ &\leq \|u\|_{p^*}^p + \frac{1}{S}\|\Delta v_m\|_p^p + o(1), \end{aligned}$$

where we also use the fact that $\|u\|_{p^*}$ and $\|v_m\|_{p^*}$ do not exceed 1. Since $\Lambda_\lambda \geq 0$ for every $0 < \lambda < \lambda_1$, the last inequality gives

$$\Lambda_\lambda \leq \Lambda_\lambda \|u\|_{p^*}^p + \frac{\Lambda_\lambda}{S}\|\Delta v_m\|_p^p + o(1).$$

By combining this inequality with (3.9), we obtain

$$\begin{aligned} \|\Delta u\|_p^p - \lambda \int_\Omega |u|^p\,dx &= \Lambda_\lambda - \|\Delta v_m\|_p^p + o(1) \\ &\leq \Lambda_\lambda \|u\|_{p^*}^p + (\Lambda_\lambda/S - 1)\|\Delta v_m\|_P^p + o(1) \\ &\leq \Lambda_\lambda \|u\|_{p^*}^p + o(1), \end{aligned}$$

which shows that $u \neq 0$ is minimizer for (3.1) (we will denote it by u_λ).

Notice that if $m = 1$ then the best Sobolev constant for the embedding $W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega)$, for all $p \leq q \leq \frac{Np}{N-mp}$, is equal to

$$\pi^{p/2} N\{(p-1)(N-p)^{-1}\}^{(-p-1/p)} \left\{ \frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)} \right\}^{p/N}$$

(see [Lio2, Sw] and also similar results in [Ta]). Let u^* be a positive eigenfunction of (1.2) and

$$\lambda_1^* = \frac{\|\Delta u^*\|_p^p - S\|u^*\|_{p^*}^p}{\|u^*\|_p^p}\ .$$

Thus $\lambda_1^* < \lambda_1$ and for $\lambda > \lambda_1^*$, we have

$$\Lambda_\lambda \leq \frac{\|\Delta u^*\|_p^p - \lambda \|u^*\|_p^p}{\|u^*\|_{p^*}^p} < S.$$

We now prove the first part of (3.2). Indeed, in view of the characterization of u^* in (3.1), we have

$$\Lambda_\lambda \le \frac{\|\Delta u^*\|_p^p - \lambda \|u^*\|_{L^p(\Gamma)}^p}{\|u^*\|_{p^*}^p} = \frac{1-\lambda/\lambda_1}{\|u^*\|_{p^*}^p} \to 0 \quad \text{as } \lambda \to \lambda_1. \tag{3.10}$$

Since u_λ is a least energy solution of (1.1), we have

$$\frac{\|\Delta u_\lambda\|_p^p - \lambda \|u_\lambda\|_p^p}{\|u_\lambda\|_{p^*}^p} = \Lambda_\lambda. \tag{3.11}$$

In fact, u_λ is the principal eigenfunction associated to Λ_λ. Moreover, by taking $v = u_\lambda$ in (3.1), we get

$$\|\Delta u_\lambda\|_p^p - \lambda \|u_\lambda\|_p^p = \|u_\lambda\|_{p^*}^{p^*}. \tag{3.12}$$

Identities (3.11)–(3.12) readily imply that $\|u_\lambda\|_{p^*} = \Lambda_\lambda^{(N-2p)/(2p^2)}$. In turn, this and (3.9) show that

$$u_\lambda \to 0 \quad \text{in } L^{p^*}(\Omega) \text{ as } \lambda \to \lambda_1. \tag{3.13}$$

Moreover, by (3.10) and (3.11) we obtain

$$\|\Delta u_\lambda\|_p^p - \lambda \|u_\lambda\|_p^p \le \frac{1-\lambda/\lambda_1}{\|u^*\|_{p^*}^p} \|u_\lambda\|_{p^*}^p.$$

Then in view of (2.2), we get

$$\|\Delta u_\lambda\|_p^p \le \frac{\lambda}{\lambda_1} \|\Delta u_\lambda\|_p^p + \frac{1-\lambda/\lambda_1}{\|u_1\|_{p^*}^p} \|u_\lambda\|_{p^*}^p.$$

Hence

$$(1 - \frac{\lambda}{\lambda_1}) \|\Delta u_\lambda\|_p^p \le \frac{1-\lambda/\lambda_1}{\|u^*\|_{p^*}^p} \|u_\lambda\|_{p^*}^p.$$

Consequently, using (3.13), the last inequality implies that

$$\|\Delta u_\lambda\|_p^p \le \|u^*\|_{p^*}^{-p} \|u_\lambda\|_{p^*}^p \to 0 \quad \text{as } \lambda \to \lambda_1. \tag{3.14}$$

Finally, we conclude that

$$u_\lambda \to 0 \quad \text{in } W_0^{2,p}(\Omega) \text{ as } \lambda \to \lambda_1. \tag{3.15}$$

From [Lin], the inequality

$$|t_1 - t_2|^p \le c\{(|t_1|^{p-2} t_1 - |t_2|^{p-2} t_2).(t_1 - t_2)\}^{\gamma/2} (|t_1|^p + |t_2|^2)^{1-\gamma/2}$$

holds true for any $t_1, t_2 \in \mathbb{R}$ with $\gamma = p$ if $1 < p < 2$ and $\gamma = 2$ if $p \ge 2$. By applying Hölder's inequality, we have

$$\begin{aligned}\left\|\frac{\Delta u_\lambda}{\|\Delta u_\lambda\|_p}-\Delta u^*\right\|_p^p &= \frac{1}{\|\Delta u_\lambda\|_p^p}\big\|\Delta u_\lambda-\|\Delta u_\lambda\|_p\Delta u^*\big\|_p^p\\ &\le \frac{c}{\|\Delta u_\lambda\|_p^{p-1}}\{G(u_\lambda,u^*)\}^{\gamma/2}(\|\Delta u_\lambda\|_p^p+\|\Delta u_\lambda\|_p^p\|\Delta u^*\|^p)^{1-\gamma/2}\\ &\le \frac{c}{\|\Delta u_\lambda\|_p^{p\gamma/2-1}}\{G(u_\lambda,u^*)\}^{\gamma/2}(1+\|\Delta u^*\|^p)^{1-\gamma/2},\end{aligned}$$

where

$$\begin{aligned}G(u_\lambda,u^*) &= \int_\Omega \big(|\Delta u_\lambda|^{p-2}\Delta u_\lambda-\big|\|\Delta u_\lambda\|_p\Delta u^*|^{p-2}(\|\Delta u_\lambda\|_p\Delta u^*)\big)\\ &\qquad\qquad\cdot(\Delta u_\lambda-\|\Delta u_\lambda\|_p\Delta u^*)\,dx\\ &\le \int_\Omega \big(|\Delta u_\lambda|^p+\|\Delta u_\lambda\|_p|\Delta u_\lambda|^{p-1}|\Delta u^*|\\ &\qquad +\|\Delta u_\lambda\|_p^{p-1}|\Delta u^*|^{p-1}|\Delta u_\lambda|+\|\Delta u_\lambda\|_p^p|\Delta u^*|^p\big)\,dx.\end{aligned}$$

Then using Hölder's inequality again, we obtain

$$\begin{aligned}&G(u_\lambda,u^*)\\ &\quad\le \|\Delta u_\lambda\|_p^p+\|\Delta u_\lambda\|_p^p\|\Delta u^*\|_p^p+\|\Delta u_\lambda\|_p^{2p-1}\|\Delta u^*\|_p^{p-1}+\|\Delta u_\lambda\|_p^p\|\Delta u^*\|_p^p\\ &\quad\le \|\Delta u_\lambda\|_p^p\big(1+\|\Delta u^*\|_p^p+\|\Delta u_\lambda\|_p^{p-1}\|\Delta u^*\|_p^{p-1}+\|\Delta u^*\|_p^p\big).\end{aligned}$$

Hence

$$\begin{aligned}&\left\|\frac{\Delta u_\lambda}{\|\Delta u_\lambda\|_p}-\Delta u^*\right\|_p^p\\ &\le \|\Delta u_\lambda\|_p\big(1+\|\Delta u^*\|_p^p+\|\Delta u_\lambda\|_p^{p-1}\|\Delta u^*\|_p^{p-1}+\|\Delta u^*\|_p^p\big)^{\gamma/2}(1+\|\Delta u^*\|^p)^{1-\gamma/2}.\end{aligned}$$

Finally, we conclude that

$$\left\|\frac{\Delta u_\lambda}{\|\Delta u_\lambda\|_p}-\Delta u^*\right\|_p^p\to 0\quad\text{as } \lambda\to\lambda_1,$$

which proves (3.2) and completes the proof of Theorem 3.4. ∎

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