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UNBIASED ESTIMATION OF RELIABILITY FOR TWO-PARAMETER EXPONENTIAL DISTRIBUTION UNDER TIME CENSORED SAMPLING

Abstract. The problem considered is that of unbiased estimation of reliability for a two-parameter exponential distribution under time censored sampling. We give necessary and sufficient conditions for the existence of uniformly minimum variance unbiased estimator and also provide a characterization of a complete class of unbiased estimators in situations where unbiased estimators exist.

1. Introduction. Let the life-length $X$ of an item follow a two-parameter exponential distribution with unknown real parameters $\mu$ and $\lambda$ ($\lambda > 0$), to be denoted hereafter as $\text{exp} (\mu, \lambda)$ distribution, defined by the probability function (p. f.)

$$f(x | \mu, \lambda) = \frac{1}{\lambda} e^{-(x-\mu)/\lambda}, \quad x > \mu.$$  

An important characteristic of the life distribution is its reliability function viz.

$$R(t) = P(X > t) = \begin{cases} 1, & t \leq \mu, \\ e^{-(t-\mu)/\lambda}, & t > \mu, \end{cases}$$

and a problem of interest in reliability theory is to estimate $R(t)$ at a given finite time point $t$ ($> 0$) through a life testing experiment.

In this paper we consider the problem of unbiased estimation of $R(t)$ under a time censored sampling plan wherein a random collection of $n$ identical items are put on test and the experiment is terminated after a pre-assigned finite time $T$ ($> 0$). For $t \leq T$, an unbiased estimator of $R(t)$

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mator

\[ \hat{R}(t) = 1 - D_0/n \]

where \( D_0 \) is the number of items failed up to time \( t \). Sengupta [3] showed
that the condition \( t \leq T \) is necessary as well for the existence of an unbi-
ased estimator and also obtained an alternative unbiased estimator of \( R(t) \)
based on the sufficient statistic for \( t \leq T \). Bartoszewicz [1] showed that the
sufficient statistic is not, however, complete except for \( n = 1, 2 \) (see also
Section 4) and as such the well known Lehmann–Scheffe theorem can not
generally be applied to obtain the uniformly minimum variance unbiased
estimator (UMVUE) of \( R(t) \) in situations where unbiased estimators exist.

Our main purpose in this article is to study the existence of the UMVUE
of \( R(t) \) under time censored sampling for an \( \text{exp} (\mu, \lambda) \) distribution. It is
proved that for \( n > 2 \), there does not exist UMVUE of \( R(t) \) for \( t \leq T \). We
also provide a characterization of a complete class of unbiased estimators of
\( R(t) \) for values of \( t \) for which \( R(t) \) is unbiasedly estimable.

2. Preliminaries. For a time censored sample, the data consist of \( D \)
and \( X_0, X_1, \ldots, X_D \), where \( D \) is the number of items failed up to pre-
assigned time \( T \) (\( > 0 \)) out of \( n \) test items, \( X_i \) being the life-length of the
\( i \)th failed item, \( 1 \leq i \leq D \) and \( X_0 = 0 \). Let \( p = R(T) \) and note that \( D \)
follows a binomial distribution with mean \( nq, q = 1-p \). The joint p.f. of \( D \)
and \( X_0, X_1, \ldots, X_D \) is (see Bartoszewicz [1])

\[
p(d, x(0), x(1), \ldots, x(d)) = \begin{cases} 
p^n, & d = 0, \\
d! \binom{n}{d} p^{n-d} I(T > \mu) \frac{1}{\lambda^d} e^{-\sum_{i=1}^{d}(x(i) - \mu)/\lambda} I(x(1) > \mu), & 1 \leq d \leq n, x(1) < \cdots < x(d) \leq T,
\end{cases}
\]

where \( I(A) \) is the indicator function of the set \( A \). Clearly a sufficient statistic
is \( V = (D, Z_D) \) where

\[
Z_d = \begin{cases} 
X_d, & d = 0, 1, \\
(X_1, S_d = \sum_{i=2}^{d}(X(i) - X(1))), & 2 \leq d \leq n.
\end{cases}
\]
The p.f. of $V$ is given by (see Bartoszewicz [1])

$$p(d, z_d) = \begin{cases} p^n, & d = 0, \\
np^{n-1}qI(T > \mu)p(x_{(1)} | d), & d = 1, \\
\binom{n}{d}p^{n-d}q^dI(T > \mu)p(x_{(1)} | d)p(s_d | d, x_{(1)}), & 2 \leq d \leq n,
\end{cases}$$

where for $d \geq 1$,

$$p(x_{(1)} | d) = \text{the conditional p.f. of } X_{(1)} \text{ given } D = d$$

$$= \frac{d}{\lambda q^d} e^{-(x_{(1)}-\mu)/\lambda} [e^{-(x_{(1)}-\mu)/\lambda} - e^{-(T-\mu)/\lambda}]^{d-1}, \quad \mu < x_{(1)} \leq T,$$

and for $d \geq 2$,

$$p(s_d | d, x_{(1)}) = \text{the conditional p.f. of } S_d \text{ given } D = d \text{ and } X_{(1)} = x_{(1)}$$

$$= \frac{1}{\chi^{d-1}(1 - e^{-(T-x_{(1)})/\lambda})} e^{-s_d/\lambda} f_d(s_d, T-x_{(1)}), \quad 0 < s_d \leq (d-1)(T-x_{(1)}),$$

with

$$f_d(u, w) = \frac{1}{\Gamma(d-1)} \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} (u-jw)^{d-2} I(u > jw).$$

In particular, for $d = 2, 3$,

$$f_2(u, w) = 1, \quad 0 < u \leq w,$$

$$f_3(u, w) = \begin{cases} u, & 0 < u \leq w, \\
2w-u, & w < u \leq 2w.
\end{cases}$$

In view of sufficiency of $V$ it is enough to restrict to estimators based on $V$ to study unbiased estimation of $R(t)$. In the following lemma we obtain a representation of the expectation of an estimator based on $V$ which plays an important role in the derivation of the results in the subsequent sections.

**Lemma 1.** Let $g(V)$ be an estimator based on the sufficient statistic $V$ with $g(0, 0) = g(0)$. Then

$$E[g(V)] = \begin{cases} g(0), & \mu \geq T, \\
g(0)p^n + \frac{n}{\lambda} \int_{\mu}^{\infty} e^{-n(x_{(1)}-\mu)/\lambda} g^*(x_{(1)}, \lambda)I(x_{(1)} \leq T) \, dx_{(1)}, & \mu < T,
\end{cases}$$
where

\begin{align}
g^*(x(1), \lambda) &= \frac{1}{\lambda^{n-1}} \int_0^\infty e^{-s/\lambda} \left\{ \frac{g(1, x(1)) (s - (n - 1)(T - x(1)))^{n-2} I(s > (n - 1)(T - x(1)))}{\Gamma(n - 1)} ight. \\
&\quad + \sum_{d=2}^{n-1} \frac{(n-1)}{(d-1)} u_d(x(1), s) I(s > (n - d)(T - x(1)))}{\Gamma(n - d)} \\
&\quad + g(n, x(1), s) f_n(s, T - x(1)) I(s \leq (n - 1)(T - x(1))) \right\} ds \\
\end{align}

and for $2 \leq d \leq n - 1$,

\begin{align}
u_d(x(1), s) = \left\{ \begin{array}{ll}
& s - (n-d)(T-x(1)) \\
& 0 \quad s \leq (n-1)(T-x(1)), \\
& \sum_{c=0}^{n-d-1} a_{cd} s^c, \quad (n-1)(T-x(1)) < s < \infty.
\end{array} \right.
\end{align}

with

\begin{align}
a_{cd} &= (-1)^{n-d-c-1} \binom{n-d-1}{c} \\
&\times \int_0^{(n-d)(T-x(1))} g(d, x(1), s_d) (s_d + (n-d)(T-x(1)))^{n-d-c-1} f_d(s_d, T - x(1)) ds_d.
\end{align}

**Proof.** For $\mu \geq T$, we have $D = 0$ with probability 1 and hence (12) is obvious. For $\mu < T$, it can be readily verified using (6)–(8) that $E[g(V)]$ is of the form (12) where

\begin{align}
g^*(x(1), \lambda) &= e^{-(n-1)(T-x(1))/\lambda} g(1, x(1)) \\
&\quad + \sum_{d=2}^{n} \binom{n-1}{d-1} \frac{1}{\lambda^{d-1}} e^{-(n-d)(T-x(1))/\lambda} \\
&\quad \times \int_0^{(d-1)(T-x(1))} g(d, x(1), s_d) e^{-s_d/\lambda} f_d(s_d, T - x(1)) ds_d.
\end{align}
Now for $1 \leq d \leq n - 1$,
\begin{equation}
(17) \quad e^{-(n-d)(T-x(1))/\lambda}
= \frac{1}{\Gamma(n-d)\lambda^{n-d}} \int_{0}^{\infty} e^{-y/\lambda}(y-(n-d)(T-x(1)))^{n-d-1}I(y > (n-d)(T-x(1))) dy
\end{equation}
so that for $2 \leq d \leq n - 1$, the $d$th term in the sum on the RHS of (16), on substituting $s = y + s_d$, can be expressed as
\begin{equation}
(18) \quad \left(\frac{n-1}{d-1}\right) \frac{1}{\Gamma(n-d)\lambda^{n-1}} \int_{0}^{\infty} e^{-s/\lambda}u_d(x(1), s)I(s > (n-d)(T-x(1))) ds
\end{equation}
where
\begin{equation}
(19) \quad u_d(x(1), s) = \int_{0}^{\infty} g(d, x(1), s_d)(s - s_d - (n-d)(T-x(1)))^{n-d-1}f_d(s_d, T-x(1))
\times I(s_d \leq (d-1)(T-x(1)))I(s - s_d > (n-d)(T-x(1))) ds_d.
\end{equation}
Since
\begin{equation}
I(s_d \leq (d-1)(T-x(1)))I(s - s_d > (n-d)(T-x(1)))
= \begin{cases} 
I(s_d \leq (d-1)(T-x(1))) & \text{for } (n-1)(T-x(1)) < s < \infty, \\
I(s - s_d > (n-d)(T-x(1))) & \text{for } (n-d)(T-x(1)) < s \leq (n-1)(T-x(1)),
\end{cases}
\end{equation}
it readily follows that the RHS of (19) is equal to the RHS of (14). Thus for $\mu < T$, (12) follows from (16)–(18).

3. Complete class of unbiased estimators. We recall that under time censored sampling, $R(t)$ is unbiasedly estimable if and only if $t \leq T$. For $t \leq T$, we obtain a characterization of a complete class of unbiased estimators of $R(t)$ in the following theorem.

**Theorem 1.** For $t \leq T$, $g(V)$ is an unbiased estimator of $R(t)$ based on the sufficient statistic $V$ if and only if it satisfies the following:
\begin{align}
(20) & \quad g(0) = g(0,0) = 1, \\
(21) & \quad g(1, x(1)) = \begin{cases} 
\frac{1 - (n-1)/n}{n-1} & \text{if } x(1) \geq t, \\
\frac{1}{n} & \text{if } x(1) < t,
\end{cases}
\end{align}
is a complete class of unbiased estimators of $R(26)$.

Further, a subclass of unbiased estimators for $x(24)$ is given by

$$
\sum_{c=2}^{d} \frac{(-1)^{d-c} \binom{n-1}{c-1} \binom{n-c-1}{d-c}}{\Gamma(n - c)} \times \int_{0}^{(c-1)(T-x(1))} g(c, x(1), s_c)(s_c + (n - c)(T - x(1)))^{d-c} f_c(s_c, T - x(1)) ds_c
$$

\[
= \begin{cases} 
\frac{(-1)^{d-2} \binom{n-2}{d-1} (n-1)^{d-1}(T - x(1))^{d-1}}{\Gamma(n-1)} & \text{if } x(1) \geq t, \\
\frac{n-1}{n} \frac{(-1)^{d-2} \binom{n-2}{d-1} (n-1)^{d-1}(T - x(1))^{d-1} - 1 - (t - x(1))^{d-1}}{\Gamma(n-1)} & \text{if } x(1) < t,
\end{cases}
\]

for $2 \leq d \leq n - 1$. For $x(1) \geq t$, $m(T - x(1)) < s_n \leq (m + 1)(T - x(1))$, $m = 0, 1, \ldots, n - 2$.

\[
g(n, x(1), s_n) f_n(s_n, T - x(1)) + \sum_{d=n-m}^{n-1} \frac{(n-1)}{(d-1)} \int_{0}^{s_n-(n-d)(T-x(1))} g(d, x(1), s_d) \times (s_n - s_d - (n - d)(T - x(1)))^{n-d-1} f_d(s_d, T - x(1)) ds_d = \frac{s_n^{n-2}}{\Gamma(n-1)}.
\]

For $x(1) < t$, $0 < s_n \leq t - x(1)$,

\[
g(n, x(1), s_n) = 0.
\]

For $x(1) < t$, $\max(t - x(1), m(T - x(1))) < s_n \leq (m + 1)(T - x(1))$, $m = 0, 1, \ldots, n - 2$.

\[
g(n, x(1), s_n) f_n(s_n, T - x(1)) + \sum_{d=n-m}^{n-1} \frac{(n-1)}{(d-1)} \int_{0}^{s_n-(n-d)(T-x(1))} g(d, x(1), s_d) \times (s_n - s_d - (n - d)(T - x(1)))^{n-d-1} f_d(s_d, T - x(1)) ds_d
\]

\[
= \frac{n-1}{n} \frac{1}{\Gamma(n-1)} (s_n - (t - x(1)))^{n-2}.
\]

Further, a subclass of unbiased estimators $g(V)$ with

\[
g(V) = 1 \quad \text{if } X(1) \geq t
\]

is a complete class of unbiased estimators of $R(t)$.
Proof. Let \( t \leq T \) and \( g(V) \) be an unbiased estimator of \( R(t) \). Since \( R(t) = 1 \) for \( \mu \geq T \), (12) implies (20). Also since for \( \mu < T \),

\[
R(t) = p^n + \frac{n}{\lambda} \int_{0}^{\infty} g_0(x(1), \lambda) e^{-n(x(1) - \mu)/\lambda} I(x(1) \leq T) \, dx(1)
\]

with

\[
g_0(x(1), \lambda) = \begin{cases} 
1 & \text{for } x(1) \geq t, \\
\frac{n-1}{n} e^{-(t-x(1))/\lambda} & \text{for } x(1) < t,
\end{cases}
\]

and the distribution of the smallest order statistic is complete for a complete random sample of size \( n \) from an \( \exp(\mu, \lambda) \) distribution with known \( \lambda \), we have, by (12) and (20), \( g^*(x(1), \lambda) = g_0(x(1), \lambda) \) for all \( \lambda > 0 \), where \( g^*(x(1), \lambda) \) is given by (13). Note that (28) can also be expressed as

\[
g_0(x(1), \lambda) = \begin{cases} 
\frac{1}{\Gamma(n-1)\lambda^{n-1}} \int_{0}^{\infty} e^{-s/\lambda} s^{n-2} \, ds & \text{for } x(1) \geq t, \\
\frac{n-1}{n} \frac{1}{\Gamma(n-1)\lambda^{n-1}} \int_{0}^{\infty} e^{-s/\lambda} (s - (t - x(1)))^{n-2} I(s > t - x(1)) \, ds & \text{for } x(1) < t.
\end{cases}
\]

Hence, by the completeness of the exponential distribution, (13) and (29) imply for \( 0 < s < \infty \),

\[
\sum_{d=2}^{n-1} \frac{(n-1)}{\sum_{d=2}^{n-1}} u_d(x(1), s) I(s > (n-d)(T - x(1)))
\]

\[
+ g(n, x(1), s)f_n(s, T - x(1)) I(s \leq (n-1)(T - x(1)))
\]

\[
= \frac{1}{\Gamma(n-1)} \left\{ w(x(1), s)
\right.
\]

\[
- g(1, x(1))(s - (n-1)(T - x(1)))^{n-2} I(s > (n-1)(T - x(1))) \}
\]

with

\[
w(x(1), s) = \begin{cases} 
s^{n-2} & \text{for } x(1) \geq t, \\
\frac{n-1}{n} (s - (t - x(1)))^{n-2} I(s > t - x(1)) & \text{for } x(1) < t,
\end{cases}
\]
and also for \((n - 1)(T - x(1)) < s < \infty\),

\[
(32) \quad \sum_{d=2}^{n-1} \frac{(n-1)}{(d-1)} \sum_{c=0}^{n-d-1} \frac{a_{cd}}{\Gamma(n-d)} s^c = \sum_{c=0}^{n-3} \frac{(n-1)}{(d-1)} \frac{a_{cd}}{\Gamma(n-d)} \sum_{d=2}^{n-c-1} s^c
\]

\[
= \begin{cases} 
\frac{1}{\Gamma(n-1)} \left\{ s^{n-2} - g(1, x(1))(s - (n - 1)(T - x(1)))^{n-2} \right\} & \text{for } x(1) \geq t, \\
\frac{1}{\Gamma(n-1)} \left\{ \frac{n-1}{n} (s - (t - x(1)))^{n-2} - g(1, x(1))(s - (n - 1)(T - x(1)))^{n-2} \right\} & \text{for } x(1) < t,
\end{cases}
\]

where \(u_d(x(1), s)\) and \(a_{cd}\) are defined respectively by (14) and (15). Equating the coefficients of \(s^{n-c-2}\), \(c = 0, 1, \ldots, n - 2\), on both sides of (32) we get (21) and

\[
(33) \quad \sum_{d=2}^{c+1} \frac{(n-1)}{(d-1)} \frac{a_{(n-c-2)d}}{\Gamma(n-d)}
\]

\[
= \begin{cases} 
\frac{(-1)^{c-1} n_c (n-1)^c (T - x(1))^c}{\Gamma(n-1)} & \text{for } x(1) \geq t, \\
\frac{n-1}{n} \frac{(-1)^{c-1} n_c (n-1)^c (T - x(1))^c}{\Gamma(n-1)} & \text{for } x(1) < t,
\end{cases}
\]

which yields (22). Finally, (23)–(25) are obtained from (30). This completes the proof of the first part of the theorem.

Now let \(g(V)\) be an unbiased estimator of \(R(t)\) not satisfying (26) and let

\[
g'(V) = \begin{cases} 
1 & \text{if } X(1) \geq t, \\
g(V) & \text{if } X(1) < t.
\end{cases}
\]

It can then be verified using (12) and (16) that \(g'(V)\) is an unbiased estimator of \(R(t)\) satisfying (26) and further \(E[g(V)]^2 - E[g'(V)]^2 = 0\) for \(\mu \geq T\), while for \(\mu < T\),

\[
E[g(V)]^2 - E[g'(V)]^2 = \frac{n}{\lambda} \int_{\max(\mu, t)}^{T} g^{**}(x(1), \lambda)e^{-n(x(1)-\mu)/\lambda} dx(1) \geq 0
\]

with strict inequality for \(\mu < t\) where

\[
g^{**}(x(1), \lambda) = \sum_{d=2}^{n} \frac{n-1}{(d-1)} \frac{1}{\lambda^d - 1} e^{-(n-d)(T-x(1))/\lambda} \left[ g(d, x(1), s_d) - 1 \right]^2 e^{-s_d/\lambda} f_d(s_d, T - x(1)) ds_d.
\]

Thus given any unbiased estimator of \(R(t)\) based on \(V\) not satisfying (26),
there exists a better unbiased estimator based on $V$ satisfying (26), and this proves the second part of the theorem.

### 4. Existence of UMVUE

We finally study the existence of UMVUE of $R(t)$ for $t \leq T$. The following characterization of the class of unbiased estimators of zero based on the sufficient statistic $V$ is useful for this purpose.

**Theorem 2.** An estimator $h(V)$ based on the sufficient statistic $V$ is an unbiased estimator of zero if and only if it satisfies the following:

1. $h(0) = h(0,0) = 0$,
2. $h(1,x(1)) = 0$,
3. $\sum_{c=2}^{d} \frac{(-1)^{d-c} c^{-1} (n-1)^{-1} (c-1)^{-1} d^{-1}}{\Gamma(n-c)} (c-1)(T-x(1))$ 

\begin{align*}
&\times \int_{0}^{s_{c}} h(c, x(1), s_{c})(s_{c} + (n-c)(T-x(1)))^{d-c} f_{c}(s_{c}, T-x(1)) ds_{c} = 0
\end{align*}

for $2 \leq d \leq n-1$,

4. $h(n, x(1), s_{n}) = 0$ for $0 < s_{n} \leq T-x(1)$,

5. $h(n, x(1), s_{n}) f_{n}(s_{n}, T-x(1))$

\begin{align*}
&+ \sum_{d=n-m}^{n-1} \frac{(n-1)(d-1)}{\Gamma(n-d)}
\end{align*}

\begin{align*}
&s_{n}-(n-d)(T-x(1))
\end{align*}

\begin{align*}
&\times \int_{0}^{s_{d}} h(d, x(1), s_{d})(s_{d} - (n-d)(T-x(1)))^{n-d-1} \times f_{d}(s_{d}, T-x(1)) ds_{d} = 0
\end{align*}

for $m(T-x(1)) < s_{n} \leq (m+1)(T-x(1))$, $m = 1, 2, \ldots, n-2$.

The proof of the theorem is similar to that of Theorem 1. As an immediate corollary we have the following result also obtained by Bartoszewicz [1].

**Corollary 1.** The sufficient statistic $V$ is complete if and only if $n = 1, 2$.

**Proof.** For $n = 1, 2$, the corollary follows trivially from (34), (35) and (37). For $n > 2$, a non-trivial unbiased estimator of zero is $h_{0}(V)$ satisfying (34)–(38), with

\begin{align*}
\text{for } &m(T-x(1)) < s_{n} \leq (m+1)(T-x(1))
\end{align*}

\begin{align*}
\text{with } &c
\end{align*}

\begin{align*}
h_{0}(2, x(1), s_{2}) = \begin{cases}
-c, & 0 < s_{2} \leq (T-x(1))/2, \\
c, & (T-x(1))/2 < s_{2} \leq T-x(1),
\end{cases}
\end{align*}

where $c$ is a non-zero real constant.
Thus for \( n = 1, 2 \) and \( t \leq T \), \( \hat{R}(t) \) defined in (3) is the unique unbiased estimator based on \( V \) and is the UMVUE of \( R(t) \). To study the existence of UMVUE for \( n > 2 \), we make use of the following result given in Rao ([2], p. 317).

**Theorem 3.** An unbiased estimator \( g(V) \) is the UMVUE of \( R(t) \) if and only if

\begin{align}
E[g(V)h(V)] = 0 \quad \text{for every } \mu, \lambda \\
\end{align}

for every unbiased estimator \( h(V) \) of zero.

In fact, in what follows we prove that for \( n > 2 \) and \( t \leq T \) there does not exist UMVUE of \( R(t) \). Not to obscure the essential steps of the reasoning, we first prove some necessary results in the following lemmas.

**Lemma 2.** For \( t \leq T \), \( g(V) \) is an unbiased estimator of \( R(t) \) satisfying (40) for every unbiased estimator \( h(V) \) of zero only if for \( x(1) < t \),

\begin{align}
g(2, x(1), s_2) = \frac{n - k - 1}{n} \\
\end{align}

for \( n > 2 \) and further

\begin{align}
g(3, x(1), s_3) = \frac{(n - k - 1)(n - k - 2)}{n(n - 2)} \\
\end{align}

for \( n > 3 \), where \( k = k(x(1)) = (t - x(1))/(T - x(1)) \).

**Proof.** Let \( t \leq T \), \( n > 2 \) and \( g(V) \) be an unbiased estimator of \( R(t) \) such that \( h^*(V) = g(V)h(V) \) satisfies (34)–(38) with \( h \) replaced by \( h^* \) for every \( h(V) \) satisfying (34)–(38). Then for \( x(1) < t \) and \( d = 2 \), (22) and (36) imply

\begin{align}
\int_0^{T-x(1)} g(2, x(1), s_2)f_2(s_2, T - x(1)) \, ds_2 = \frac{(n - k - 1)(T - x(1))}{n},
\end{align}

and

\begin{align}
\int_0^{T-x(1)} h^*(2, x(1), s_2)f_2(s_2, T - x(1)) \, ds_2 = 0
\end{align}

for every \( h \) satisfying

\begin{align}
\int_0^{T-x(1)} h(2, x(1), s_2)f_2(s_2, T - x(1)) \, ds_2 = 0.
\end{align}

By the same arguments used to prove Theorem 3, (43)–(45) and (10) imply

\begin{align}
g(2, x(1), s_2) = \frac{(n - k - 1)(T - x(1))}{n} = \frac{n - k - 1}{n}.
\end{align}
and this proves (41). If further \( n > 3 \), then for \( t < x(1), \ d = 3 \) and \( h(2, x(1), s_2) = 0 \), (22) and (36) along with (41) imply

\[
2(T-x(1)) \int_0^0 g(3, x(1), s_3)f_3(s_3, T-x(1)) \, ds_3 = \frac{(n-k-1)(n-k-2)(T-x(1))^2}{n(n-2)}
\]

and

\[
2(T-x(1)) \int_0^0 h^*(3, x(1), s_3)f_3(s_3, T-x(1)) \, ds_3 = 0
\]

for every \( h \) satisfying

\[
2(T-x(1)) \int_0^0 h(3, x(1), s_3)f_3(s_3, T-x(1)) \, ds_3 = 0.
\]

As before, (46)–(48) and (11) imply

\[
g(3, x(1), s_3) = \frac{(n-k-1)(n-k-2)(T-x(1))^2}{n(n-2) \int_0^2(T-x(1)) f_3(s_3, T-x(1)) \, ds_3}
\]

\[
= \frac{(n-k-1)(n-k-2)}{n(n-2)}
\]

which proves (42).

**Lemma 3.** For \( n > 2 \) and \( t \leq T \), an unbiased estimator \( g(V) \) of \( R(t) \) satisfying (41) and (42) does not satisfy (40) for \( h(V) = h_0(V) \), where \( h_0(V) \) satisfies (34)–(39).

**Proof.** Let \( t \leq T \) and suppose that an unbiased estimator \( g(V) \) of \( R(t) \) satisfying (41) and (42) also satisfies (40) for \( h(V) = h_0(V) \).

Consider first \( n = 3 \). By (10), (11), (25) and (38), it then follows that for \( x(1) < t \) and \( T-x(1) < s_3 < 2(T-x(1)) \),

\[
h_0(3, x(1), s_3) \neq 0,
\]

\[
g(3, x(1), s_3)h_0(3, x(1), s_3)f_3(s_3, T-x(1))
\]

\[
= \frac{(2-k)(2(T-x(1))-s_3)}{3}h_0(3, x(1), s_3),
\]

\[
g(3, x(1), s_3)f_3(s_3, T-x(1)) = \frac{2(1-k)(2(T-x(1))-s_3)}{3}.
\]

Clearly (49) and (50) contradict (51) and hence \( g(V) \) cannot satisfy (40) for \( h(V) = h_0(V) \).
Consider now $n > 3$ and assume $x(1) < t$. For $d = 3$, (36) then implies
\[
\sum_{c=2}^{3} \frac{(-1)^{3-c}(n-1)(n-c-1)}{\Gamma(n-c)} \frac{(c-1)(T-x(1))}{(c-1)(3-c)} \int_{0}^{T-x(1)} \frac{h_0(c,x(1),s_c)}{h_0(c,x(1),s_c)}
\times g(c,x(1),s_c)(s_c + (n-c)(T-x(1)))^{3-c} f_c(s_c,T-x(1)) ds_c
\]
\[
= \frac{(n-k-1)(n-k-2)}{n(n-2)} \sum_{c=2}^{3} \frac{(-1)^{3-c}(n-1)(n-c-1)}{\Gamma(n-c)} \frac{(c-1)(T-x(1))}{(c-1)(3-c)} \int_{0}^{T-x(1)} \frac{h_0(c,x(1),s_c)}{h_0(c,x(1),s_c)}
\times g(c,x(1),s_c)(s_c + (n-c)(T-x(1)))^{3-c} f_c(s_c,T-x(1)) ds_c
\]
\[
- \frac{k(n-1)(n-3)(n-k-1)}{n(n-2)\Gamma(n-2)}
\times \int_{0}^{T-x(1)} h_0(c,x(1),s_c)(s_2 + (n-2)(T-x(1)))f_2(s_2,T-x(1)) ds_2
\]
\[
= - \frac{k(n-1)(n-3)(n-k-1)}{n(n-2)\Gamma(n-2)}
\times \int_{0}^{T-x(1)} h_0(c,x(1),s_c)(s_2 + (n-2)(T-x(1))) ds_2,
\]
which is not zero, and this contradicts (36) with $h(V)$ replaced by $g(V)h_0(V)$ for $d = 3$. Hence, $g(V)$ can not satisfy (40) for $h(V) = h_0(V)$. This completes the proof of the lemma.

It follows from Lemmas 2 and 3 that for $n > 2$ and $t \leq T$, there does not exist any unbiased estimator $g(V)$ of $R(t)$ satisfying (40) for every unbiased estimator $h(V)$ of zero, and hence by Theorem 3, there does not exist UMVUE of $R(t)$. The results obtained above are summarized in the following theorem.

**Theorem 4.** For exp($\mu, \lambda$) distribution, there exists UMVUE of $R(t)$ under time censored sampling if and only if $n = 1, 2$ and $t \leq T$. Also for $n = 1, 2$ and $t \leq T$, $\hat{R}(t)$ defined in (3) is the UMVUE of $R(t)$.

**References**


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