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A QUASISTATIC CONTACT PROBLEM WITH ADHESION AND FRICTION FOR VISCOELASTIC MATERIALS

Abstract. We consider a mathematical model which describes the contact between a deformable body and a foundation. The contact is frictional and is modelled by a version of normal compliance condition and the associated Coulomb's law of dry friction in which adhesion of contact surfaces is taken into account. The evolution of the bonding field is described by a first order differential equation and the material's behaviour is modelled by a nonlinear viscoelastic constitutive law. We derive a variational formulation of the mechanical problem and prove the existence and uniqueness of a weak solution if the friction coefficient is sufficiently small. The proof is based on time-dependent variational inequalities, differential equations and the Banach fixed point theorem.

1. Introduction. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of these processes a considerable effort has been made in their modelling and numerical simulations. A first study of frictional contact problems within the framework of variational inequalities was made in [6]. The mathematical, mechanical and numerical state of the art can be found in [16]. The quasistatic contact problem with normal compliance and friction for viscoelastic materials was studied in [13].

In this paper we continue the study of this model in which moreover the adhesion of contact surfaces is taken into account. Models for dynamic or quasistatic process of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 15, 16, 18]. In [2] a model

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of a contact problem with adhesion and friction was studied in which β represents a continuous transition between total adhesive and pure frictional states. Also in [5] a quasistatic unilateral contact problem with local friction and adhesion for elastic materials was studied and an existence result for a friction coefficient small enough was established.

As in [8, 9], we use the bonding field as an additional state variable β , defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$. When $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [10, 12, 14–18]. In this work we derive a variational formulation of the mechanical problem for which we prove the existence and uniqueness of a weak solution if the friction coefficient is sufficiently small, and obtain a partial regularity result for the solution.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 2.1.

2. Problem statement and variational formulation. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a domain initially occupied by a viscoelastic body. Ω is supposed to be open, bounded, with a sufficiently regular boundary Γ partitioned into three measurable parts, $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are disjoint open sets and $\text{meas } \Gamma_1 > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2 . On Γ_3 the body is in adhesive and frictional contact with a foundation.

Thus, the classical formulation of the mechanical problem is as follows.

PROBLEM P_1 . Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \rightarrow [0, 1]$ such that

$$(2.1) \quad \text{div } \sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0, T),$$

$$(2.2) \quad \sigma = A\varepsilon(\dot{u}) + B\varepsilon(u) \quad \text{in } \Omega \times (0, T),$$

$$(2.3) \quad u = 0 \quad \text{on } \Gamma_1 \times (0, T),$$

$$(2.4) \quad \sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0, T),$$

$$(2.5) \quad -\sigma\nu = p(u_\nu) - c_\nu\beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0, T),$$

$$(2.6) \quad \begin{cases} |\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)| \leq \mu p(u_\nu) \\ |\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)| < \mu p(u_\nu) \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)| = \mu p(u_\nu) \Rightarrow \\ \exists \lambda \geq 0 : \dot{u}_\tau = -\lambda(\sigma_\tau + c_\tau\beta^2 R_\tau(u_\tau)), \end{cases} \quad \text{on } \Gamma_3 \times (0, T),$$

$$(2.7) \quad \dot{\beta} = -[\beta(c_\nu |R_\nu(u_\nu)|^2 + c_\tau |R_\tau(u_\tau)|^2) - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0, T),$$

$$(2.8) \quad u(0) = u_0 \quad \text{in } \Omega,$$

$$(2.9) \quad \beta(0) = \beta_0 \quad \text{on } \Gamma_3.$$

We denote by u the displacement field, by σ the stress field and by $\varepsilon(u)$ the linearized strain tensor. Equation (2.1) is the equilibrium equation. Equation (2.2) represents the viscoelastic constitutive law of the material in which A and B are given nonlinear constitutive functions. Here and below, a dot above a variable represents a time derivative. We recall that in linear viscoelasticity the stress tensor $\sigma = (\sigma_{ij})$ is given by

$$\sigma_{ij} = a_{ijkh} \varepsilon_{kh}(\dot{u}) + b_{ijkh} \varepsilon_{kh}(u),$$

where $A = (a_{ijkh})$ is the viscosity tensor and $B = (b_{ijkh})$ is the elasticity tensor, for $i, j, k, h = 1, \dots, d$. (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ , and $\sigma\nu$ represents the Cauchy stress vector. Condition (2.5) represents the normal compliance condition with adhesion and (2.6) is the associated Coulomb's law of dry friction on the contact surface Γ_3 . Here p is a given function, μ is the friction coefficient and the parameters c_ν , c_τ and ε_a are adhesion coefficients which may depend on $x \in \Gamma_3$. As in [18], R_ν , R_τ are truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases} \quad R_\tau(v) = \begin{cases} v & \text{if } |v| \leq L, \\ Lv/|v| & \text{if } |v| > L, \end{cases}$$

where $L > 0$ is a characteristic length of the bonds. Equation (2.7) is an ordinary differential equation which describes the evolution of the bonding field and it was already used in [17]; here $[s]_+ = \max(s, 0)$ for $s \in \mathbb{R}$. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs, bonding cannot be reestablished. Also we wish to make it clear that from [11] it follows that the model does not allow for a complete debonding field in finite time. Finally, (2.8) and (2.9) represent respectively the initial displacement field and the initial bonding field. We recall that the inner products and the corresponding norms on \mathbb{R}^d and S_d are given by

$$\begin{aligned} u \cdot v &= u_i v_i, & |v| &= (v \cdot v)^{1/2} \quad \forall u, v \in \mathbb{R}^d, \\ \sigma \cdot \tau &= \sigma_{ij} \tau_{ij}, & |\tau| &= (\tau \cdot \tau)^{1/2} \quad \forall \sigma, \tau \in S_d, \end{aligned}$$

where S_d is the space of second order symmetric tensors on \mathbb{R}^d ($d = 2, 3$). Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted.

Now, to proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \quad H_1 = (H^1(\Omega))^d, \\ Q = \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad Q_1 = \{\sigma \in Q; \operatorname{div} \sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$\langle u, v \rangle_H = \int_{\Omega} u_i v_i \, dx, \quad \langle \sigma, \tau \rangle_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx.$$

The linearized strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2} (u_{i,j} + u_{j,i});$$

$\operatorname{div} \sigma = (\sigma_{ij,j})$ is the divergence of σ . For every element $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ , given by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular function $\sigma \in Q_1$, we define its normal and tangential components by

$$\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma \nu - \sigma_\nu \nu$$

and we recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + \langle \operatorname{div} \sigma, v \rangle_H = \int_{\Gamma} \sigma_\nu \cdot v \, da \quad \forall v \in H_1,$$

where da is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1; v = 0 \text{ on } \Gamma_1\}.$$

Since $\operatorname{meas} \Gamma_1 > 0$, the following Korn's inequality holds [6]:

$$(2.9) \quad \|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V,$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1 . We equip V with the inner product

$$(u, v)_V = \langle \varepsilon(u), \varepsilon(v) \rangle_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V . Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which depends only on the domain Ω , Γ_1 and Γ_3 such that

$$(2.10) \quad \|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V.$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1, \infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1, \infty}(0, T; V)} = \|v\|_{L^\infty(0, T; V)} + \|\dot{v}\|_{L^\infty(0, T; V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X ; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$(2.11) \quad \varphi_1 \in C([0, T]; H), \quad \varphi_2 \in C([0, T]; (L^2(\Gamma_2))^d),$$

and we denote by $f(t)$ the element of V defined by

$$(2.12) \quad (f(t), v)_V = \int_{\Omega} \varphi_1(t) \cdot v \, dx + \int_{\Gamma_2} \varphi_2(t) \cdot v \, da \quad \forall v \in V, t \in [0, T].$$

Using (2.11) and (2.12) we obtain

$$f \in C([0, T]; V).$$

Also we define the functional $j : V \times V \rightarrow \mathbb{R}$ by

$$j(v, w) = \int_{\Gamma_3} \mu p(v_\nu) |w_\tau| \, da \quad \forall v, w \in V,$$

where the normal compliance p is assumed to satisfy

$$(2.13) \quad \left\{ \begin{array}{l} \text{(a) } p : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+; \\ \text{(b) there exists } L_p > 0 \text{ such that } |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \\ \quad \text{for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \Gamma_3; \\ \text{(c) } x \mapsto p(x, r) \text{ is measurable on } \Gamma_3 \text{ for any } r \in \mathbb{R}; \\ \text{(d) } p(x, r) = 0 \text{ for all } r \leq 0 \text{ and a.e. } x \in \Gamma_3. \end{array} \right.$$

μ is the friction coefficient and we assume that

$$(2.14) \quad \mu \in L^\infty(\Gamma_3) \quad \text{and} \quad \mu \geq 0 \text{ a.e. on } \Gamma_3.$$

In the study of problem P_1 we assume that the viscosity operator A satisfies

$$(2.15) \quad \left\{ \begin{array}{l} \text{(a) } A : \Omega \times S_d \rightarrow S_d; \\ \text{(b) there exists } M_A > 0 \text{ such that} \\ \quad |A(x, \varepsilon_1) - A(x, \varepsilon_2)| \leq M_A |\varepsilon_1 - \varepsilon_2| \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d \text{ and a.e. } x \text{ in } \Omega; \\ \text{(c) there exists } m_A > 0 \text{ such that} \\ \quad (A(x, \varepsilon_1) - A(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_A |\varepsilon_1 - \varepsilon_2|^2 \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d \text{ and a.e. } x \text{ in } \Omega; \\ \text{(d) } x \mapsto A(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \varepsilon \text{ in } S_d; \\ \text{(e) } x \mapsto A(x, 0) \in Q. \end{array} \right.$$

The elasticity operator B satisfies

$$(2.16) \quad \left\{ \begin{array}{l} \text{(a) } B : \Omega \times S_d \rightarrow S_d \\ \text{(b) there exists } M_B > 0 \text{ such that} \\ \quad |B(x, \varepsilon_1) - B(x, \varepsilon_2)| \leq M_B |\varepsilon_1 - \varepsilon_2|, \\ \quad \text{for all } \varepsilon_1, \varepsilon_2 \text{ in } S_d \text{ and a.e. } x \text{ in } \Omega; \\ \text{(c) } x \mapsto B(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \varepsilon \text{ in } S_d; \\ \text{(e) } x \mapsto B(x, 0) \in Q. \end{array} \right.$$

As in [18] we suppose that the adhesion coefficients c_ν , c_τ and ε_a satisfy

$$(2.17) \quad c_\nu, c_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^\infty(\Gamma_3), \quad c_\nu, c_\tau, \varepsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3.$$

We assume that the initial data satisfy

$$(2.18) \quad u_0 \in V,$$

$$(2.19) \quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{a.e. on } \Gamma_3.$$

Next, we define the functional $r : L^2(\Gamma_3) \times V \times V \rightarrow \mathbb{R}$ by

$$r(\beta, u, v) = \int_{\Gamma_3} (p(u_\nu)v_\nu - c_\nu\beta^2 R_\nu(u_\nu)v_\nu + c_\tau\beta^2 R_\tau(u_\tau).v_\tau) da.$$

Finally, we need the following set for the bonding fields:

$$\mathcal{O} = \{\theta : [0, T] \rightarrow L^2(\Gamma_3); 0 \leq \theta(t) \leq 1 \quad \forall t \in [0, T], \text{ a.e. on } \Gamma_3\}.$$

Now, assuming the solution to be sufficiently regular, and using Green's formula, we see that Problem P_1 has the following variational formulation.

PROBLEM P_2 . Find a displacement field $u \in C^1([0, T]; V)$ and a bonding field $\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{O}$ such that

$$(2.20) \quad \begin{aligned} \langle A\varepsilon(\dot{u}(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q + \langle B\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)) \rangle_Q \\ + j(u(t), v) - j(u(t), \dot{u}(t)) + r(\beta(t), u(t), v - \dot{u}(t)) \\ \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V, t \in [0, T], \end{aligned}$$

$$(2.21) \quad \begin{aligned} \dot{\beta}(t) = -[\beta(t)(c_\nu|R_\nu(u_\nu(t))|^2 \\ + c_\tau|R_\tau(u_\tau(t))|^2) - \varepsilon_a]_+ \quad \text{a.e. } t \in (0, T), \end{aligned}$$

$$(2.22) \quad u(0) = u_0 \quad \text{in } \Omega,$$

$$(2.23) \quad \beta(0) = \beta_0 \quad \text{on } \Gamma_3.$$

Our main result, which will be established in the next section, is the following theorem.

THEOREM 2.1. *Let $T > 0$ and assume that (2.11) and (2.13)–(2.19) hold. Then there exists a constant $\mu_0 > 0$ such that if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0,$$

then Problem P_2 has a unique solution.

3. Existence and uniqueness result. The proof of Theorem 2.1 will be carried out in several steps. In the first step, for given $\eta \in C([0, T]; V)$ and $g \in C([0, T]; V)$ we consider the following variational problem.

PROBLEM $P_{\eta g}$. Find $v_{\eta g} : [0, T] \rightarrow V$ such that

$$(3.1) \quad \langle A\varepsilon(v_{\eta g}(t)), \varepsilon(w) - \varepsilon(v_{\eta g}(t)) \rangle_Q + (\eta(t), w - v_{\eta g}(t))_V + j(g(t), w) - j(g(t), v_{\eta g}(t)) \geq (f(t), w - v_{\eta g}(t))_V \quad \forall w \in V, t \in [0, T].$$

We show the following result.

LEMMA 3.1. *Problem $P_{\eta g}$ has a unique solution and the solution satisfies $v_{\eta g} \in C([0, T]; V)$.*

Proof. We define the operator $C : V \rightarrow V$ by

$$(Cv, w)_V = \langle A\varepsilon(v), \varepsilon(w) \rangle_Q \quad \forall v, w \in V.$$

It follows from assumption (2.15) that C is a strongly monotone and Lipschitz continuous operator. Next, let $t \in [0, T]$. The functional $j(g(t), \cdot)$ is a continuous seminorm on V , so by a classical argument of elliptic variational inequalities [1], there exists a unique element $v_{\eta g}(t) \in V$ such that

$$(3.2) \quad \langle A\varepsilon(v_{\eta g}(t)), \varepsilon(w) - \varepsilon(v_{\eta g}(t)) \rangle_Q + j(g(t), w) - j(g(t), v_{\eta g}(t)) \geq (f(t) - \eta(t), w - v_{\eta g}(t))_V \quad \forall w \in V.$$

Thus, we use (3.2) to see that $v_{\eta g}(t)$ is the unique element which solves (3.1), for each $t \in [0, T]$. Now, let $t_1, t_2 \in [0, T]$. We write (3.2) for $t = t_1$ and $w = v_{\eta g}(t_2)$. Then for $t = t_2$ and $w = v_{\eta g}(t_1)$, by adding the resulting inequalities we obtain

$$\begin{aligned} & \langle A\varepsilon(v_{\eta g}(t_1)) - A\varepsilon(v_{\eta g}(t_2)), \varepsilon(v_{\eta g}(t_1)) - \varepsilon(v_{\eta g}(t_2)) \rangle_Q \\ & \leq (f(t_1) - f(t_2), v_{\eta g}(t_1) - v_{\eta g}(t_2))_V - (\eta(t_1) - \eta(t_2), v_{\eta g}(t_1) - v_{\eta g}(t_2))_V \\ & \quad + j(g(t_1), v_{\eta g}(t_2)) - j(g(t_1), v_{\eta g}(t_1)) + j(g(t_2), v_{\eta g}(t_1)) - j(g(t_2), v_{\eta g}(t_2)). \end{aligned}$$

Using (2.15)(c), (2.13)(b), (2.17), and (2.10), we see that

$$\begin{aligned} m_A \|v_{\eta g}(t_1) - v_{\eta g}(t_2)\|_V^2 & \leq d_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)} \|v_{\eta g}(t_1) - v_{\eta g}(t_2)\|_V^2 \\ & \quad + \|\eta(t_1) - \eta(t_2)\|_V \|v_{\eta g}(t_1) - v_{\eta g}(t_2)\|_V \\ & \quad + d_\Omega^2 \|g(t_1) - g(t_2)\|_V \|v_{\eta g}(t_1) - v_{\eta g}(t_2)\|_V \\ & \quad + \|f(t_1) - f(t_2)\|_V \|v_{\eta g}(t_1) - v_{\eta g}(t_2)\|_V. \end{aligned}$$

Then, if we take

$$\mu_0 = m_A/d_\Omega^2,$$

it follows that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, there exists a constant $c_1 > 0$ such that

$$\begin{aligned} & \|v_{\eta g}(t_1) - v_{\eta g}(t_2)\|_V \\ & \leq c_1(\|f(t_1) - f(t_2)\|_V + \|g(t_1) - g(t_2)\|_V + \|\eta(t_1) - \eta(t_2)\|_V). \end{aligned}$$

As $f \in C([0, T]; V)$, $g \in C([0, T]; V)$ and $\eta \in C([0, T]; V)$, we deduce that $v_{\eta g} \in C([0, T]; V)$.

Now, let us consider the operator $A_\eta : C([0, T]; V) \rightarrow C([0, T]; V)$ defined by

$$(3.3) \quad A_\eta g = g_\eta, \quad g \in C([0, T]; V),$$

where

$$(3.4) \quad g_\eta(t) = u_0 + \int_0^t v_{\eta g}(s) ds \quad \text{for } t \in [0, T].$$

We have

LEMMA 3.2. *The operator A_η has a unique fixed point $g_\eta^* \in C([0, T]; V)$.*

Proof. We refer the reader to [13, Proposition 4.2].

Next, for $\eta \in C([0, T]; V)$, we denote by g_η^* the fixed point given in Lemma 3.2. Let $v_\eta \in C([0, T]; V)$ be the function defined by

$$(3.5) \quad v_\eta = v_{\eta g_\eta^*}.$$

Using (3.3) and (3.4), let $u_\eta : [0, T] \rightarrow V$ be the function

$$(3.6) \quad u_\eta(t) = g_\eta^*(t) = u_0 + \int_0^t v_\eta(s) ds \quad \text{for } t \in [0, T].$$

Now, we consider the following problem.

PROBLEM $P_{\eta\beta}$. Find a bonding field $\beta_\eta : [0, T] \rightarrow L^2(\Gamma_3)$ such that

$$(3.7) \quad \begin{aligned} \dot{\beta}_\eta(t) = & -[\beta_\eta(t)(c_\nu |R_\nu(u_{\eta\nu}(t)))|^2 \\ & + c_\tau |R_\tau(u_{\eta\tau}(t))|^2] - \varepsilon_a]_+ \quad \text{a.e. } t \in (0, T), \end{aligned}$$

$$(3.8) \quad \beta_\eta(0) = \beta_0 \quad \text{on } \Gamma_3.$$

We have the following result.

LEMMA 3.3. *There exists a unique solution to Problem $P_{\eta\beta}$ and it satisfies*

$$\beta_\eta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{O}.$$

Proof. Let $k > 0$ and let

$$X = \{\beta \in C([0, T]; L^2(\Gamma_3)); \sup_{t \in [0, T]} [\exp(-kt) \|\beta(t)\|_{L^2(\Gamma_3)}] < +\infty\}.$$

X is a Banach space with the norm

$$\|\beta\|_X = \sup_{t \in [0, T]} [\exp(-kt) \|\beta(t)\|_{L^2(\Gamma_3)}],$$

which is equivalent to the standard norm $\|\cdot\|_{C([0, T]; L^2(\Gamma_3))}$. Consider the mapping $T : X \rightarrow X$ defined by

$$T\beta(t) = \beta_0 - \int_0^t [\beta(s)(c_\nu |R_\nu(u_{\eta\nu}(s))|^2 + c_\tau |R_\tau(u_{\eta\tau}(s))|^2) - \varepsilon_a]_+ da.$$

Using that $|R_r(u_{\eta r})| \leq L$, $r = \nu, \tau$, it follows that there exists a constant $c_2 > 0$ such that

$$\|T\beta_1(t) - T\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_2 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds.$$

Since

$$\begin{aligned} \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds &= \int_0^t e^{ks} (e^{-ks} \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}) ds \\ &\leq \|\beta_1 - \beta_2\|_{X_1} \frac{e^{kt}}{k}, \end{aligned}$$

this inequality implies that

$$(3.9) \quad \|T\beta_1 - T\beta_2\|_X \leq \frac{c_2}{k} \|\beta_1 - \beta_2\|_X.$$

The inequality (3.9) shows that for $k > c_2$, T is a contraction. Thus, by the Banach fixed point theorem, T has a unique fixed point β_η which satisfies (3.7) and (3.8). The regularity $\beta_\eta \in \mathcal{O}$ is a consequence of (3.8) and (2.19); see [18] for details. ■

Moreover, using the Riesz representation theorem we define the function $\Lambda : [0, T] \rightarrow V$ by

$$(3.10) \quad (\Lambda\eta(t), w)_V = \langle B\varepsilon(u_\eta(t)), \varepsilon(w) \rangle_Q + r(\beta_\eta(t), u_\eta(t), w), \quad \forall w \in V, t \in [0, T].$$

LEMMA 3.3. *For each $\eta \in C([0, T]; V)$ the function $\Lambda\eta : [0, T] \rightarrow V$ belongs to $C([0, T]; V)$. Moreover, there exists a unique $\eta^* \in C([0, T]; V)$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let $\eta \in C([0, T]; V)$ and $t_1, t_2 \in [0, T]$. Using (3.10), it follows that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \|\Lambda\eta(t_1) - \Lambda\eta(t_2)\|_V &\leq \|B\varepsilon(u_\eta(t_1)) - B\varepsilon(u_\eta(t_2))\|_Q \\ &\quad + C_1 (\|\beta_\eta^2(t_1)R_\tau(u_{\eta\tau}(t_1)) - \beta_\eta^2(t_2)R_\tau(u_{\eta\tau}(t_2))\|_{L^2(\Gamma_3)} \\ &\quad + \|p(u_{\eta\nu}(t_1)) - p(u_{\eta\nu}(t_2))\|_{L^2(\Gamma_3)}) \\ &\quad + C_1 \|\beta_\eta^2(t_1)R_\nu(u_{\eta\nu}(t_1)) - \beta_\eta^2(t_2)R_\nu(u_{\eta\nu}(t_2))\|_{L^2(\Gamma_3)}. \end{aligned}$$

Using the properties (see [16]) of the operators R_ν , R_τ such that

$$\begin{aligned} |R_r(u_{\eta r})| &\leq L, \quad r = \nu, \tau, \quad |R_\nu(a) - R_\nu(b)| \leq |a - b| \quad \forall a, b \in \mathbb{R}, \\ |R_\tau(a) - R_\tau(b)| &\leq |a - b| \quad \forall a, b \in \mathbb{R}^d, \end{aligned}$$

(2.13)(b), (2.16), and that $0 \leq \beta_\eta(t) \leq 1$ for all $t \in [0, T]$, it follows that there exists a constant $C_2 > 0$ such that

$$(3.11) \quad \begin{aligned} \|\Lambda\eta(t_1) - \Lambda\eta(t_2)\|_V \\ \leq C_2(\|u_\eta(t_1) - u_\eta(t_2)\|_V + \|\beta_\eta(t_1) - \beta_\eta(t_2)\|_{L^2(\Gamma_3)}). \end{aligned}$$

Since $u_\eta \in C^1([0, T]; V)$ and $\beta_\eta \in W^{1,\infty}(0, T; V)$ we deduce from inequality (3.11) that $\Lambda\eta \in C([0, T]; V)$.

Let now $\eta_1, \eta_2 \in C([0, T]; V)$. For $t \in [0, T]$ we integrate (3.7) with the initial condition (3.8) to obtain

$$\beta_{\eta_i}(t) = \beta_0 - \int_0^t [\beta_{\eta_i}(s)(c_\nu |R_\nu(u_{\eta_i\nu}(s))|^2 + c_\tau |R_\tau(u_{\eta_i\tau}(s))|^2) - \varepsilon_a]_+ da.$$

Hence there exists a constant $C_3 > 0$ such that

$$\begin{aligned} \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \\ \leq C_3 \int_0^t \|\beta_{\eta_1}(s) |R_\nu(u_{\eta_1\nu}(s))|^2 - \beta_{\eta_2}(s) |R_\nu(u_{\eta_2\nu}(s))|^2\|_{L^2(\Gamma_3)} ds \\ + C_3 \int_0^t \|\beta_{\eta_1}(s) |R_\tau(u_{\eta_1\tau}(s))|^2 - \beta_{\eta_2}(s) |R_\tau(u_{\eta_2\tau}(s))|^2\|_{L^2(\Gamma_3)} ds. \end{aligned}$$

We use the definition of the truncation operators R_ν , R_τ and write

$$\beta_{\eta_1}(s) = \beta_{\eta_1}(s) - \beta_{\eta_2}(s) + \beta_{\eta_2}(s).$$

After some elementary calculations we find that there exists a constant $C_4 > 0$ such that

$$\begin{aligned} \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} &\leq C_4 \int_0^t \|\beta_{\eta_1}(s) - \beta_{\eta_2}(s)\|_{L^2(\Gamma_3)} ds \\ &+ C_4 \int_0^t \|u_{\eta_1\nu}(s) - u_{\eta_2\nu}(s)\|_{L^2(\Gamma_3)} ds + C_4 \int_0^t \|u_{\eta_1\tau}(s) - u_{\eta_2\tau}(s)\|_{(L^2(\Gamma_3))^d} ds. \end{aligned}$$

Using (2.10), it follows that

$$\begin{aligned} \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} &\leq C_4 \int_0^t \|\beta_{\eta_1}(s) - \beta_{\eta_2}(s)\|_{L^2(\Gamma_3)} ds \\ &+ 2C_4 d_\Omega \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds. \end{aligned}$$

Using a Gronwall-type inequality, we deduce that there exists a constant $C_5 > 0$ such that

$$(3.12) \quad \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} \leq C_5 \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds.$$

On the other hand, using arguments similar to those in the proof of (3.12), we find that there exists a constant $C_6 > 0$ such that

$$\|A\eta_1(t) - A\eta_2(t)\|_V \leq C_6(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + \|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)}).$$

Hence, by (3.12) we have

$$(3.13) \quad \begin{aligned} \|A\eta_1(t) - A\eta_2(t)\|_V \\ \leq C_6\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + C_5C_6 \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds. \end{aligned}$$

On the other hand, the function u_{η_i} satisfies the inequality

$$(3.14) \quad \begin{aligned} \langle A\varepsilon(v_{\eta_i}(t)), \varepsilon(w - v_{\eta_i}(t)) \rangle_Q + (\eta_i(t), w - v_{\eta_i}(t))_V + j(u_{\eta_i}(t), w) \\ - j(u_{\eta_i}(t), v_{\eta_i}(t)) \geq (f(t), w - v_{\eta_i}(t))_V \quad \forall w \in V, \end{aligned}$$

where $i = 1, 2$ and $t \in [0, T]$. It follows from (3.14) and the estimate in the proof of Lemma 3.1 that there exists a constant $C > 0$ such that

$$\begin{aligned} \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V &\leq \int_0^t \|v_{\eta_1}(s) - v_{\eta_2}(s)\|_V ds \\ &\leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds + C \int_0^t \|u_{\eta_1}(s) - u_{\eta_2}(s)\|_V ds \quad \text{for } t \in [0, T]. \end{aligned}$$

Using now a Gronwall-type inequality we get

$$(3.15) \quad \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \quad \forall t \in [0, T].$$

From (3.13) and (3.15) it follows that there exists a constant $C' > 0$ such that

$$(3.16) \quad \|A\eta_1(t) - A\eta_2(t)\|_V \leq C' \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds.$$

Let now $k > 0$, and denote

$$\|\eta\|_k = \sup_{t \in [0, T]} [\exp(-kt)\|\eta(t)\|_V] \quad \forall \eta \in C([0, T]; V).$$

Clearly $\|\cdot\|_k$ defines a norm on the space $C([0, T]; V)$ which is equivalent to the standard norm $\|\cdot\|_{C([0, T]; V)}$. Using (3.16) and arguments similar to those in the proof of (3.9), after some calculations we find that there exists

a constant $c > 0$ such that

$$\|A\eta_1 - A\eta_2\|_k \leq \frac{c}{k} \|\eta_1 - \eta_2\|_k \quad \forall \eta_1, \eta_2 \in C([0, T]; V).$$

So for $k > c$, the operator A is a contraction on the space $C([0, T]; V)$ endowed with the norm $\|\cdot\|_k$. Then by the Banach fixed point theorem, A has a unique fixed point $\eta^* \in C([0, T]; V)$, which concludes the proof.

Now, we have all the ingredients to prove Theorem 2.1.

Proof of Theorem 2.1. Existence. Let $\eta^* \in C([0, T]; V)$ be the fixed point of A and let v_{η^*} and u_{η^*} be the functions given by (3.5) and (3.6) for $\eta = \eta^*$. Let β_{η^*} the solution of Problem $P_{\eta\beta}$ for $\eta = \eta^*$. We show that $(u_{\eta^*}, \beta_{\eta^*})$ is a solution of Problem P_2 . Indeed, choosing $\eta = \eta^*$, $g = g_{\eta^*}^*$ in (3.1) and using (3.5), we obtain

$$(3.17) \quad \langle A\varepsilon(v_{\eta^*}(t)), \varepsilon(w) - \varepsilon(v_{\eta^*}(t)) \rangle_Q + (\eta^*(t), w - v_{\eta^*}(t))_V + j(g_{\eta^*}^*(t), w) - j(g_{\eta^*}^*(t), v_{\eta^*}(t)) \geq (f(t), w - v_{\eta^*}(t))_V, \quad \forall w \in V, t \in [0, T].$$

Let β denote the solution of Problem $P_{\eta\beta}$ for $\eta = \eta^*$, i.e., $\beta = \beta_{\eta^*}$. As $\eta^* = A\eta^*$, the inequality (2.20) follows from (3.4), (3.6) and (3.17), since $v_{\eta^*} = \dot{u}_{\eta^*}$ and $g_{\eta^*}^* = u_{\eta^*}$. The equality (2.22) follows from (3.6), and the regularity $u_{\eta^*} \in C^1([0, T]; V)$ is a consequence of Lemma 3.1, (2.18) and (3.6). Clearly, equalities (2.21) and (2.23) hold by Problem $P_{\eta\beta}$. Also the regularity of the bonding field $\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{O}$ follows from Lemma 3.3.

Uniqueness. Let $(u, \beta) \in C^1([0, T]; V) \times W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{O}$ be a solution of Problem P_2 and denote by $\eta \in C([0, T]; V)$ the function defined by

$$(3.18) \quad (\eta(t), w)_V = \langle B\varepsilon(u(t)), \varepsilon(w) \rangle_Q + r(\beta(t), u(t), w) \quad \forall w \in V, t \in [0, T],$$

and let

$$(3.19) \quad v = \dot{u}.$$

Using (2.20) we infer that v is a solution of the variational problem $P_{\eta u}$ and since this problem has a unique solution $v_{\eta u} \in C([0, T]; V)$, we conclude that

$$(3.20) \quad v = v_{\eta u}.$$

Hence, from (2.22), (3.19) and (3.20) we obtain

$$u(t) = u_0 + \int_0^t v_{\eta u}(s) ds, \quad t \in [0, T],$$

i.e., u is a fixed point of A_η . It follows from Lemma 3.1 that $u = g_\eta^*$ and by (3.20) we have

$$(3.21) \quad v = v_{\eta g_\eta^*}.$$

Then (3.5) and (3.21) imply

$$(3.22) \quad v = v_\eta.$$

So, it follows from (2.22), (3.6), (3.19) and (3.22) that

$$(3.23) \quad u = u_\eta.$$

Next, (2.21) and the initial condition $\beta(0) = \beta_0$ imply that β is a solution of Problem $P_{\eta\beta}$, and since this problem admits a unique solution β_η , we conclude that

$$(3.24) \quad \beta = \beta_\eta.$$

Using now (3.10), (3.18), (3.23), and (3.24) we deduce that $\Lambda\eta = \eta$, and as the operator Λ admits a unique fixed point guaranteed by Lemma 3.3, it follows that

$$(3.25) \quad \eta = \eta^*.$$

The uniqueness of the solution is now a consequence of (3.23)–(3.25).

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