APPLICATIONES MATHEMATICAE
37,1 (2010), pp. 1–12

Joanna Rencławowicz and Wojciech M. Zajączkowski (Warszawa)

EXISTENCE OF SOLUTIONS TO THE POISSON EQUATION IN L_p-WEIGHTED SPACES

Abstract. We examine the Poisson equation with boundary conditions on a cylinder in a weighted space of L_p, $p \geq 3$, type. The weight is a positive power of the distance from a distinguished plane. To prove the existence of solutions we use our result on existence in a weighted L_2 space.

1. Introduction. In the paper, we continue the study of the Poisson equation in weighted spaces formulated in [RZ] but here we consider the more general situation, i.e. spaces based on L_p, $p \geq 2$. We consider the following problem:

$$
\begin{aligned}
-\Delta\varphi &= f' \quad \text{in } \Omega',\\
\bar{n}\cdot\nabla\varphi|_{S_*} &= 0,\\
\varphi|_{S_1} &= 0,\\
\varphi|_{S_0} &= 0,
\end{aligned}
\tag{1.1}
$$

where $\Omega' \subset \mathbb{R}^3$ is a cylindrical domain, $\partial\Omega' = S_0 \cup S_1 \cup S_* = S$ (see Fig. 1).

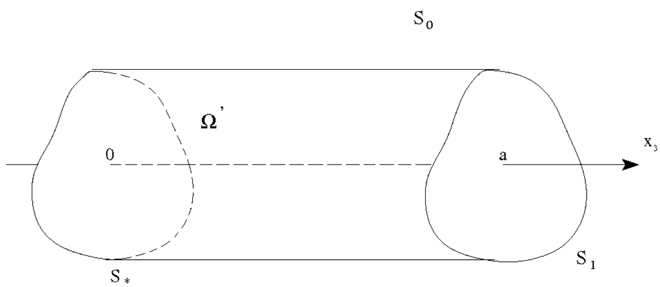

Fig. 1. Domain Ω'

2010 *Mathematics Subject Classification*: Primary 35J05; Secondary 31A30, 35J25.
Key words and phrases: Poisson equation, weighted Sobolev spaces, Neumann boundary-value problem, Dirichlet boundary-value problem.

DOI: 10.4064/am37-1-1

Here, S_0 is parallel and S_1 and S_* are perpendicular to the x_3 axis, and S_1 meets the x_3 axis at the point $x_3 = a$ (while S_* meets the x_3 axis at the point $x_3 = 0$).

We assume that $f' \in L_{p,\mu}(\Omega')$ where

$$\|f'\|_{L_{p,\mu}(\Omega')} = \Big(\int\limits_{\Omega'} |f'(x)|^p x_3^{p\mu}\, dx\Big)^{1/p}, \quad p \in [2,\infty),\ \mu \in (0,1).$$

The motivation of the problem is the analysis of the inflow-outflow motion described by the Navier–Stokes equations. In order to avoid some restrictions on the boundary inflow for the Navier–Stokes system, we will use the weighted space estimates derived here. Namely, the proof of global existence in [Z2] requires that the inflow flux must vanish sufficiently fast as $t \to \infty$. To show the existence of solutions with nonvanishing inflow flux or of periodic solutions we need estimates in weighted spaces derived in this paper and in [RZ].

To proceed, we first reformulate the problem (1.1). We extend the solutions φ to $x_3 < 0$ using the zero Neumann boundary conditions $(1.1)_2$. Consequently, we construct an even function v by setting:

$$v(x_3) = \begin{cases} \varphi(x_3) & \text{for } x_3 \geq 0, \\ \varphi(-x_3) & \text{for } x_3 < 0. \end{cases}$$

Then $u = v - \varphi(0)$ is a solution to the following problem on Ω:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u|_{S_0\cup S_1} &= 0, \end{aligned} \tag{1.2}$$

where $f = f' + \Delta\varphi(0)$, with the domain Ω described by Fig. 2.

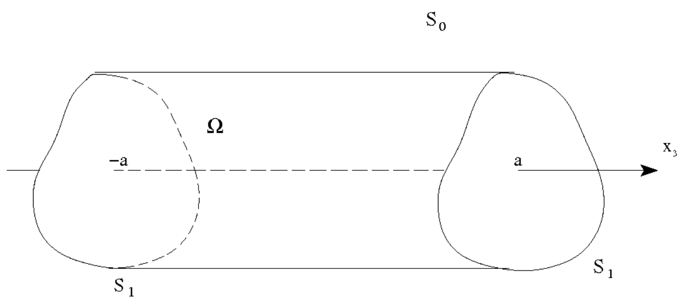

Fig. 2. Domain Ω

Here $\partial\Omega = S_0 \cup S_1$ and $f \in L_{p,\mu}(\Omega)$, where

$$\|f\|_{L_{p,\mu}(\Omega)} = \Big(\int\limits_{\Omega} |f(x)|^p |x_3|^{p\mu}\, dx\Big)^{1/p}, \quad p \in [2,\infty),\ \mu \in (0,1).$$

The technique of weighted Sobolev spaces close the one presented in this paper was developed in [Z1, Z3, Z4, ZS].

We organize the paper as follows. In Section 2, we collect the definitions and notation used, as well as some auxiliary facts. In Section 3, we prove the main estimates and the existence theorem.

2. Notation and auxiliary results. We use the weighted space $L_{p,\mu}$ and define the following spaces and norms. We introduce the weighted spaces $V^l_{p,\beta}(Q)$, $Q \subset \mathbb{R}^3$:

$$\|u\|_{V^l_{p,\beta}}(Q) = \Big(\sum_{|\alpha|\le l} \int_Q dx'\, dx_3\, |D^\alpha_x u|^p |x_3|^{p(\beta+|\alpha|-l)} \Big)^{1/p}, \qquad \beta \in \mathbb{R},\ l \in \mathbb{N} \cup \{0\},$$

where $x' = (x_1, x_2)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index. We observe

$$V^0_{p,\beta}(Q) = L_{p,\beta}(Q), \qquad V^l_{2,\beta}(Q) = H^l_\beta(Q).$$

For a function $g : \mathbb{R}^3 \to \mathbb{R}^3$, we denote by $\hat{g}$ its (partial) Fourier transform:

$$\hat{g}(\xi, x_3) = \int_{\mathbb{R}^2} e^{-ix'\cdot\xi} g(x', x_3)\, dx',$$

where $\xi = (\xi_1, \xi_2)$, $x' \cdot \xi = x_1\xi_1 + x_2\xi_2$. We will use the following local regularity result in Sobolev spaces W^2_p ([Mo]).

PROPOSITION 2.1 (local regularity). *Let $B_r = \{x' \in \mathbb{R}^2 : |x'| < r\}$, $\xi_i = \xi_i(x_3)$, $\xi_i \in C^\infty_0(\mathbb{R}), i = 1, 2, \xi_1\xi_2 = \xi_1, \operatorname{supp} \xi_2 \subset \{x_3 : c_1 < |x_3| < c_2\}$. Then for a function $u \in W^2_p(B_2 \times \mathbb{R})$ the following inequality holds:*

$$\|\xi_1 u\|_{W^2_p(B\times\mathbb{R})} \le c(\|\xi_2 \varDelta u\|_{L_p(B_2\times\mathbb{R})} + \|\xi_2 u\|_{L_{p_1}(B_2\times\mathbb{R})}),$$

for some $p_1 \in [1, \infty]$ and the constant c independent of u.

We will also need the following Marcinkiewicz–Mikhlin type result (see [DS, Part 2, Ch. 11, Theorem 28]).

PROPOSITION 2.2. *Let $L_p(\mathbb{R}^d; H)$ be the space of functions with the finite norm*

$$\|f\|_{L_p(\mathbb{R}^d;H)} = \Big(\int_{\mathbb{R}^d} \|f(z, \cdot)\|^p_H\, dz \Big)^{1/p} < \infty,$$

where H is a Hilbert space. Let $M(\xi)$, $\xi \in \mathbb{R}^d$, be a bounded linear operator in H. Assume that for $s = 0, \dots, d, i_k \ne i_l$,

$$|\xi|^s \left\| \frac{\partial^s M}{\partial\xi_{i_1} \dots \partial\xi_{i_s}}(\varphi) \right\|_{H\to H} \le \text{const}.$$

Then, if F is the Fourier transform in $\mathbb{R}^d$, $F^{-1}_{\xi\to z} M(\varphi) F_{z\to\xi}$ is a continuous operator in $L_p(\mathbb{R}^d; H)$.

To prove the main estimates, we will need the construction of some partitions of unity. Namely, we introduce the following families of functions.

DEFINITION 2.3 (partitions of unity). Consider families $\{\zeta_j\}_{j=-\infty}^{\infty}$, $\{\sigma_j\}_{j=-\infty}^{\infty}$ where $\zeta_j, \sigma_j \in C^\infty(\mathbb{R})$ satisfy

- $\operatorname{supp} \zeta_j \subset \{x_3 : 2^{j-1} < |x_3| < 2^{j+1}\}$,
- $\operatorname{supp} \sigma_j \subset \{x_3 : 2^{j-2} < |x_3| < 2^{j+2}\}$,
- $\zeta_j \sigma_j = \zeta_j$,
- $|D^\alpha \zeta_j| + |D^\alpha \sigma_j| \le c_\alpha 2^{-j|\alpha|}$ for all multiindices α.

Properties of these partitions make it possible to show the following statement:

LEMMA 2.4. *Let $\beta \in \mathbb{R}$. Then for any function*

$$u \in W_p^2(\mathbb{R}^2 \times \{2^{j-2} < |x_3| < 2^{j+2}\})$$

the following inequality holds:

$$\|\zeta_j u\|_{V^2_{p,\beta}(\mathbb{R}^2\times\mathbb{R})} \le c\|\sigma_j \Delta u\|_{L_{p,\beta}(\mathbb{R}^2\times\mathbb{R})} + c\|\sigma_j u\|_{L_{p,\beta-2}(\mathbb{R}^2\times\mathbb{R})}. \tag{2.1}$$

Proof. We define

$$\begin{aligned} B &= \{x' : |x'| < 2\}, \\ B_\mu &= \{x' : |x'| < 2^{\mu+1}\}, \\ K &= \{x_3 : 1 < |x_3| < 2\}, \\ K_\mu &= \{x_3 : 2^\mu < |x_3| < 2^{\mu+1}\}. \end{aligned}$$

We can apply Proposition 2.1 with $p_1 = p$ to obtain

$$\sum_{|\alpha|=0}^{2} \|D^\alpha(\zeta_j u)\|_{L_p(B\times K)} \le c\|\sigma_j \Delta u\|_{L_p(B_1\times 2K)} + c\|\sigma_j u\|_{L_p(B_1\times 2K)}$$

with $2K = \{x_3 : 1/2 < |x_3| < 4\}$.

In view of scaling $x \mapsto 2^\mu x$ we have

$$\sum_{|\alpha|=0}^{2} 2^{\mu(|\alpha|-2)} \|D^\alpha(\zeta_j u)\|_{L_p(B_\mu\times K_\mu)} \le c\|\sigma_j \Delta u\|_{L_p(B_{\mu+1}\times 2K_\mu)} + c2^{-2\mu}\|\sigma_j u\|_{L_p(B_{\mu+1}\times 2K_\mu)}.$$

Now we multiply this formula by $2^{\beta\mu}$ and then raise the resulting inequality to the power p. Next, we note that $\rho = |x_3| \sim 2^\mu$ and we sum over μ to obtain

$$\sum_{|\alpha|=0}^{2} \|\rho^{\beta+|\alpha|-2} D^\alpha(\zeta_j u)\|_{L_p(\mathbb{R}^3)} \le c\|\rho^\beta \sigma_j \Delta u\|_{L_p(\mathbb{R}^3)} + c\|\rho^{\beta-2}\sigma_j u\|_{L_p(\mathbb{R}^3)}.$$

Applying the definition of the spaces $V^2_{p,\beta}$ and $L_{p,\beta}$ to this estimate yields (2.1). ∎

COROLLARY 2.5. *For $\beta \in \mathbb{R}$ and u as in Lemma 2.4,*

$$\|u\|_{V^2_{p,\beta}(\mathbb{R}^3)} \le c\|\Delta u\|_{L_{p,\beta}(\mathbb{R}^3)} + c\|u\|_{L_{p,\beta-2}(\mathbb{R}^3)}. \tag{2.2}$$

Proof. We sum up inequality (2.1) with respect to j to obtain the result. ∎

We apply Definition 2.3 to formulate a result on operators in Banach spaces. Let $\mathcal{E}_0(\mathbb{R}^3), \mathcal{E}_1(\mathbb{R}^3)$ be Banach spaces of functions defined on $\mathbb{R}^3$, closed under pointwise multiplication by functions from $C_0^\infty(\mathbb{R}^2 \times \mathbb{R} \setminus \{0\})$.

Let $\{\zeta_j(x_3)\}_{j=-\infty}^{\infty}$ be the partition of unity described in Definition 2.3.

Assume that there exist p and q, $1 \le p \le q \le \infty$, such that for all $u \in \mathcal{E}_0, v \in \mathcal{E}_1$ the following inequalities hold:

$$\|u\|_{\mathcal{E}_0(\mathbb{R}^3)} \le c\Big(\sum_{j=-\infty}^{\infty} \|\zeta_j u\|^q_{\mathcal{E}_0(\mathbb{R}^3)}\Big)^{1/q}, \tag{2.3}$$

$$\|v\|_{\mathcal{E}_1(\mathbb{R}^3)} \ge c\Big(\sum_{j=-\infty}^{\infty} \|\zeta_j v\|^p_{\mathcal{E}_1(\mathbb{R}^3)}\Big)^{1/p}, \tag{2.4}$$

where $\|\cdot\|_{\mathcal{E}_i(\mathbb{R}^3)}$ is the norm of $\mathcal{E}_i(\mathbb{R}^3)$.

PROPOSITION 2.6 ([MP], [Z2]). *Let $\mathcal{O} : \mathcal{E}_1(\mathbb{R}^3) \to \mathcal{E}_0(\mathbb{R}^3)$ be a linear operator defined on functions with compact support such that for some $\varepsilon > 0$ and arbitrary $\mu, \nu \in \mathbb{Z}$,*

$$\|\zeta_\mu \mathcal{O} \zeta_\nu v\|_{\mathcal{E}_0(\mathbb{R}^3)} \le e^{-\varepsilon|\mu-\nu|}\|\zeta_\nu v\|_{\mathcal{E}_1(\mathbb{R}^3)} \tag{2.5}$$

where $v \in \mathcal{E}_1(\mathbb{R}^3)$. Assume that (2.3)–(2.4) *are satisfied. Then, for v with a compact support,*

$$\|\mathcal{O}v\|_{\mathcal{E}_0(\mathbb{R}^3)} \le c\|v\|_{\mathcal{E}_1(\mathbb{R}^3)}, \tag{2.6}$$

where c does not depend on v.

3. Main estimates. Our goal in this part will be an estimate of the form

$$\|u\|_{V^2_{p,\alpha}(\mathbb{R}^3)} \le c\|f\|_{V^0_{p,\alpha}(\mathbb{R}^3)} \tag{3.1}$$

with $p \ge 2$ and for u, f defined by (1.2).

To show the result, we start with the estimate (2.2) in Corollary 2.5 and next, to estimate the function u on the right hand side of (2.2)—in another weighted space—we will need to examine the related functions and equations. Namely, we need to consider an auxiliary problem and derive an estimate that is of the form of condition (2.5) in Proposition 2.6. The statement of Proposition 2.6 with a suitably defined operator transforming the auxiliary problem into the original one will give the desired estimate for u. Then we collect these facts and obtain the inequality for an appropriate choice of the weight parameter α.

Let

$$P(\partial_{x'},\partial_{x_3}) = -\Delta \quad \text{and} \quad P(\xi,\partial_{x_3}) = -\partial_{x_3}^2 + \xi^2.$$

Let $A(\xi)$ denote the operator of the problem

$$\begin{aligned} P(\xi,\partial_{x_3})\hat{u} &= \hat{f} \quad \text{in } \mathbb{R}, \\ \hat{u}|_{x_3=\pm a} &= 0, \end{aligned} \tag{3.2}$$

where $\hat{g}$ denotes the Fourier transform of g. We show the following simple property of A:

Lemma 3.1. *For $A(\xi)$ introduced in* (3.2), $\ker A(\xi) = 0$.

Proof. We take $\xi \neq 0$. Every solution of the homogeneous equation $(3.2)_1$ has the form

$$\hat{u} = \alpha \sinh(|\xi|x_3) + \beta \cosh(|\xi|x_3)$$

and $(3.2)_2$ implies

$$\begin{aligned} -\alpha \sinh(|\xi|a) + \beta \cosh(|\xi|a) &= 0, \\ \alpha \sinh(|\xi|a) + \beta \cosh(|\xi|a) &= 0, \end{aligned}$$

so that $\alpha = \beta = 0$.

If $\xi = 0$, then any solution to $(3.2)_1$ has the form

$$\hat{u} = \alpha x_3 + \beta$$

and now $(3.2)_2$ gives

$$\alpha a + \beta = 0, \qquad -\alpha a + \beta = 0,$$

so $\alpha = \beta = 0$. This concludes the proof. ■

Corollary 3.2. *There exists an inverse operator $A^{-1}(\xi)$ to problem* (3.2) *such that*

$$\hat{u}(\xi,x_3) = A^{-1}(\xi)\hat{f}(\xi,x_3).$$

Next, we prove a useful property of the operator A^{-1}.

Lemma 3.3. *For any $\xi \in \mathbb{R}^2\backslash\{0\}$ and for any $f \in V^0_{2,\beta}(\mathbb{R})$, $\gamma \in \mathbb{N} \cup \{0\}$, $\beta \neq \mathbb{Z}$,*

$$\sum_{\nu=0}^{2} |\xi|^\nu \|\partial_\xi^\gamma A(\xi)^{-1}\hat{f}\|_{V^{2-\nu}_{2,\beta}(\mathbb{R})} \le c|\xi|^{-\gamma}\|\hat{f}\|_{V^0_{2,\beta}(\mathbb{R})}. \tag{3.3}$$

Proof. This is a special case of Lemma 7.1 in Chapter 7 of [MP] for a general operator A. ■

Lemma 3.4. *Let the assumptions of Lemma* 3.3 *be satisfied. Then the operator P of the problem*

$$P(\partial_{x'},\partial_{x_3})u = f \tag{3.4}$$

is an isomorphism

$$P : V^2_{2,\beta}(\mathbb{R}^3) \to V^0_{2,\beta}(\mathbb{R}^3).$$

Proof. The previous lemma and (3.3) with $\nu = 0$ give

$$\|\hat{u}\|_{V^2_{2,\beta}(\mathbb{R})} = \|A(\xi)^{-1}\hat{f}\|_{V^2_{2,\beta}(\mathbb{R})} \le c\|\hat{f}\|_{V^0_{2,\beta}(\mathbb{R})}.$$

To get a similar result in $\mathbb{R}^3$ we use the L_2-estimates from [RZ]. ■

Let us now consider an auxiliary problem. We take functions ζ_ν, σ_μ with properties described in Definition 2.3 and set $f_\nu = f\zeta_\nu$. We examine the problem

$$P(\xi, \partial_{x_3})u = f_\nu \text{ in } \mathbb{R},$$
$$\hat{u}\big|_{x_3=\pm a} = 0.$$

Then

$$u_\mu = \sigma_\mu A^{-1} f_\nu \tag{3.5}$$

where $u_\mu = u\sigma_\mu$. For the operator of the above problem we prove the following estimate:

Lemma 3.5. *With the assumptions of Lemma* 3.3 *and* $\xi \ne 0$, *we have*

$$\|\sigma_\mu A^{-1}(\xi)\zeta_\nu\|_{V^0_{2,\beta}(\mathbb{R}) \to V^0_{2,\beta}(\mathbb{R})} \le c2^{-\varepsilon|\mu-\nu|+2\mu} \tag{3.6}$$

for $\beta \notin \mathbb{Z}$ *and* ε *sufficiently small.*

Proof. We use the estimate (3.3) and properties of the partition of unity defined in Definition 2.3 to obtain

$$\|u_\mu\|_{V^2_{2,\beta}(\mathbb{R}^2)} \le 2^{\varepsilon\mu}\|u_\mu\|_{V^2_{2,\beta-\varepsilon}(\mathbb{R}^2)} \le 2^{\varepsilon\mu}\|f_\nu\|_{V^0_{2,\beta-\varepsilon}(\mathbb{R}^2)} \le 2^{\varepsilon|\mu-\nu|}\|f_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)}.$$

Since

$$\|u_\mu\|_{V^2_{2,\beta}(\mathbb{R}^2)} \ge 2^{-2\mu}\|u_\mu\|_{V^0_{2,\beta}(\mathbb{R}^2)}$$

we conclude

$$\|u_\mu\|_{V^0_{2,\beta}(\mathbb{R}^2)} \le 2^{\varepsilon|\mu-\nu|+2\mu}\|f_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)}. \tag{3.7}$$

Taking $-\varepsilon$ instead of ε we get similarly

$$\|u_\mu\|_{V^0_{2,\beta}(\mathbb{R}^2)} \le 2^{-\varepsilon(\mu-\nu)+2\mu}\|f_\nu\|_{V^0_{2,\beta}(\mathbb{R}^2)}. \tag{3.8}$$

From (3.7) and (3.8), we apply (3.5) to derive the desired estimate (3.6). This concludes the proof. ■

We need to improve this result: we derive an L_p estimate using the Marcinkiewicz–Mikhlin theorem (Proposition 2.2). We now examine the original problem in the form (3.4) with the right hand side f_ν.

Lemma 3.6. *Let the assumptions of Lemma* 3.3 *be satisfied and* $u_\nu \in V^2_{2,\beta}(\mathbb{R}^3)$ *and*

$$P(\partial_{x'}, \partial_{x_3}) u_\nu = f_\nu. \tag{3.9}$$

Then

$$\begin{aligned}\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} |x_3|^{2\beta} |\sigma_\mu(x_3) u_\nu(x', x_3)|^2 \, dx_3 \Big)^{p/2} dx' \\ \le c 2^{-p\varepsilon|\mu-\nu|+2\mu p} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} |x_3|^{2\beta} |\zeta_\nu(x_3) f(x', x_3)|^2 \, dx_3 \Big)^{p/2} dx'.\end{aligned} \tag{3.10}$$

Proof. We have

$$u_\nu = F^{-1}_{\xi\to x'} A(\xi)^{-1} F_{x'\to\xi} \zeta_\nu f$$

where F denotes the Fourier transform in $\mathbb{R}^2$. On the other hand, we can apply Proposition 2.2 to find that $F^{-1} M(\xi) F$, where $M(\xi) = \sigma_\mu A(\xi)^{-1} \zeta_\nu$, is a continuous operator in $L_p(\mathbb{R}^2; V^0_{2,\beta}(\mathbb{R}^1))$. Thus, using estimate (3.6) we will derive the result. ∎

We work with the bound above to obtain

Lemma 3.7. *Let the assumptions of Lemma* 3.3 *be satisfied and* $u_\nu \in V^2_{2,\beta}$. *Then for* $p \ge 2$ *and some* $\varepsilon_1 > 0$,

$$\begin{aligned}\int_{\mathbb{R}^3} |x_3|^{p(\beta-1)-2} |\zeta_\mu(x_3) u_\nu(x', x_3)|^p \, dx_3 \, dx' \\ \le c 2^{-|\mu-\nu|\varepsilon_1 p} \int_{\mathbb{R}^3} |x_3|^{p(\beta+1)-2} |\zeta_\nu(x_3) f(x', x_3)|^p \, dx_3 \, dx'.\end{aligned} \tag{3.11}$$

Proof. By the Hölder inequality we estimate the integral on the right hand side of (3.10) as follows:

$$\begin{aligned}&\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} |x_3|^{2\beta} |\zeta_\nu(x_3) f(x', x_3)|^2 \, dx_3 \Big)^{p/2} dx' \\ &\le \int_{\mathbb{R}^2} \Big[\Big(\int_{\operatorname{supp}\zeta_\nu} 1^{p/(p-2)} \, dx_3 \Big)^{(p-2)/p} \cdot \Big(\int_{\mathbb{R}} |x_3|^{p\beta} |\zeta_\nu(x_3) f(x', x_3)|^p \, dx_3 \Big)^{2/p} \Big]^{p/2} dx' \\ &\le \int_{\mathbb{R}^2} 2^{(\nu-1)(p-2)/2} \int_{\mathbb{R}} |x_3|^{p\beta} |\zeta_\nu(x_3) f(x', x_3)|^p \, dx_3 \, dx' \\ &\le c \int_{\mathbb{R}^2} \int_{\mathbb{R}} |x_3|^{p(\beta+1/2)-1} |\zeta_\nu(x_3) f(x', x_3)|^p \, dx_3 \, dx'\end{aligned}$$

because on $\operatorname{supp} \zeta_\nu$ we have $|x_3| \in (2^{\nu-1}, 2^{\nu+1})$.

Next, we deal with the left hand side of (3.10). We cover $\mathbb{R}^2$ with balls Q_j of radius $2^{\mu-1}$. By $2Q_j$ we denote a ball of radius $2^{\mu+1}$ which contains Q_j. By the Hölder inequality we have

$$I_1 \equiv \int\limits_{2Q_j}\int\limits_{\mathbb{R}} |\sigma_\mu u_\nu|\,dx'\,dx_3 \le \int\limits_{2Q_j}\Big(\int\limits_{\operatorname{supp}\sigma_\mu} 1^2\Big)^{1/2}\Big(\int\limits_{\mathbb{R}}|\sigma_\mu u_\nu|^2\Big)^{1/2} dx'$$
$$\le \int\limits_{2Q_j} 2^{\mu/2+1}\Big(\int\limits_{\mathbb{R}}|\sigma_\mu u_\nu|^2\,dx_3\Big)^{1/2} dx' \equiv I_2$$

where we have used the fact that $\sigma_\mu \subset \{x_3 : 2^{\mu-2} < |x_3| < 2^{\mu+2}\}$. Next, we obtain

$$I_2 \le c\int\limits_{2Q_j} 2^{\mu/2-\beta\mu}\Big(\int\limits_{\mathbb{R}}|x_3|^{2\beta}|\sigma_\mu u_\nu|^2\,dx_3\Big)^{1/2} dx'$$

where we have applied again the support of σ_μ. Therefore,

$$(3.12)\qquad (I_1)^p = \Big(\int\limits_{2Q_j}\int\limits_{\mathbb{R}}|\sigma_\mu u_\nu|\,dx'\,dx_3\Big)^p$$
$$\le c2^{(1/2-\beta)\mu p}\Big[\int\limits_{2Q_j}\Big(\int\limits_{\mathbb{R}}|x_3|^{2\beta}|\sigma_\mu u_\nu|^2\,dx_3\Big)^{1/2} dx'\Big]^p$$
$$\le c2^{(1/2-\beta)\mu p}\Big[\Big(\int\limits_{2Q_j} 1^{p'}\,dx'\Big)^{1/p'}$$
$$\times\Big(\int\limits_{2Q_j}\Big(\int\limits_{\mathbb{R}}|x_3|^{2\beta}|\sigma_\mu u_\nu|^2\,dx_3\Big)^{p/2} dx'\Big)^{1/p}\Big]^p$$
$$\le c2^{(1/2-\beta)\mu p+2\mu(p-1)}\int\limits_{2Q_j}\Big(\int\limits_{\mathbb{R}}|x_3|^{2\beta}|\sigma_\mu u_\nu|^2\,dx_3\Big)^{p/2} dx'$$

where $p' = p/(p-1)$, $p/p' = p-1$. In view of the properties of the partition of unity, we apply the local regularity result (Proposition 2.1) and scaling $x \mapsto 2^\mu x$ to obtain the following inequality for $|\mu-\nu| > 3$:

$$(3.13)\qquad \int\limits_{Q_j}\int\limits_{\mathbb{R}}|\zeta_\mu u_\nu|^p\,dx'\,dx_3 \le c2^{3\mu(1-p)}\Big(\int\limits_{2Q_j}\int\limits_{\mathbb{R}}|\sigma_\mu u_\nu|\,dx'\,dx_3\Big)^p$$
$$\equiv c2^{3\mu(1-p)}(I_1)^p.$$

Therefore, combining (3.12) and (3.13), for $|\mu-\nu|>3$ we derive the inequality

$$(3.14)\qquad \int\limits_{\mathbb{R}^3}|\zeta_\mu u_\nu|^p\,dx'\,dx_3 \le c2^{-\mu[p(\beta+1/2)-1]}\int\limits_{\mathbb{R}^2}\Big(\int\limits_{\mathbb{R}}|x_3|^{2\beta}|\sigma_\mu u_\nu|^2\,dx_3\Big)^{p/2} dx'.$$

Next, we apply the inequality (3.10) and the bound of the r.h.s. shown above to obtain

$$2^{\mu[p(\beta-3/2)-1]}\int\limits_{\mathbb{R}^3}|\zeta_\mu u_\nu|^p\,dx_3\,dx' \le c2^{-p\varepsilon|\mu-\nu|}\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta+1/2)-1}|\zeta_\nu f|^p\,dx_3\,dx'.$$

We multiply both sides by $2^{\mu(p/2-1)}$, use the support of ζ_μ on the left hand side and the support of ζ_ν on the right hand side. Therefore, we get

$$\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta-1)-2}|\zeta_\mu(x_3)u_\nu(x',x_3)|^p\,dx_3\,dx' \le c2^{|\mu-\nu|(-p\varepsilon+p/2-1)}\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta+1)-2}|\zeta_\nu(x_3)f(x',x_3)|^p\,dx_3\,dx'. \tag{3.15}$$

We note that in the case $|\mu-\nu|\le 3$ we add on the r.h.s. of (3.13) the expression

$$c2^{2\mu p}\int\limits_{2Q_j}\int\limits_{\mathbb{R}}|\zeta_\nu f|^p\,dx'\,dx_3$$

and consequently also on the r.h.s. of (3.14). Therefore, taking $\varepsilon>1/2$, we set $\varepsilon_1=\varepsilon-1/2+1/p$ to conclude the proof. ■

Let us observe that we have some bounds for u_μ, the solution of the auxiliary problem defined in (3.9). Now we are going to prove the analogous result but for the original problem with the functions u and f.

Lemma 3.8. *Let u solve*

$$P(\partial_{x'},\partial_{x_3})u=f$$

and let the assumptions of Lemma 3.3 be satisfied, $p\ge 2$. Then

$$\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta-1)-2}|u(x',x_3)|^p\,dx_3\,dx' \le c\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta+1)-2}|f(x',x_3)|^p\,dx_3\,dx'. \tag{3.16}$$

Proof. To deal with the problem for u and f, we apply Proposition 2.6 to the situation of Lemma 3.7. Namely, we note that estimate (3.11), which we recall here:

$$\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta-1)-2}|\zeta_\mu(x_3)u_\nu(x',x_3)|^p\,dx_3\,dx' \le c2^{-|\mu-\nu|\varepsilon_1 p}\int\limits_{\mathbb{R}^3}|x_3|^{p(\beta+1)-2}|\zeta_\nu(x_3)f(x',x_3)|^p\,dx_3\,dx',$$

is of the form of condition (2.5), i.e.

$$\|\zeta_\mu\mathcal{O}\zeta_\nu v\|_{\mathcal{E}_0(\mathbb{R}^3)}\le e^{-\varepsilon|\mu-\nu|}\|\zeta_\nu v\|_{\mathcal{E}_1(\mathbb{R}^3)}$$

with

$$q=p,\qquad \mathcal{E}_0(\mathbb{R}^3)=V^0_{p,\beta-1-2/p}(\mathbb{R}^3),\qquad \mathcal{E}_1(\mathbb{R}^3)=V^0_{p,\beta+1-2/p}(\mathbb{R}^3)$$

and $\mathcal{O} = A^{-1}$ is such that

$$\mathcal{O} : V^0_{2,\beta}(\mathbb{R}^3) \to V^0_{2,\beta}(\mathbb{R}^3).$$

Therefore, the conclusion (2.6) of the form

$$\|\mathcal{O}v\|_{\mathcal{E}_0(\mathbb{R}^3)} \le c\|v\|_{\mathcal{E}_1(\mathbb{R}^3)}$$

yields (3.16). ■

Our final estimate combines the previous result with the local regularity estimates as follows.

Lemma 3.9. *Let the assumptions of Lemma* 3.3 *be satisfied,* $f \in V^0_{2,\kappa} \cap V^0_{p,\kappa}(\mathbb{R}^3)$, $p \ge 2$. *Then, for* u *solving the problem* (3.4) *and* $u \in V^2_{2,\kappa}(\mathbb{R}^3)$,

$$\|u\|_{V^2_{p,\kappa}(\mathbb{R}^3)} \le c\|f\|_{V^0_{p,\kappa}(\mathbb{R}^3)}. \tag{3.17}$$

Proof. We apply the local regularity statement and from (2.2) with κ in place of β we derive

$$\|u\|_{V^2_{p,\kappa}(\mathbb{R}^3)} \le c(\|f\|_{V^0_{p,\kappa}(\mathbb{R}^3)} + \|u\|_{V^0_{p,\kappa-2}(\mathbb{R}^3)}). \tag{3.18}$$

Since $f \in V^0_{p,\kappa}$ we set $p(\beta+1) - 2 = p\kappa$, so we calculate $\beta = \kappa - 1 + 2/p$ and substitute to find that $p(\beta-1) - 2 = p(\kappa-2)$. Therefore, we apply the formula (3.16) to derive $u \in V^0_{p,\kappa-2}$ and

$$\|u\|_{V^0_{p,\kappa-2}(\mathbb{R}^3)} \le \|f\|_{V^0_{p,\kappa}(\mathbb{R}^3)}.$$

We combine this with (3.18) to get (3.17). This concludes the proof. ■

Lemma 3.10. *Let the assumptions of Lemma* 3.3 *be satisfied,* $f \in V^0_{2,\kappa} \cap V^0_{p,\kappa}(\mathbb{R}^3)$, $p \ge 2$. *Then there exists a solution* u *of the problem* (3.4) *such that* $u \in V^2_{p,\kappa}(\mathbb{R}^3)$ *and the estimate* (3.17) *holds.*

Proof. We take a sequence f_ν of smooth functions with compact support such that $f_\nu \to f$ in $V^0_{p,\kappa}(\mathbb{R}^3)$. According to Lemma 3.4, there exists a solution $u_\nu \in V^2_{2,\kappa}$ of the problem (3.4) with the right hand side f_ν. Applying the estimate (3.17) we infer the convergence of the sequence u_ν in $V^2_{p,\kappa}$ to the limit u. ■

Using the standard regularizer methods, we apply the above result to deduce the existence theorem for the problem (1.2) in Ω and, consequently, for (1.1) in Ω'.

Theorem 1. *Let the assumptions of Lemma* 3.3 *be satisfied,* $f \in V^0_{2,\mu} \cap V^0_{p,\mu}(\Omega')$, $p \ge 2$. *Then there exists a solution* φ *of the problem* (1.1) *such that* $\varphi - \varphi(0) \in V^2_{p,\mu}(\Omega')$ *and*

$$\|\varphi - \varphi(0)\|_{V^2_{p,\mu}(\Omega')} \le c\|f\|_{V^0_{p,\mu}(\Omega')},$$

where $f = f' + \Delta\varphi(0)$.

Corollary 1. *Denote by* $L^2_{p,\mu}(\Omega')$ *the following space:*

$$L^2_{p,\mu}(\Omega') = \Big\{ u : \|u\|_{L^2_{p,\mu}(\Omega')} = \Big(\sum_{|\alpha|=2} \int_{\Omega'} |D^\alpha_x u|^p x_3^{p\mu}\, dx \Big)^{1/p} < \infty \Big\}.$$

Then under the assumptions of Theorem 1 *we have the inequality*

$$\|\varphi\|_{L^2_{p,\mu}(\Omega')} \le c\|f\|_{L^2_{p,\mu}(\Omega')}.$$

Acknowledgements. Research partially supported by MNiSW grant no. 1 P03A 021 30 and EC FP6 Marie Curie ToK programme SPADE2.

References

[DS] N. Dunford and J. T. Schwartz, *Linear Operators*, Interscience, 1966.

[MP] V. G. Maz'ya and B. A. Plamenevskiĭ, *L_p-estimates for solutions to elliptic boundary value problems in domain with edges*, Trudy Moskov. Mat. Obshch. 37 (1978), 49–93 (in Russian).

[Mo] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer, 1966.

[RZ] J. Rencławowicz and W. M. Zajączkowski, *Existence of solutions to the Poisson equation in L_2-weighted spaces*, Appl. Math. (Warsaw), to appear.

[Z1] W. M. Zajączkowski, *Existence and regularity of solutions of some elliptic system in domain with edges*, Dissertationes Math. 274 (1988), 91 pp.

[Z2] —, *Global regular nonstationary flow for the Navier–Stokes equations in a cylindrical pipe*, Topol. Methods Nonlinear Anal. 26 (2005), 221–286.

[Z3] —-, *Existence of solutions to the* (rot-div)*-system in L_p-weighted spaces*, Appl. Math. (Warsaw), to appear.

[Z4] —, *Existence of solutions to the* (rot-div)*-system in L_2-weighted spaces*, Appl. Math. (Warsaw) 36 (2009), 83–106.

[ZS] W. M. Zajączkowski and V. A. Solonnikov, *On the Neumann problem for elliptic equations of second order in domains with edges on the boundary*, Zap. Nauchn. Sem. LOMI 127 (1983), 7–48 (in Russian).

Joanna Rencławowicz
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: jr@impan.pl

Wojciech M. Zajączkowski
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: wz@impan.pl
and
Institute of Mathematics and Cryptology
Military University of Technology
Kaliskiego 2
00-908 Warszawa, Poland

Received on 5.1.2009;
revised version on 15.10.2009 (1985)