APPLICATIONES MATHEMATICAE
37,1 (2010), pp. 113–126

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INEXACT NEWTON METHODS AND RECURRENT FUNCTIONS

Abstract. We provide a semilocal convergence analysis for approximating a solution of an equation in a Banach space setting using an inexact Newton method. By using recurrent functions, we provide under the same or weaker hypotheses: finer error bounds on the distances involved, and an at least as precise information on the location of the solution as in earlier papers. Moreover, if the splitting method is used, we show that a smaller number of inner/outer iterations can be obtained.

Furthermore, numerical examples are provided using polynomial, integral and differential equations.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding solutions of certain equations. For example, dynamical systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The

2000 *Mathematics Subject Classification*: 65H10, 65J15, 65G99, 47H17, 49M15.
Key words and phrases: inexact Newton method, recurrent functions, Banach space, majorizing sequence, residual, semilocal convergence, Lipschitz/center-Lipschitz conditions.

DOI: 10.4064/am37-1-8

unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We shall use the inexact Newton method (INM)

$$x_{n+1} = x_n + s_n \quad (n \geq 0), \tag{1.2}$$

where step s_n satisfies

$$F'(x_n)\, s_n = -F(x_n) + r_n \quad (n \geq 0) \tag{1.3}$$

for some null residual sequence $\{r_n\} \subseteq \mathcal{Y}$, to generate a sequence $\{x_n\}$ approximating the solution $x^\star$.

A convergence analysis of (INM) has been given by many authors and under various assumptions [1]–[7], [13], [15]–[17].

If $r_n = 0$ $(n \geq 0)$, we obtain the ordinary Newton method for solving nonlinear equations. Otherwise, iterative procedure (1.2) is called the inexact Newton method. By semilocal convergence we mean that we are seeking a solution $x^\star$ inside a ball centered at the initial guess x_0, and of a certain finite radius. We recommend the reading of Chapter XVIII on Newton's method of Kantorovich and Akilov's book [18], especially Theorem 6 in Subsection 1.5, along with the proof, to see how the majorizing function is constructed there (whose least zero plays an important role); see also the relevant Section 4.2 in [6].

There are two kinds of methods for the solution of linear equations. The first kind is the so-called direct methods, elimination methods. In this case the exact solution is determined through a finite number of arithmetic operations in real arithmetic without considering the round-off errors. For a list of difficulties and how to handle them we refer the reader to [9].

Another kind of methods are the iterative ones, which result in a two-stage method, sometimes termed as inner/outer iterations for solving nonlinear equation (1.1). In the two-stage method, Newton's method is the outer iteration, while an iterative method which is used to solve the Newton iteration is the inner iteration. In the case $\mathcal{X} = \mathcal{Y} = \mathbb{R}^j$ (j fixed in $\mathbb{N}$), we can split the matrix $F'(x_n)$ into $F'(x_n) = B_n - C_n$, to obtain the inner-outer iteration

$$x_{n+1} = x_n - (H_n^{m_n-1} + \cdots + H_n + I)B_n^{-1}F(x_n) \quad (n \geq 0), \tag{1.4}$$

$$H_n = B_n^{-1}C_n, \tag{1.5}$$

where m_n is the number of inner iterations. We usually let $m_n = m$, or choose any sequence in advance, like $m_n = n+1$ $(n \geq 0)$. As an example, consider the case when the nonlinear mapping $F(x)$ is mildly continuous, i.e.,

$$F(x) = Ax - \phi(x),$$

where $A \in \mathbb{R}^{j\times j}$ is nonsingular, and $\phi : \mathbb{R}^j \to \mathbb{R}^j$ is a nonlinear diagonal function with certain local smoothness properties [9]. Another case is when F is an affine mapping, i.e.,

$$F(x) = Ax - b,$$

where $A \in \mathbb{R}^{j\times j}$ is a large sparse, maybe ill-conditioned symmetric positive definite matrix, and b is fixed vector in $\mathbb{R}^j$.

In this study, we are motivated by optimization considerations and the elegant works by Guo [13]. Guo provided semilocal convergence analysis for (INM) using Lipschitz conditions on the Fréchet derivative F' of the operator F. He also provided bounds on the number of inner iteration steps. Using a combination of Lipschitz and center-Lipschitz conditions, we provided in [8] a semilocal convergence analysis (under the same or weaker hypotheses) with the following advantages over the work in [13]: finer error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$ $(n \geq 0)$, and an at least as precise information on the location of the solution $x^\star$.

We also show that the above advantages simplify our sufficient convergence conditions given in [8]. To achieve this goal, we use our new idea of recurrent polynomials in Section 2, where the semilocal convergence of (INM) is examined. Finally, in Section 3, numerical examples are provided, using a polynomial equation, an integral equation of Chandrasekhar type [11], and a differential equation involving Green's function. As a last application, we also provide a result (see Theorem 3.6) that simplifies the convergence condition for our Theorem 3.7 in [8], concerning the number of inner iterations under the conditions of Theorem 2.2.

2. Semilocal convergence analysis of (INM). We need the following result on majorizing sequences for (INM).

Lemma 2.1. *Let $\beta, \gamma_0, \gamma, \eta > 0$ be given constants. Assume that*

$$(2\gamma_0 + \gamma)\beta < 2 \tag{2.1}$$

and the quadratic polynomials

$$f_1(s) = 2\gamma_0\beta s^2 - (2 - (2\gamma_0 + \gamma)\beta)s + 2\eta, \tag{2.2}$$

$$\bar{f}_\infty(s) = s^2 - (1 + \eta - \gamma_0\beta)s + \eta \tag{2.3}$$

have minimal positive zeros in $(0, 1)$ *denoted by* s_1, s_∞, *respectively. Moreover, suppose that for*

$$\delta_0 = \frac{\gamma\beta + 2\eta}{1 - \gamma_0\beta}, \tag{2.4}$$

$$\delta_+ = \frac{-\gamma + \sqrt{\gamma^2 + 8\gamma_0\gamma}}{4\gamma_0}, \tag{2.5}$$

$$\delta_\infty = 2s_\infty, \tag{2.6}$$

the following hold:

$$\delta_0 \le \delta_\infty, \tag{2.7}$$

$$s_1 \le \delta_+. \tag{2.8}$$

Choose

$$\delta \in [\delta_\infty,\, 2\delta_+]. \tag{2.9}$$

Then the scalar sequence $\{t_n\}$ $(n \ge 0)$ *generated by*

$$t_0 = 0, \qquad t_1 = \beta, \qquad t_{n+2} = t_{n+1} + \frac{\gamma(t_{n+1} - t_n) + 2\eta}{2(1 - \gamma_0 t_{n+1})}\,(t_{n+1} - t_n) \tag{2.10}$$

is increasing, bounded above by

$$t^{\star\star} = \frac{2\beta}{2 - \delta}, \tag{2.11}$$

and converges to its unique least upper bound $t^\star$ *such that*

$$t^\star \in [0, t^{\star\star}]. \tag{2.12}$$

Moreover, the following estimates hold for all $n \ge 0$*:*

$$0 \le t_{n+2} - t_{n+1} \le \frac{\delta}{2}\,(t_{n+1} - t_n) \le \left(\frac{\delta}{2}\right)^{n+1}\beta. \tag{2.13}$$

Note that the most appropriate choice for δ *seems to be* $\delta = \delta_\infty$.

Proof. We shall show by induction on m that

$$0 < t_{m+2} - t_{m+1} = \frac{\gamma(t_{m+1} - t_m) + 2\eta}{2(1 - \gamma_0 t_{m+1})}\,(t_{m+1} - t_m) \le \frac{\delta}{2}\,(t_{m+1} - t_m), \tag{2.14}$$

and

$$\gamma_0 t_{m+1} < 1. \tag{2.15}$$

If (2.14) and (2.15) hold, then (2.13) holds, and

$$
\begin{aligned}
t_{m+2} &\le t_{m+1} + \frac{\delta}{2}(t_{m+1} - t_m) \\
&\le t_m + \frac{\delta}{2}(t_m - t_{m-1}) + \frac{\delta}{2}(t_{m+1} - t_m) \\
&\le \beta + \left(\frac{\delta}{2}\right)\beta + \cdots + \left(\frac{\delta}{2}\right)^{m+1}\beta \\
&= \frac{1-(\delta/2)^{m+2}}{1-\delta/2}\beta \\
&< \frac{2\beta}{2-\delta} = t^{\star\star} \qquad (\text{by } (2.11)).
\end{aligned} \tag{2.16}
$$

It will then also follow that the sequence $\{t_m\}$ is increasing, bounded above by $t^{\star\star}$, and hence converges to some $t^\star$ satisfying (2.12).

Estimates (2.14) and (2.15) hold for $m = 0$, by the initial conditions, (2.1), and the choice of δ, δ_0:

$$
\frac{\gamma(t_1 - t_0) + 2\eta}{1 - \gamma_0 t_1} = \frac{\gamma\beta + 2\eta}{1 - \gamma_0\beta} = \delta_0 \le \delta,
$$
$$
\gamma_0 t_1 = \gamma_0 \beta < 1.
$$

Let us assume (2.13)–(2.15) hold for all $m \le n + 1$. Estimate (2.14) can be rewritten as

$$
\gamma(t_{m+1} - t_m) + 2\eta + \gamma_0 \delta t_{m+1} - \delta \le 0,
$$

or

$$
\gamma\left(\frac{\delta}{2}\right)^m \beta + \gamma_0 \delta\, \frac{1-(\delta/2)^{m+1}}{1-\delta/2}\, \beta + 2\eta - \delta \le 0. \tag{2.17}
$$

By letting $m \to \infty$ in (2.17), and using the definition of $\overline{f}_\infty$, we see that s_∞ solves the equation

$$
\eta + \frac{\gamma_0 \beta s}{1-s} - s = 0,
$$

or equivalently, s_∞ is a zero of the polynomial $\overline{f}_\infty$.

Estimate (2.17) motivates us to introduce functions f_m on $[0, +\infty)$ $(m \ge 1)$ for $s = \delta/2$ by:

$$
f_m(s) = \gamma s^m \beta + 2\gamma_0 s(1 + s + s^2 + \cdots + s^m)\beta - 2s + 2\eta. \tag{2.18}
$$

Estimate (2.17) certainly holds if

$$
f_m(s) \le 0 \quad \text{for all } s \in [s_\infty, \delta_+] \ (m \ge 1). \tag{2.19}
$$

We need to find a relationship between two consecutive polynomials f_m:

$$
\begin{aligned}
f_{m+1}(s) &= \gamma s^{m+1}\beta + 2\gamma_0 s(1+s+s^2+\cdots+s^m+s^{m+1})\beta - 2s + 2\eta \\
&= \gamma s^m\beta - \gamma s^m\beta + \gamma s^{m+1}\beta \\
&\quad + 2\gamma_0 s(1+s+s^2+\cdots+s^m)\beta + 2\gamma_0 s^{m+2}\beta - 2s + 2\eta \\
&= f_m(s) + g(s)\beta s^m,
\end{aligned}
\tag{2.20}
$$

where

$$
g(s) = 2\gamma_0 s^2 + \gamma s - \gamma. \tag{2.21}
$$

Note that g has a positive zero δ_+ given by (2.5).

By hypothesis, the function f_1 has a minimal positive zero s_1. Using (2.20) for $m = 1$, and (2.8), we have

$$
f_2(s_1) = f_1(s_1) + g(s_1)\beta s_1^m = g(s_1)\beta s_1^m < 0. \tag{2.22}
$$

From (2.18) we also have

$$
f_m(0) = 2\eta > 0 \quad (m \geq 1). \tag{2.23}
$$

It follows from (2.22), (2.23), and the intermediate value theorem that there exists $s_2 \in (0, s_1]$ such that $f_2(s_2) = 0$. Note that s_2 is the unique positive zero of f_2 in $(0, s_1]$, since

$$
f_m'(s) > 0 \quad (s > 0). \tag{2.24}
$$

Assume that there exists $s_m \in (0, s_{m-1}]$ such that $f_m(s_m) = 0$. Then we have as above

$$
f_{m+1}(s_m) = f_m(s_m) + g(s_m)\beta s_m^m = g(s_m)\beta s_m^m < 0. \tag{2.25}
$$

since $f_m(s_m) = 0$ and $s_m \leq \delta_+$.

Estimates (2.23)–(2.25) establish the existence of a unique zero s_{m+1} of f_{m+1} in $(0, s_m)$.

The sequence $\{s_m\}$ is nonincreasing, bounded below by zero, and so it converges to its unique maximum lowest bound $s^\star$ satisfying $s^\star \geq s_\infty$. Hence, we showed (2.17) holds, since $\delta/2 \in [s_\infty, \delta_+]$. That completes the induction for (2.14) and (2.15). ∎

If

$$
(4\gamma_0 + \gamma)\beta + 2\eta < 2,
$$

then it follows from the intermediate value theorem applied to the function f_1 for $s \in (0, 1)$ that

$$
f_1(0)f_1(1) = 2\eta((4\gamma_0 + \gamma)\beta + 2\eta - 2) < 0.
$$

Hence, s_1 exists in $(0, 1)$.

By the induction hypotheses s_∞ exists, and if

$$
\eta < 1 + \gamma_0\beta,
$$

then $s_\infty \in (0, 1)$.

Hence, the above conditions can replace the hypotheses on the functions f_1 and $\overline{f}_\infty$ given in Lemma 2.1.

We can now provide a semilocal convergence theorem for (INM) using information on the differences $r_n - r_{n-1}$, and $\|x_n - x_{n-1}\|$ $(n \geq 1)$:

THEOREM 2.2. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be a Fréchet differentiable operator. Assume that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ for some $x_0 \in \mathcal{D}$, and there exist constants $\beta, \gamma_0, \gamma > 0$ and $\eta \in [0, 1)$ such that for all $x, y \in \mathcal{D}$,*

$$
\begin{aligned}
\|F'(x_0)^{-1}[F(x_0) - r_0]\| &\leq \beta, \\
\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| &\leq \gamma_0 \|x - x_0\|, \\
\|F'(x_0)^{-1}[F'(x) - F'(y)]\| &\leq \gamma \|x - y\|, \\
\|F'(x_0)^{-1}(r_n - r_{n-1})\| &\leq \eta_n \|x_n - x_{n-1}\|, \qquad \eta = \max_n \{\eta_n\}, \\
U = \overline{U}(x_0, t^\star) &\subseteq \mathcal{D},
\end{aligned}
\tag{2.26}
$$

and the hypotheses of Lemma 2.1 hold. Then the sequence $\{x_n\}$ generated by (INM) is well defined, remains in U for all $n \geq 0$, and converges to a solution $x^\star$ of the equation $F(x) = 0$.

Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{2.27}$$

$$\|x^\star - x_n\| \leq t^\star - t_n, \tag{2.28}$$

where the scalar sequence $\{t_n\}$ and $t^\star$ are given in Lemma 2.1.

Proof. If $\|x_k - x_{k-1}\| \leq t_k - t_{k-1}$, using (2.26), and the estimate

$$\|F'(x_k)^{-1} F'(x_0)\| \leq \frac{1}{1 - \gamma_0 \|x_k - x_0\|} \leq \frac{1}{1 - \gamma t_k} \qquad \text{(see [8])},$$

we can get in turn

$$
\begin{aligned}
\|x_{k+1} - x_k\| &= \|F'(x_k)^{-1}[F(x_k) - r_k]\| \\
&\leq \|F'(x_k)^{-1} F'(x_0)\| \{ \|F'(x_0)^{-1}[F(x_k) - F(x_{k-1}) \\
&\qquad - F'(x_{k-1})(x_k - x_{k-1})]\| + \|F'(x_0)^{-1}(r_{k-1} - r_k)\| \} \\
&\leq \frac{1}{1 - \gamma_0 \|x_k - x_0\|} \left[\frac{\gamma}{2} \|x_k - x_{k-1}\|^2 + \eta_k \|x_k - x_{k-1}\| \right] \\
&\leq \frac{1}{1 - \gamma_0 t_k} \left[\frac{\gamma}{2} (t_k - t_{k-1}) + \eta \right] (t_k - t_{k-1}) = t_{k+1} - t_k.
\end{aligned}
\tag{2.29}
$$

That is, we have shown estimate (2.27) for all $k \geq 0$.

Estimate (2.28) follows from (2.27) by using standard majorization techniques [6], [7], [18].

In view of the fact that the sequence $\{t_n\}$ is Cauchy, it follows that $\{x_n\}$ is also a Cauchy sequence in the Banach space $\mathcal{X}$ and so it converges to some $x^\star \in U$ (since U is a closed set). By letting $k \to \infty$ in (2.29) and since $\lim_{k\to\infty} r_k = 0$, we get $F(x^\star) = 0$. ∎

REMARK 2.3. Let us define a related majorizing sequence $\{\bar{t}_n\}$ by simply replacing γ_0 by γ in the definition of $\{t_n\}$. Then, under the hypotheses of Theorem 2.7 in [13] (see, e.g., the hypothesis on h_G in Application 3.1), and our Theorem 2.2, since $\gamma_0 \le \gamma$ we have

$$t_n \le \bar{t}_n \quad (n \ge 2),$$
$$t_n - t_{n-1} \le \bar{t}_n - \bar{t}_{n-1} \quad (n \ge 2),$$
$$t^\star - t_n \le \bar{t}^\star - \bar{t}_n \quad (n \ge 0),$$

where

$$\bar{t}^\star = \lim_{n\to\infty} \bar{t}_n.$$

Note that strict inequality holds in the first two error estimates provided that $\gamma_0 < \gamma$.

Hence, the claims made in the introduction of this study are now justified.

3. Special cases and applications .

APPLICATION 3.1 (Newton's method). That is, set $\eta = 0$. The hypothesis

$$h_G = \beta\gamma \le \frac{1-\eta^2}{2} \quad \text{(see [13])} \tag{3.1}$$

reduces to the famous Newton–Kantorovich hypothesis [6], [7], [18] for solving nonlinear equations:

$$h_K:\ \gamma\beta \le 1/2. \tag{3.2}$$

Note that in this case, the polynomials f_m $(m \ge 1)$ should be

$$f_m(s) = (\gamma s^{m-1} + 2\gamma_0(1 + s + s^2 + \cdots + s^m))\beta - 2, \tag{3.3}$$

and

$$f_{m+1}(s) = f_m(s) + g(s)s^{m-1}\beta. \tag{3.4}$$

But this time,

$$s_\infty = 1 - \gamma_0\beta, \qquad \delta_\infty = 2s_\infty, \tag{3.5}$$

and the conditions corresponding to Lemma 2.1 should be

$$\delta_1 = \max\{\delta_0/2,\ \delta_+\} \le s_\infty, \tag{3.6}$$

whereas

$$\delta/2 \in [\delta_1, \delta_\infty]. \tag{3.7}$$

It is then simple algebra to show that conditions (3.6)–(3.7) reduce to

$$h_A = \alpha\beta \le 1/2, \tag{3.8}$$

where

$$\alpha = \tfrac{1}{8}\left(\gamma + 4\gamma_0 + \sqrt{\gamma^2 + 8\gamma_0\gamma}\right). \tag{3.9}$$

In view of (3.2), (3.8), and (3.9), we get

$$h_K \le 1/2 \;\Rightarrow\; h_A \le 1/2, \tag{3.10}$$

but not necessarily vice versa unless $\gamma = \gamma_0$.

We provide examples where $\gamma_0 \le \gamma$ or (3.8) holds but (3.2) is violated.

EXAMPLE 3.2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$, $U_0 = \{x : |x - x_0| \le 1 - p\}$, $p \in [0, 1/2)$, and define a function F on U_0 by

$$F(x) = x^3 - p. \tag{3.11}$$

CASE 1: $\eta = 0$. Using the hypotheses of Theorem 2.2, we get

$$\beta = \tfrac{1}{3}(1 - p), \qquad \gamma_0 = 3 - p, \quad \text{and} \quad \gamma = 2(2 - p).$$

The Kantorovich condition (3.2) is violated, since

$$\tfrac{4}{3}(1 - p)(2 - p) > 1 \qquad \text{for all } p \in [0, 1/2).$$

Hence, there is no guarantee that Newton's method converges to $x^\star = \sqrt[3]{p}$, starting at $x_0 = 1$.

However, our condition (3.8) is true for all $p \in I = [.450339002, 1/2)$. Hence, the conclusions of our Theorem 2.2 can apply to solve equation (3.11) for all $p \in I$.

CASE 2: $0 \ne \eta = 0.01$. Choose $p = .49$; then using (2.2)–(2.5) and the above, we get

$$\begin{aligned} &\gamma_0 = 2.51 < \gamma = 3.02, \qquad \beta = .17, \\ &s_1 = .033058514 < \delta_+ = .53112045, \\ &\delta_0 = .3347085 < s_\infty = .03587956. \end{aligned}$$

Note also that condition (3.1) is violated no matter how η is chosen in $(0, 1)$.

EXAMPLE 3.3. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with the norm

$$\|x\| = \max_{0 \le s \le 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the "cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s, t) u(t)\, dt + y(s) - \theta. \tag{3.12}$$

Here the kernel $q(s,t)$ is a continuous function of two variables defined on $[0,1]\times[0,1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0,1]$; and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (3.12) arise in the kinetic theory of gases [7], [11]. For simplicity, we choose $u_0(s)=y(s)=1$, and $q(s,t)=s/(s+t)$ for all $s,t\in[0,1]$ with $s+t\neq 0$. If we let $\mathcal{D}=U(u_0,1-\theta)$, and define the operator F on $\mathcal{D}$ by

$$F(x)(s)=x^3(s)-x(s)+\lambda\, x(s)\int_0^1 q(s,t)x(t)\,dt+y(s)-\theta \tag{3.13}$$

for all $s\in[0,1]$, then every zero of F satisfies equation (3.12). We have the estimates

$$\max_{0\le s\le 1}\left|\int\frac{s}{s+t}\,dt\right|=\ln 2.$$

Therefore, if we set $\xi=\|F'(u_0)^{-1}\|$, then it follows from the hypotheses of Theorem 2.2 that

$$\beta=\xi(|\lambda|\ln 2+1-\theta),$$
$$\gamma=2\xi(|\lambda|\ln 2+3(2-\theta)),\qquad \gamma_0=\xi(2|\lambda|\ln 2+3(3-\theta)).$$

It follows from Theorem 2.2 that if condition (3.8) holds, then problem (3.12) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (3.2).

Note also that $\gamma_0<\gamma$ for all $\theta\in[0,1]$.

Example 3.4. Consider the following nonlinear boundary value problem [7]:

$$\begin{cases} u''=-u^3-\gamma u^2,\\ u(0)=0,\quad u(1)=1.\end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s)=s+\int_0^1 Q(s,t)(u^3(t)+\gamma u^2(t))\,dt \tag{3.14}$$

where Q is the Green function

$$Q(s,t)=\begin{cases} t(1-s), & t\le s,\\ s(1-t), & s<t.\end{cases}$$

We observe that

$$\max_{0\le s\le 1}\int_0^1|Q(s,t)|\,dt=\frac{1}{8}.$$

Let $\mathcal{X}=\mathcal{Y}=\mathcal{C}[0,1]$, with norm $\|x\|=\max_{0\le s\le 1}|x(s)|$. Then problem

(3.14) is in the form (1.1), where $F : \mathcal{D} \to \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \gamma x^2(t))\,dt.$$

It is easy to verify that the Fréchet derivative of F is

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s,t)(3x^2(t) + 2\gamma x(t))v(t)\,dt.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. It follows that $2\gamma < 5$. Then

$$\begin{aligned}
\|I - F'(u_0)\| &\le \frac{3\|u_0\|^2 + 2\gamma\|u_0\|}{8} = \frac{3+2\gamma}{8}\,\|F'(u_0)^{-1}\| \\
&\le \frac{1}{1 - \frac{3+2\gamma}{8}} = \frac{8}{5-2\gamma}, \\
\|F(u_0)\| &\le \frac{\|u_0\|^3 + \gamma\|u_0\|^2}{8} = \frac{1+\gamma}{8}, \\
\|F(u_0)^{-1}F(u_0)\| &\le \frac{1+\gamma}{5-2\gamma}.
\end{aligned}$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v](s) = -\int_0^1 Q(s,t)(3x^2(t) - 3y^2(t) + 2\gamma(x(t) - y(t)))v(t)\,dt.$$

Consequently,

$$\begin{aligned}
\|F'(x) - F'(y)\| &\le \frac{\|x-y\|(2\gamma + 3(\|x\| + \|y\|))}{8} \\
&\le \frac{\|x-y\|(2\gamma + 6R + 6\|u_0\|)}{8} \\
&= \frac{\gamma + 6R + 3}{4}\,\|x-y\|, \\
\|F'(x) - F'(u_0)\| &\le \frac{\|x-u_0\|(2\gamma + 3(\|x\| + \|u_0\|))}{8} \\
&\le \frac{\|x-u_0\|(2\gamma + 3R + 6\|u_0\|)}{8} \\
&= \frac{2\gamma + 3R + 6}{8}\,\|x-u_0\|.
\end{aligned}$$

Therefore, the conditions of Theorem 2.2 hold with

$$\beta = \frac{1+\gamma}{5-2\gamma}, \qquad \gamma = \frac{\gamma + 6R + 3}{4}, \qquad \gamma_0 = \frac{2\gamma + 3R + 6}{8}.$$

Note also that $\gamma_0 < \gamma$.

Application 3.5. Let us assume $m_n = m$ in iteration (1.4). We can obtain a result concerning the estimation of the number of inner iterations under the conditions of Theorem 2.2:

Theorem 3.6. *Under the hypotheses of Theorem* 2.2, *further assume:*

$$\|B_0^{-1}F'(x_0)\| \leq q,$$
$$a_0 h^m + m b h^{m-1} \leq \eta_n, \qquad \sup_n \|H_n\| \leq h < 1,$$

where

$$\begin{aligned} a_0 &= \frac{3-2\eta+2\beta\gamma^n}{\eta^2}, \\ b &= \frac{2-\eta}{\eta}\,\frac{q(q+1)\gamma_0}{[1-(1-\eta)\gamma_0 q]^2}\left[\frac{(1-\eta)^2}{2\gamma}+\frac{1-\eta}{\gamma}+\beta\right]. \end{aligned} \tag{3.15}$$

Moreover, suppose that

$$\|F'(x_0)^{-1}R\| \leq \|F'(x_0)^{-1}S\|$$

with R any submatrix of S, and

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D},$$

and the hypotheses of Lemma 2.1 *hold. Then the conclusions of Theorem* 2.2 *hold true for the inexact iteration* (1.4).

Proof. This follows exactly as in Corollary 3.3 of [13], and our Theorem 3.7 in [8]. Here are the changes (with γ_0 replacing γ in the proof):

$$\begin{aligned} \|F'(x_0)^{-1}F'(x_n)\| &\leq 1+\gamma_0\|x_n-x_0\|, \\ \|F'(x_n)^{-1}F'(x_0)\| &\leq \frac{1}{1-\gamma_0\|x_n-x_0\|}, \\ \|F'(x_0)^{-1}F(x_n)\| &\leq \frac{\gamma}{2}\,\|x_n-x_0\|^2+\|x_n-x_0\|+\beta, \\ \|F'(x_0)^{-1}(B_n-B_{n-1})\| &\leq \gamma\|x_n-x_{n-1}\|, \\ \|B_n^{-1}F'(x_0)^{-1}\| &\leq \frac{q}{1-\gamma_0\|x_n-x_0\|q}. \end{aligned}$$

The constant $\overline{b}$ defined in [13] (for $\gamma_0 = \gamma$) is larger than b, which is an advantage of our approach for the selection of a smaller η, when $\gamma < \gamma_0$. ∎

Note that the hypotheses of Theorem 3.6 are simpler than the hypotheses of our Theorem 3.7 in [8], and weaker than those of Corollary 3.3 in [13].

Conclusion.. We provided a semilocal convergence analysis for (INM) in order to approximate a locally unique solution of an equation in a Banach space.

Using recurrent functions, a combination of Lipschitz and center-Lipschitz conditions, instead of only Lipschitz conditions [1]–[10], [11]–[24], we provided an analysis with the following advantages over the work in [1]–[7], [13], [15]–[17]:

(a) weaker sufficient convergence conditions in some interesting cases (e.g., when $F'(x_n) = B_n - C_n$);
(b) larger convergence domain;
(c) finer majorizing sequences;
(d) an at least as precise information on the location of the solution.

Note that these advantages are obtained under the same computational cost as in [1]–[7], [13], [15]–[17], since in practice the computation of the Lipschitz constant γ requires the computation of γ_0.

Numerical examples further validating the results were also provided in this study.

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Received on 3.11.2008;
revised version on 24.2.2009 (1975)