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A CONTINUOUS-TIME MODEL FOR CLAIMS RESERVING

Abstract. Prediction of outstanding liabilities is an important problem in non-life insurance. In the framework of the Solvency II Project, the best estimate must be derived by well defined probabilistic models properly calibrated on the relevant claims experience. A general model along these lines was proposed earlier by Norberg (1993, 1999), who suggested modelling claim arrivals and payment streams as a marked point process. In this paper we specify that claims occur in $[0,1]$ according to a Poisson point process, possibly non-homogeneous, and that each claim initiates a stream of payments, which is modelled by a non-homogeneous compound Poisson process. Consecutive payment streams are i.i.d. and independent of claim arrivals. We find estimates for the total payment in an interval $(v, v+s]$, where $v \geq 1$, based upon the total payment up to time v. An estimate for Incurred But Not Reported (IBNR) losses is also given.

1. Introduction. Prediction of outstanding liabilities is an important problem in non-life insurance. Insurance companies are required to designate appropriate reserves to cover future claims. In the framework of the Solvency II Project, the best estimate must be derived by well defined probabilistic models properly calibrated on the relevant claims experience. Different methods were proposed; see Jessen et al. [6] for some critical remarks. In this study we undertake another line of research with the use of stochastic processes. A general model along these lines was proposed earlier by Norberg [11, 12], where claim arrivals and payment streams were modelled as a marked point process. A good introduction to defining such processes by Poisson measures is given in the book of Mikosch [10]. Along these lines there is a cycle of papers by Mikosch and Matsui [9] and Matsui [7, 8].

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In this paper we consider only payments initiated by claims from a fixed period (typically a fiscal year), set to be the interval $[0,1]$. We denote the point process of claim occurrences by N. Each claim consists of the occurrence time and a mark. The mark describes how the claim is settled, which is modelled by the so called payment process.

The object of interest is $S(t)$, the cumulative amount paid by time t (*risk reserve*). In the following, $v \geq 1$ is fixed. We wish to predict $S(v, v+s] = S(v+s)-S(v)$ on the basis of some history up to time v. Different knowledge may be available to the insurer: the full history up to time v, the knowledge of the number $N(1)$, or of $S(v)$, as is the case in this paper.

In Matsui and Mikosch [9, 7], at each claim arrival time from $[0,1]$, which is a point of a homogeneous Poisson process, there starts an independent copy of a payment stream modelled by a Lévy process. They propose estimators for reserves in the form $(S(v, v+s) \mid N(v))$. Recently Matsui [8] considered claims from a non-homogeneous Poisson process, with delayed payments modelled by a sequence of i.i.d. r.v.s and payment processes modelled by Lévy processes or additive processes.

In this paper we consider the following modifications to the above-described models. We consider claims occurring in $[0,1]$ according to a non-homogeneous Poisson point process and we assume that each claim initiates a stream of payments, modelled by a non-homogeneous compound Poisson process. As a result, the total amount paid for a claim is a Poisson compound r.v. (provided it is finite). We suppose that all payment processes are non-negative integer valued. Notice that in this model the first payment is also delayed, and the delays are i.i.d.

We are interested in predicting $S(v, v+s] = S(v+s) - S(v)$ conditioned on $S(v)$. Therefore we compute the moment generating function (m.g.f.) of $(S(v, v+s] \mid S(v))$. In particular we work out a formula for the estimator $\mathbb{E}[S(v, v+s] \mid S(v) = k]$. Since it is quite complicated, we propose a saddlepoint approximation for this estimator. We conjecture that this approximation is asymptotically consistent as $k \to \infty$ but no proof is presented. Numerical experiments are discussed, where we compute

$$\mathbb{E}[S(v, v+s] \mid S(v) = k] = \frac{\mathbb{E}[S(v, v+s];\, S(v) = k]}{\mathbb{P}(S(v) = k)}.$$

The denominator above is a Poisson compound r.v., for which there are many numerical methods available (see the survey by Embrecht and Frei [4]). On the other hand, we show that for the numerator the fast Fourier transform (FFT) method can be applied. We check the conjecture of saddlepoint approximation using the proposed numerical method.

In this study we also consider estimates for losses in the interval $(v, v+s]$ caused by IBNR (incurred but not reported) claims. We identify the first payment with the reported time and consequently we have two types of claims: those reported by time v or not. We denote the point process of reported claims (by time v) by N^{o} and of non-reported ones by N^*. Notice that $N(1) = N^{\mathrm{o}}(1) + N^*(1)$. Correspondingly the cumulative payment up to time s of the reported claims is denoted by $S^{\mathrm{o}}(s)$ and of the non-reported ones by $S^*(s)$. Notice that $S^{\mathrm{o}}(s) = S(s)$ for $s \le v$.

2. Model and assumption. In this section the basic random objects are defined on a common measurable space $(\Omega, \mathcal{F})$ with probability measure $\mathbb{P}$. Let N be a non-homogeneous Poisson process in $[0,1]$ with intensity function $a(t)$. It is known that $N(1)$ is a Poissonian r.v. with mean $\bar{a} = \int_0^1 a(s)\,ds$ and if there are n points of N in $[0,1]$, then they are obtained by drawing n i.i.d. points $T_1, \ldots, T_n$ with the common probability density function (p.d.f.)

$$f_T(t) = \frac{a(t)}{\bar{a}} \mathbb{1}(0 < t < 1). \tag{2.1}$$

Hence the joint distribution of the random vector $(N(1), T_1, \ldots, T_{N(1)})$ is

$$\begin{aligned} \mathbb{P}(N(1) = n, T_1 \in dt_1, \ldots, T_{N(1)} \in dt_n) \\ = \frac{\bar{a}^n}{n!} e^{-\bar{a}} f_T(t_1) \cdots f_T(t_n)\, dt_1 \cdots dt_n \end{aligned} \tag{2.2}$$

(see textbooks on point processes by Daley and Vere-Jones [3] and Cinlar [2]).

At each claim occurrence time T_i we start a process X_i which will be interpreted as the claim payment stream. We suppose that $(X_i)_{i=1,2,\ldots}$ is a sequence of i.i.d. mixed compound Poisson processes independent of N,

$$X_i(t) = \sum_{j=1}^{M_i(t)} C_{ij},$$

where (M_i) is a sequence of i.i.d. non-homogeneous Poisson processes with intensity function $b(t)$, independent of N and with a double array of i.i.d. r.v.s C_{ij} with generic r.v. C. The distribution of C is denoted by B and the corresponding moment generating function (m.g.f.) by $\hat{B}$. Throughout this paper, C is non-negative integer valued. We tacitly assume that $M_i(t) = 0$ for $t < 0$, so $X_i(t) = 0$ there.

In this paper we are interested in the risk reserve process

$$S(t) = \sum_{i=1}^{N(1)} X_i\,(t - T_i) =_{\mathrm{d}} \sum_{i=1}^{N(1)} X_i\left(t - T_{(i)}\right) \tag{2.3}$$

where $T_{(1)} < T_{(2)} < \cdots < T_{(N(1))}$.

We obtain results featuring expressions of the form

$$\mathbb{E}\Big[\Phi(N(1),T_1,\ldots,T_{N(1)})\;\Big|\;\sum_{j=1}^{Z}C_j=k\Big], \tag{2.4}$$

where Z is a r.v. independent of the i.i.d. sequence $C_1, C_2, \ldots$, Poissonian distributed with parameter $\sum_{j=1}^{N(1)} h(T_j)$ for some function $h(x)$ (to be defined in (3.2)) and $\Phi(n, t_1, \ldots, t_n)$ is symmetric for each n. We can rewrite (2.4) as

$$\mathbb{E}\Big[\Phi(N(1),T_1,\ldots,T_{N(1)})\;\Big|\;\sum_{i=1}^{N(1)}U_i=k\Big], \tag{2.5}$$

where

$$U_i=\sum_{j=1}^{W_i}C_{ij} \tag{2.6}$$

and $W_1, W_2, \ldots$ are independent r.v.s, mixed Poissonian with mixing distribution defined by $\xi_i = h(T_i)$.

We now give special cases of symmetric functionals of interest in this paper:

- If $\Phi(n, t_1, \ldots, t_n) = \sum_{i=1}^{n} g(t_i)$, then (2.4) reduces to

$$\mathbb{E}\Big[N(1)g(T_1)\;\Big|\;\sum_{j=1}^{Z}C_j=k\Big]. \tag{2.7}$$

- If $\Phi(n, t_1, \ldots, t_n) = n$, then (2.4) reduces to

$$\mathbb{E}\Big[N(1)\;\Big|\;\sum_{j=1}^{Z}C_j=k\Big]. \tag{2.8}$$

- $$\Phi(n,t_1,\ldots,t_n))=G_\alpha(n,t_1,\ldots,t_n)=\exp\Big\{\sum_{i=1}^{n}g(t_i)(\hat{B}(\alpha)-1)\Big\}.$$

Analysing formula (2.8) we notice that

$$\mathbb{E}\Big[N(1)\;\Big|\;\sum_{j=1}^{Z}C_j=k\Big]=\mathbb{E}\Big[N(1)\;\Big|\;\sum_{i=1}^{N(1)}U_i=k\Big], \tag{2.9}$$

where U_i are as defined in (2.6). We remark that some cases of (2.9) were studied in [15].

3. Prediction of cumulative claim amount in a future time interval. Suppose $v \geq 1$ and define, for $s > 0$ and $x \in [0,1]$,

$$g_{v,s}(x) = g(x) = \int_{v-x}^{v+s-x} b(w)\,dw, \tag{3.1}$$

$$h_v(x) = h(x) = \int_{0}^{v-x} b(w)\,dw. \tag{3.2}$$

Let $\{S(t)\}_{t\geq 0}$ be the stochastic process defined in (2.3) and $S(v, v+s] = S(v+s) - S(v)$. Further, we denote by $(C_{ij})_{ij}$ a stochastic array of i.i.d. r.v.s distributed as C_1. Define

$$p_k(n, t_1, \ldots, t_n) = \mathbb{P}\Big(\sum_{j=1}^{Z} C_j = k\Big), \tag{3.3}$$

where Z is independent of the sequence $C_1, C_2, \ldots$, Poissonian distributed with mean $\sum_{j=1}^{n} h(t_j)$.

We now study the conditional moment generating function $\mathbb{E}[e^{\alpha S(v,v+s]} \mid S(v) = k]$.

Proposition 1. *We have*

$$\mathbb{E}[e^{\alpha S(v,v+s]} \mid S(v) = k] = \frac{\mathbb{E}[G_\alpha(N(1), T_1, \ldots, T_{N(1)}) \cdot p_k(N(1), T_1, \ldots, T_{N(1)})]}{\mathbb{E}[p_k(N(1), T_1, \ldots, T_{N(1)})]}, \tag{3.4}$$

where

$$G_\alpha(n, t_1, \ldots, t_n) = \exp\Big\{\sum_{i=1}^{n} \int_{v-t_i}^{v+s-t_i} b(v)\,dv \cdot (\hat{B}(\alpha) - 1)\Big\}.$$

Proof. We write

$$\mathbb{E}[e^{\alpha S(v,v+s]} \mid S(v) = k] = \frac{\mathbb{E}[e^{\alpha S(v,v+s]};\ S(v) = k]}{\mathbb{P}(S(v) = k)} = \frac{n_k(v,s)}{d_k(v,s)}.$$

We have to compute

$$n_k(v,s) = \sum_{n\geq 1} \int_0^1 \ldots \int_0^1 \mathbb{E}\Big[\prod_{i=1}^{n} \exp\Big\{\alpha \sum_{j=M_i(v-t_i)+1}^{M_i(v+s-t_i)} C_{ij}\Big\} \mathbb{1}(S(v) = k)\Big] \times \frac{(\int_0^1 a(w)\,dw)^n}{n!} e^{-\int_0^1 a(w)\,dw}\, \mathbb{P}(T_1 \in dt_1, \ldots, T_n \in dt_n)$$

$$= \sum_{n\ge 1} \int_0^1 \dots \int_0^1 \mathbb{E}\Big[\prod_{i=1}^n \exp\Big\{\alpha \sum_{j=M_i(v-t_i)+1}^{M_i(v+s-t_i)} C_{ij}\Big\}\Big] p_k(n,t_1,\dots,t_n)$$
$$\times \frac{(\int_0^1 a(w)\,dw)^n}{n!} e^{-\int_0^1 a(w)\,dw}\, \mathbb{P}(T_1\in dt_1,\dots,T_n\in dt_n)$$

Now

$$\mathbb{E}\Big[\prod_{i=1}^n \exp\Big\{\alpha \sum_{j=M_i(v-t_i)+1}^{M_i(v+s-t_i)} C_{ij}\Big\}\Big] = \prod_{i=1}^n \mathbb{E}[\exp\{\alpha \sum_{j=M_i(v-t_i)+1}^{M_i(v+s-t_i)} C_{ij}\}]$$
$$= \prod_{i=1}^n \exp\Big\{\int_{v-t_i}^{v+s-t_i} b(w)\,dw \cdot (\hat{B}(\alpha)-1)\Big\}$$
$$= \exp\Big\{\sum_{i=1}^n \int_{v-t_i}^{v+s-t_i} b(w)\,dw \cdot (\hat{B}(\alpha)-1)\Big\}$$
$$= G_\alpha(n,t_1,\dots,t_n).$$

Thus we can rewrite the above as

$$n_k(v,s) = \mathbb{E}[G_\alpha(N(1),T_1,\dots,T_{N(1)})\cdot p_k(N(1),T_1,\dots,T_{N(1)})],$$

which finally yields (3.4). ■

Corollary 1. *We have*

$$\text{(3.5)}\qquad \mathbb{E}[S(v,v+s]\,|\,S(v)=k]$$
$$= \mathbb{E}\, C_{ij}\cdot \frac{\mathbb{E}[N(1)g(T_1)p_k(N(1),T_1,\dots,T_{N(1)})]}{\mathbb{E}[p_k(N(1),T_1,\dots,T_{N(1)})]}$$
$$= \mathbb{E}\, C_{ij}\cdot \mathbb{E}\Big[N(1)g(T_1)\ \Big|\ \sum_{j=1}^{Z} C_j = k\Big].$$

Proof. Compute the first derivative of the conditional moment generating function in (3.4) at $\alpha = 0$. ■

Example 1. In the special case when $v=1$, $a(t)\equiv a$, $b(t)\equiv b$ we have $\xi_i = b(1-T_i)$, $g(x)\equiv sb$, and under the condition $N(1)=n$ we have

$$\sum_{j=1}^n \int_0^{1-T_i} b(v)\,dv = b\sum_{j=1}^n (1-T_j).$$

Thus $h(x) = b(1-x)$. In this case $T_1,\dots,T_n$ are i.i.d. uniformly distributed on $[0,1]$ (under $\mathbb{P}$). Then, if $N(1)=n$,

$$\sum_{j=1}^{Z} C_j =_{\mathrm{d}} \sum_{i=1}^n U_i,$$

where $U_i = \sum_{j=1}^{W_i} C_{ij}$ and each W_i is mixed Poisson$(b(1-T_i))$. Supposing $C_{ij} \equiv 1$ we see that (3.5) reduces to

$$\mathbb{E}\Big[N(1) \,\Big|\, \sum_{j=1}^{N(1)} U_j = k\Big].$$

4. A hypothesis of saddlepoint approximation. In this section we prove a fundamental identity, which will serve as a starting point for saddlepoint approximations of

$$\mathbb{E}\Big[\Phi(N(1), T_1, \ldots, T_{N(1)}) \,\Big|\, \sum_{j=1}^{Z} C_j = k\Big].$$

Recall that B is the distribution of C and the corresponding m.g.f. is $\hat{B}(s)$. We will assume that $\hat{B}(s)$ is *steep*, that is, either (i) $\hat{B}(s) < \infty$ for all $s > 0$, or (ii) there exists $s_0 > 0$ such that $\hat{B}(s_0) < \infty$ for all $s < s_0$ and $\hat{B}(s) = \infty$ for all $s \geq s_0$ (see e.g. [1, p. 91]). In other words $\hat{B}$ assumes all values from $[1, \infty)$. In what follows, $N(1)$, $(T_j)_{j=1,2,\ldots}$, $(C_j)_{j=1,2,\ldots}$ will always be independent r.v.s defined on $(\Omega, \mathcal{F})$. Let s be such that $\hat{B}(s) < \infty$. We assume that:

(A.1) $N(1)$ is Poisson with mean $\int_0^1 a(x) e^{(\hat{B}(s)-1)h(x)}\,dx$,
(A.2) $T_1, T_2, \ldots$ are i.i.d. with p.d.f.

$$f_T^{(s)}(x) = \frac{a(x) e^{(\hat{B}(s)-1)h(x)}}{\int_0^1 a(y) e^{(\hat{B}(s)-1)h(y)}\,dy} \mathbb{1}(x \in (0,1)).$$

(A.3) $C_1, C_2, \ldots$ are i.i.d. with distribution $e^{sx} B(dx)/\hat{B}(s)$.

In probability theory one demonstrates that for a suitably chosen measurable space $(\Omega, \mathcal{F})$ one can find a probability measure $\mathbb{P}^{(s)}$ such that (A.1)–(A.3) are fulfilled. We will denote the corresponding expectation by $\mathbb{E}^{(s)}$. Notice that for $s = 0$ we have $\mathbb{P}^{(0)} = \mathbb{P}$, that is, the original probability measure under which we considered our model.

The conditional probability $\mathbb{P}^{(s)}$ on $N(1) = n$ and $T_1 = t_1, \ldots, T_n = t_n$ is denoted by $\mathbb{P}^{(s)}_{n,t_1,\ldots,t_n}$. Moreover, under $\mathbb{P}^{(s)}_{n,t_1,\ldots,t_n}$, the r.v. Z is Poissonian with mean $\hat{B}(s) \sum_{j=1}^{n} h(t_j)$, where the function h was defined in (3.2).

The idea of so called tilted measures (in this paper the family of measures $\mathbb{P}^{(s)}$) is used to prove some limiting theorems of probability theory. For example for Poisson compounds, the reader can find information how to define tilted measures in Jensen [5, introduction to Chapter 7]; see also Asmussen & Albrecher [1, Chapter XVI, 2a].

Proposition 2.

$$\mathbb{E}\Big[\Phi(N(1),T_1,\ldots,T_{N(1)});\ \sum_{j=1}^{Z} C_j = k\Big] = e^{sk-\bar{a}+\int_0^1 a(x)e^{(\hat{B}(s)-1)h(x)}\,dx}\cdot \mathbb{E}^{(s)}\Big[\Phi(N(1),T_1,\ldots,T_{N(1)});\ \sum_{j=1}^{Z} C_j = k\Big].$$

Proof. We have

$$\mathbb{E}\Big[\Phi(N(1),T_1,\ldots,T_{N(1)});\ \sum_{j=1}^{Z} C_j = k\Big] = \sum_{n\ge 0}\frac{\bar{a}^n}{n!}e^{-\bar{a}}\sum_{z\ge 0}\int_0^1\ldots\int_0^1 \frac{(\sum_{j=1}^n h(t_j))^z}{z!}e^{-\sum_{j=1}^n h(t_j)}\Phi(n,t_1,\ldots,t_n) \times \mathbb{P}_{n,t_1,\ldots,t_n}\Big(\sum_{j=1}^{z} C_j = k\Big) f_T(t_1)\ldots f_T(t_n)\,dt_1\ldots\,dt_n.$$

Now

$$\mathbb{P}_{n,t_1,\ldots,t_n}\Big(\sum_{j=1}^{z} C_j = k\Big) = \mathbb{E}^{(s)}_{n,t_1,\ldots,t_n}\Big[e^{-s\sum_{j=1}^{z} C_j + z\hat{\beta}(s)};\sum_{j=1}^{z} C_j = k\Big] = e^{-sk}\hat{B}(s)^z\,\mathbb{P}^{(s)}_{n,t_1,\ldots,t_n}\Big(\sum_{j=1}^{z} C_j = k\Big),$$

where $\hat{\beta}(s) = \log \hat{B}(s)$. Hence

$$\sum_{n\ge 0}\frac{\bar{a}^n}{n!}e^{-\bar{a}}\sum_{z\ge 0}\int_0^1\ldots\int_0^1 \frac{(\sum_{j=1}^n h(t_j))^z}{z!}e^{-\sum_{j=1}^n h(t_j)}\Phi(n,t_1,\ldots,t_n) \times \mathbb{P}_{n,t_1,\ldots,t_n}\Big(\sum_{j=1}^{z} C_j = k\Big) f_T(t_1)\ldots f_T(t_n)\,dt_1\ldots\,dt_n$$

$$= e^{-sk}\sum_{n\ge 0}\frac{\bar{a}^n}{n!}e^{-\bar{a}} \times \sum_{z\ge 0}\int_0^1\ldots\int_0^1 \frac{(\sum_{j=1}^n \hat{B}(s)h(t_j))^z}{z!}e^{-\sum_{j=1}^n \hat{B}(s)h(t_j)}\Phi(n,t_1,\ldots,t_n) \times \mathbb{P}^{(s)}_{n,t_1,\ldots,t_n}\Big(\sum_{j=1}^{z} C_j = k\Big)\prod_{j=1}^{n} e^{(\hat{B}(s)-1)h(t_j)} f_T(t_j)\,dt_j$$

$$
\begin{aligned}
&= e^{-sk-\bar{a}+\int_0^1 a(x)e^{(\hat{B}(s)-1)h(x)}\,dx} \\
&\qquad\times \sum_{n\ge 0} \frac{(\int_0^1 a(x)e^{(\hat{B}(s)-1)h(x)}\,dx)^n}{n!} e^{-\int_0^1 a(x)e^{(\hat{B}(s)-1)h(x)}\,dx} \\
&\qquad\times \sum_{z\ge 0} \int_0^1 \dots \int_0^1 \frac{(\sum_{j=1}^n \hat{B}(s)h(t_j))^z}{z!} e^{-\sum_{j=1}^n \hat{B}(s)h(t_j)} \Phi(n,t_1,\dots,t_n) \\
&\qquad\times \mathbb{P}^{(s)}_{n,t_1,\dots,t_n}\Big(\sum_{j=1}^z C_j = k\Big) \prod_{j=1}^n f_T^{(s)}(t_j)\,dt_j,
\end{aligned}
$$

which is the required RHS of the conclusion. ■

Observe that under the new measure $\mathbb{P}^{(s)}$,

$$
\begin{aligned}
(4.1)\qquad \mathbb{E}^{(s)}\Big[\sum_{j=1}^Z C_j\Big] &= \mathbb{E}^{(s)}\, C\, \mathbb{E}^{(s)}\, Z = \frac{\hat{B}'(s)}{\hat{B}(s)}\,\mathbb{E}^{(s)}\, N(1)\, \mathbb{E}^{(s)}\, h(T) \\
&= \frac{\hat{B}'(s)}{\hat{B}(s)} \cdot \int_0^1 a(x)e^{(\hat{B}(s)-1)h(x)}\,dx \\
&\qquad \times \int_0^1 h(x)a(x)e^{(\hat{B}(s)-1)h(x)}\,dx \Big/ \int_0^1 a(x)e^{(\hat{B}(s)-1)h(x)}\,dx \\
&= \frac{\hat{B}'(s)}{\hat{B}(s)} \int_0^1 h(x)a(x)e^{(\hat{B}(s)-1)h(x)}\,dx,
\end{aligned}
$$

where in the first equality we have used the Wald identity.

Lemma 1. *Suppose that $\hat{B}$ is steep. Then for each $k = 1, 2, \dots$ there exists a unique solution $\theta = \theta(k)$ of*

$$
\mathbb{E}^{(\theta)}\Big[\sum_{j=1}^Z C_j\Big] = k.
$$

Proof. Function $\hat{B}'(s)/\hat{B}(s)$ is nondecreasing because its derivative

$$
\frac{\hat{B}''(s)}{\hat{B}(s)} - \bigg(\frac{\hat{B}'(s)}{\hat{B}(s)}\bigg)^2
$$

is non-negative as the variance of a r.v. with distribution $(e^{sx}/\hat{B}(s))\,B(dx)$. Thus from formula (4.1) we see that

$$
\frac{\hat{B}'(s)}{\hat{B}(s)} \int_0^1 h(x)a(x)e^{(\hat{B}(s)-1)h(x)}\,dx
$$

is increasing and it tends continuously to infinity. ■

The following corollary to Proposition 2 is crucial for our approximations.

COROLLARY 2. *For all* $k = 1, 2, \dots,$

$$\mathbb{E}\Big[\Phi(N(1), T_1, \dots, T_{N(1)}) \;\Big|\; \sum_{j=1}^{Z} C_j = k\Big] = \frac{\mathbb{E}^{(\theta)}[\Phi(N(1), T_1, \dots, T_{N(1)});\ \sum_{j=1}^{Z} C_j = k]}{\mathbb{P}^{(\theta)}(\sum_{j=1}^{Z} C_j = k)}.$$

We now formulate the following conjecture.

CONJECTURE 1 (about saddlepoint approximation). Suppose that $\hat{B}$ is steep and the distribution of C is lattice with span 1. If

$$\mathbb{P}^{(\theta)}(|\Phi(N(1), T_1, \dots, T_{N(1)}) / \mathbb{E}^{(\theta)}\, \Phi(N(1), T_1, \dots, T_{N(1)}) - 1| > \epsilon) \to 0, \tag{4.2}$$

then

$$\mathbb{E}\Big[\Phi(N(1), T_1, \dots, T_{N(1)}) \;\Big|\; \sum_{j=1}^{Z} C_j = k\Big] \sim \mathbb{E}^{(\theta)}[\Phi(N(1), T_1, \dots, T_{N(1)})] \quad \text{as } k \to \infty.$$

REMARK 1. Denote for short

$$\Phi = \Phi(N(1), T_1, \dots, T_{N(1)}),$$
$$Y = \frac{\sum_{j=1}^{Z} C_j - k}{\sqrt{\mathbb{V}\mathrm{ar}(\sum_{j=1}^{Z} C_j)}}.$$

Notice that for each k, the distribution of Y under $\mathbb{P}^{(\theta)}$ is concentrated on a lattice. Moreover $\mathbb{E}^{\theta}\, Y = 0$ and $\mathbb{V}\mathrm{ar}^{\theta}\, Y = 1$. In view of Corollary 2 we have to consider

$$\frac{\mathbb{E}^{\theta}[\Phi;\ Y = 0]}{\mathbb{P}^{(\theta)}(Y = 0)} = \mathbb{E}^{(\theta)}[\Phi] \frac{\mathbb{E}^{(\theta)}\big[\frac{\Phi}{\mathbb{E}^{(\theta)}\,\Phi};\ Y = 0\big]}{\mathbb{P}^{(\theta)}(Y = 0)}.$$

One can prove that Y under $\mathbb{P}^{\theta}$ converges in distribution to the standard normal distribution as $k \to \infty$. Hence, recalling condition (4.2), it is plausible that

$$\frac{\mathbb{E}^{(\theta)}\big[\frac{\Phi}{\mathbb{E}^{(\theta)}\,\Phi};\ Y = 0\big]}{\mathbb{P}^{(\theta)}(Y = 0)} \to 1.$$

EXAMPLE 2. We continue Example 1 and show in this case the saddlepoint approximation for the conditional m.g.f. We have

$$G_{\alpha}(n, t_1, \dots, t_n) = e^{nsb(\hat{B}(\alpha) - 1)}$$

and the saddlepoint is the solution of

$$\frac{\hat{B}'(\theta)}{\hat{B}(\theta)} ab \int_0^1 x e^{b(\hat{B}(\theta)-1)x}\, dx = k.$$

Then, under $\mathbb{P}^{(\theta)}$, the r.v. $N(1)$ is Poissonian with mean $(\hat{B}(\theta)/\hat{B}'(\theta))k$. Hence

$$\begin{aligned}&\mathbb{E}[e^{\alpha S(t,t+s]} \mid S(t) = k]\\ &\qquad \sim \mathbb{E}^{(\theta)}[e^{N(1)sb(\hat{B}(\alpha)-1)}] = \exp\biggl(\frac{\hat{B}(\theta)}{\hat{B}'(\theta)} k(e^{sb(\hat{B}(\alpha)-1)} - 1)\biggr).\end{aligned}$$

4.1. A central limit theorem. We now show a central limit theorem for $S(v, v+s]$. That is, we find a_n, b_n such that $((S(v, v+s] - a_n)/b_n \mid S(v) = k)$ converges in distribution to the standard normal $\mathcal{N}(0,1)$. Unfortunately a direct approach seems to be difficult. Therefore to work out this limit theorem we use the saddlepoint hypothesis applied to the conditional m.g.f. of $S(t, t+s]$. Thus define

$$a_k = \mathbb{E}[S(v, v+s] \mid S(v) = k], \qquad b_k^2 = \mathbb{V}\mathrm{ar}[S(v, v+s] \mid S(v) = k].$$

Since computing the above sequences can be troublesome, we use our Conjecture 1. Hence we have, for $k \to \infty$,

$$a_k \sim \mathbb{E}\, C \cdot \mathbb{E}^{(\theta)}\, N(1) \cdot \mathbb{E}^{(\theta)}\Bigl[\int_{v-T_1}^{v+s-T_1} b(x)\, dx\Bigr] \tag{4.3}$$

and

$$\begin{aligned} b_k^2 \sim{}& \mathbb{E}\, C^2\, \mathbb{E}^{(\theta)}\, N(1)\, \mathbb{E}^{(\theta)}\Bigl[\int_{v-T_1}^{v+s-T_1} b(x)\, dx\Bigr]\\ &+ (\mathbb{E}\, C)^2\, \mathbb{E}^{(\theta)}\, N(1)\, \mathbb{E}^{(\theta)}\Bigl[\int_{v-T_1}^{v+s-T_1} b(x)\, dx\Bigr]^2. \end{aligned} \tag{4.4}$$

Furthermore

$$\begin{aligned} &\mathbb{E}\bigl[e^{\alpha \frac{S(v,v+s]-a_k}{b_k}} \bigm| S(v) = k\bigr]\\ &= e^{-\alpha\frac{a_k}{b_k}}\, \mathbb{E}\bigl[e^{\frac{\alpha}{b_k} S(v,v+s]} \bigm| S(v) = k\bigr]\\ &= e^{-\alpha\frac{a_k}{b_k}} \cdot \mathbb{E}\Bigl[e^{\sum_{i=1}^{N(1)} \int_{v-T_i}^{v+s-T_i} b(x)\, dx\cdot(\hat{B}(\alpha/b_k)-1)} \Bigm| \sum_{i=1}^{N(1)} U_i = k\Bigr] && (4.5)\\ &\sim e^{-\alpha\frac{a_k}{b_k}} \cdot \mathbb{E}^{(\theta)}\bigl[e^{\sum_{i=1}^{N(1)} \int_{v-T_i}^{v+s-T_i} b(x)dx\cdot(\hat{B}(\alpha/b_k)-1)}\bigr], && (4.6) \end{aligned}$$

where (4.5) is due to Proposition 1, and (4.6) follows from the conjecture about saddlepoint approximation.

Let us now compute the expected value in (4.6). Clearly it is a moment generating function of $\sum_{i=1}^{N(1)} \int_{v-T_i}^{v+s-T_i} b(x)\,dx$ at the point $\hat{B}(\alpha)-1$. Denote $r_j = \mathbb{E}^{(\theta)}[\int_{v-T_1}^{v+s-T_1} b(x)\,dx]^j$. Then

$$
\begin{aligned}
(4.7)\qquad & e^{-\alpha\frac{a_k}{b_k}}\,\mathbb{E}^{(\theta)}\big[e^{\sum_{i=1}^{N(1)}\int_{v-T_i}^{v+s-T_i} b(x)\,dx\cdot(\hat{B}(\alpha/b_k)-1)}\big]\\
&= e^{-\alpha\frac{a_k}{b_k}}\exp\big\{\mathbb{E}^{(\theta)}\,N(1)\big(\mathbb{E}^{(\theta)}\big[e^{(\hat{B}(\alpha/b_k)-1)\int_{v-T_i}^{v+s-T_i} b(x)\,dx}\big]-1\big)\big\}\\
&\sim e^{-\alpha\frac{a_k}{b_k}}\exp\bigg\{\mathbb{E}^{(\theta)}\,N(1)\bigg(r_1(\hat{B}(\alpha/b_k)-1)+r_2\frac{(\hat{B}(\alpha/b_k)-1)^2}{2}+\cdots\bigg)\bigg\}\\
&\sim e^{-\alpha\frac{a_k}{b_k}}\exp\bigg\{\mathbb{E}^{(\theta)}\,N(1)\bigg(r_1\bigg(\mathbb{E}\,C\frac{\alpha}{b_k}+\mathbb{E}\,C^2\frac{\alpha^2}{2b_k^2}\bigg)+r_2\frac{(\mathbb{E}\,C\frac{\alpha}{b_k})^2}{2}+\cdots\bigg)\bigg\}\\
&= e^{\frac{\alpha^2}{2}\big(\frac{\mathbb{E}^{(\theta)}\,N(1)(\mathbb{E}\,C^2 r_1+(\mathbb{E}\,C)^2 r_2)}{b_k^2}\big)}\cdot e^{\alpha\big(\frac{\mathbb{E}^{(\theta)}\,N(1)\,\mathbb{E}\,C r_1-a_k}{b_k}\big)}\cdots.
\end{aligned}
$$

We conclude now from (4.7), using (4.3) and (4.4), that

$$
\mathbb{E}\big[e^{\alpha\frac{S(v,v+s]-a_k}{b_k}}\;\big|\;S(v)=k\big]\to e^{\alpha^2/2}\qquad\text{as } k\to\infty.
$$

4.2. Numerical experiments. Here our purpose is to check Conjecture 1 taking into account some examples and by doing some numerical calculations. We will focus on

$$
(4.8)\qquad \varPhi(n,t_1,\ldots,t_n)\equiv\phi(n)g(t_1).
$$

We suppose that $g(x)\ge 0$ and $0<\int g(x)\,dx<\infty$. Then

$$
(4.9)\qquad \mathbb{E}\Big[\phi(N(1))g(T_1)\;\Big|\;\sum_{j=1}^{Z} C_j=k\Big]=\mathbb{E}\Big[\phi(N(1))g(T_1)\;\Big|\;\sum_{j=1}^{N} H_j=k\Big],
$$

where $H_1,H_2,\ldots$ are i.i.d. r.v.s distributed as $\sum_{j=1}^{W} C_j$, and W, independent of (C_j) and $N(1)$, is mixed Poisson ($W=_{\rm d} Poi(\xi)$, where $\xi=\int_0^{v-T} b(x)\,dx$). We point out that the choice of $\varPhi$ in (4.8) is equivalent to $\varPhi(n,t_1,\ldots,t_n)=\sum_{j=1}^{n} t_j$, which for each n is a symmetric function. Unfortunately it is not easy to numerically compute expressions like (4.9). To do that, we propose to use the fast Fourier transform (FFT) (see e.g. the book by Rolski et al. [13] or [4]).

For this we need to compute the Fourier transforms of the following sequences:

$$n_k = \mathbb{E}\Big[\phi(N(1))g(T_1);\ \sum_{j=1}^{Z} C_j = k\Big] \tag{4.10}$$
$$= \sum_{n\ge 0} \frac{\bar{a}^n}{n!} e^{-\bar{a}} \phi(n)\, \mathbb{E}[g(T_1) p_k(n, T_1, \ldots, T_n)],$$

$$d_k = \mathbb{E}\Big[\mathbb{1}\Big(\sum_{j=1}^{Z} C_j = k\Big)\Big], \qquad k = 0, 1, \ldots. \tag{4.11}$$

Then

$$i_k = \mathbb{E}\Big[\phi(N(1))g(T_1);\ \sum_{j=1}^{Z} C_j = k\Big] = \frac{n_k}{d_k}.$$

PROPOSITION 3. *For n_k defined in* (4.10) *and d_k defined in* (4.11) *we have*

$$\sum_{k\ge 0} n_k e^{ixk} = e^{-\bar{a}}\phi(0) + \bar{a} e^{-\bar{a}}\, \mathbb{E}\, g(T_1) e^{h(T_1)(\hat{B}(ix)-1)} \sum_{n\ge 1} \frac{\phi(n)}{n!} (\bar{a}\, \mathbb{E}\, e^{h(T)(\hat{B}(ix)-1)})^{n-1}$$

and

$$\sum_{k\ge 0} d_k e^{ixk} = e^{\bar{a}(\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}]-1)}.$$

Proof. Notice that

$$\sum_{k\ge 0} e^{ixk}\, \mathbb{P}_{n,t_1,\ldots,t_n}\Big(\sum_{j=1}^{Z} C_j = k\Big) = e^{\sum_{j=1}^{n} h(t_j)(\hat{B}(ix)-1)},$$

where $\hat{B}(x) = \mathbb{E}\, e^{xC}$. Furthermore

$$\sum_{k\ge 0} n_k e^{ixk}$$
$$= \sum_{k\ge 0}\sum_{n\ge 0} \frac{\bar{a}^n}{n!} e^{-\bar{a}} e^{ixk} \int_0^1 \cdots \int_0^1 g(t_1)\phi(n) p_k(n, t_1, \ldots, t_n) \times f_T(t_1) \cdots f_T(t_n)\, dt_1 \cdots dt_n$$
$$= e^{-\bar{a}}\phi(0) + \sum_{n\ge 1} \frac{\bar{a}^n}{n!} e^{-\bar{a}} \phi(n) \int_0^1 \cdots \int_0^1 \sum_{k\ge 0} e^{ixk} p_k(n, t_1, \ldots, t_n) \times f_T(t_1) g(t_1) f_T(t_2) \cdots f_T(t_n)\, dt_1 \cdots dt_n$$

$$= e^{-\bar a}\phi(0) + \sum_{n\ge 1}\frac{\bar a^n}{n!}e^{-\bar a}\phi(n)\int_0^1\ldots\int_0^1 e^{\sum_{j=1}^n h(t_j)(\hat B(ix)-1)} \times f_T(t_1)g(t_1)f_T(t_2)\cdots f_T(t_n)\,dt_1\cdots dt_n$$

$$= e^{-\bar a}\phi(0) + \sum_{n\ge 1}\frac{\bar a^n}{n!}e^{-\bar a}\phi(n)\int_0^1\ldots\int_0^1 g(t_1)\prod_{j=1}^n e^{h(t_j)(\hat B(ix)-1)}f_T(t_j)\,dt_1\cdots dt_n$$

$$= e^{-\bar a}\phi(0) + \sum_{n\ge 1}\frac{\bar a^n}{n!}e^{-\bar a}\phi(n) \times \Big(\int_0^1 g(t_1)e^{h(t_1)(\hat B(ix)-1)}f_T(t_1)\,dt_1\cdot\prod_{j=2}^n\int_0^1 e^{h(t_j)(\hat B(ix)-1)}f_T(t_j)dt_j\Big)$$

$$= e^{-\bar a}\phi(0) + \mathbb{E}\,g(T_1)e^{h(T_1)(\hat B(ix)-1)}\sum_{n\ge 1}\phi(n)\frac{\bar a^n}{n!}(\mathbb{E}\,e^{h(T)(\hat B(ix)-1)})^{n-1}$$

$$= e^{-\bar a}\phi(0) + \bar a e^{-\bar a}\,\mathbb{E}\,g(T_1)e^{h(T_1)(\hat B(ix)-1)}\sum_{n\ge 1}\frac{\phi(n)}{n!}(\bar a\,\mathbb{E}\,e^{h(T)(\hat B(ix)-1)})^{n-1}.$$

Setting $\varPhi\equiv 1$ in n_k we get d_k, which completes the proof. ∎

Remark 2. For some special cases we know the function $\sum_{n\ge 1}\frac{\phi(n)}{n!}z^{n-1}$ explicitly. For example, $\phi(n)=n$ yields

$$\sum_{n\ge 1}\frac{\phi(n)}{n!}z^{n-1} = e^z,$$

and $\phi(n)=n(n-1)$ gives

$$\sum_{n\ge 1}\frac{\phi(n)}{n!}z^{n-1} = ze^z.$$

Finally if $\phi(n)=e^{dn}$, then

$$\sum_{n\ge 1}\frac{\phi(n)}{n!}z^{n-1} = \frac{1}{z}(e^{ze^d}-1).$$

We will continue in Examples 3 and 4.

4.2.1. *Examples.* In this section we compute the expected value in (4.9) for some special cases. We compare the values obtained with asymptotics based on saddlepoint approximations.

Example 3. In this case we have $\phi(n)=n$. Consider $a(t)\equiv a$, $b(t)\equiv b$, $v=1$ and $s=2$. We want to calculate $\mathbb{E}[S(1,2]\,|\,S(1)=k]$. Following

Proposition 1 we get

$$\mathbb{E}[S(1,2] \mid S(1)=k] = \mathbb{E}\, C_j \cdot \mathbb{E}\Big[N(1)g(T_1) \;\Big|\; \sum_{j=1}^{Z} C_j = k\Big],$$

where $g(t)=b$ and Z under the condition $N(1)=n$ is Poissonian distributed with parameter $\sum_{j=1}^{n} h(t_j)$ for $h(t)=(1-t)b$. In addition we assume that the r.v.s $\{C_i\}$ are i.i.d. with $\mathbb{P}(C_1=1)=\mathbb{P}(C_1=2)=0.5$. Hence the characteristic function of C_1 is

$$\hat{B}(ix) = \frac{(\cos x + \cos 2x) + i(\sin x + \sin 2x)}{2}.$$

Denote

$$\begin{aligned} i_k &= \mathbb{E}\Big[N(1)g(T_1) \;\Big|\; \sum_{j=1}^{Z} C_j = k\Big] \\ &= \frac{n_k}{d_k} = \frac{\mathbb{E}[N(1)g(T_1);\, \sum_{j=1}^{Z} C_j = k]}{\mathbb{P}(\sum_{j=1}^{Z} C_j = k)}. \end{aligned}$$

In this case the Fourier transform of n_k is

$$\sum_{k\geq 0} n_k e^{itk} = abe^{-a}\, \mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}] e^{a\,\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}]}.$$

Moreover

$$\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}] = \mathbb{E}[e^{b(1-T)(\hat{B}(ix)-1)}] = \frac{e^{b(\hat{B}(ix)-1)}-1}{b(\hat{B}(ix)-1)}.$$

We can make analogous computations for the denominator $d_k = \mathbb{P}(\sum_{j=1}^{Z} C_j = k)$ to demonstrate

$$\sum_{k\geq 0} d_k e^{itk} = e^{a(\mathbb{E}[e^{b(1-T)(\hat{B}(ix)-1)}]-1)}.$$

In the next step we use the fast Fourier transform function in some software environment (we use R environment) to compute approximate values of n_k and d_k. On the other hand, we compute the asymptotics based on saddlepoint approximation. The saddlepoint equation takes the form

$$ab\hat{B}'(\theta)\frac{e^{b(\hat{B}(\theta)-1)}(b(\hat{B}(\theta)-1)-1)+1}{b^2(\hat{B}(\theta)-1)^2} = k \tag{4.12}$$

and the asymptotics is

$$i_k = \frac{n_k}{d_k} \sim ab\frac{e^{b(\hat{B}(\theta)-1)}-1}{b(\hat{B}(\theta)-1)},$$

where θ is a solution of the equation in (4.12), which can be solved numerically.

The results for $a = 1$ and $b = 5$ are shown in Figure 1.

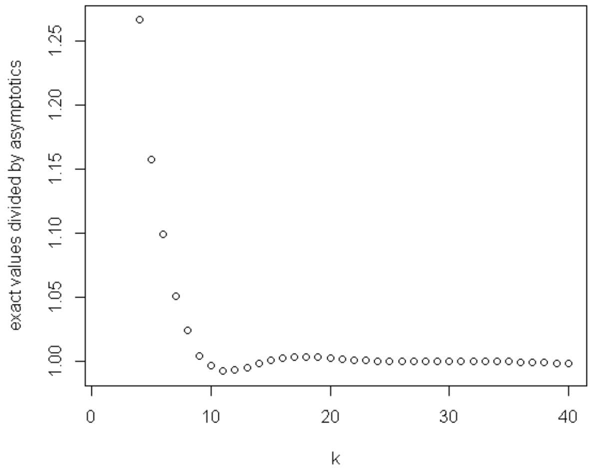

Fig. 1. Values of i_k for $a = 1$ and $b = 5$

Example 4. Our second example takes $a(t) \equiv a$, $b(t) = t$, $v = 1$ and $s = 2$. As before we want to calculate $\mathbb{E}[S(1,2] \mid S(1) = k]$.

We now have $g(t) = 3/2 - t$ and $h(t) = (1-t)^2/2$ and $\{C_i\}$ are the same, i.e. $\mathbb{P}(C_1 = 1) = \mathbb{P}(C_1 = 2) = 0.5$. The Fourier transform of n_k takes the form

$$\sum_{k\geq 1} n_k e^{ixk} = ae^{-a}\,\mathbb{E}[g(T)e^{h(T)(\hat{B}(ix)-1)}]e^{a\,\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}]}.$$

An easy computation shows that

$$\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}] = \int_0^1 e^{-\frac{t^2}{2}(1-\hat{B}(ix))}\,dt$$

and

$$\mathbb{E}[g(T)e^{h(T)(\hat{B}(ix)-1)}] = \frac{1}{2}\,\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}] + \frac{1-e^{-(1-\hat{B}(ix))/2}}{1-\hat{B}(ix)}.$$

For the denominator $d_k = \mathbb{P}(\sum_{j=1}^{Z} C_j = k)$ in this case we get

$$\sum_{k\geq 1} d_k e^{itk} = e^{a(\mathbb{E}[e^{h(T)(\hat{B}(ix)-1)}]-1)}.$$

In order to get the asymptotics we have to solve the saddlepoint equation

$$\hat{B}'(\theta)\frac{1}{2}a\int_0^1 t^2 e^{-\frac{t^2}{2}(1-\hat{B}(\theta))}\,dt = k.$$

Hence

$$i_k = \frac{n_k}{d_k} \sim a\left(\frac{1}{2}\int_0^1 e^{-\frac{t^2}{2}(1-\hat{B}(\theta))}\,dt + \frac{e^{(\hat{B}(\theta)-1)/2}-1}{\hat{B}(\theta)-1}\right)$$

as $k \to \infty$. Both n_k and d_k tend to 0 as $k \to \infty$, they are very small and their accuracy is not sufficient and therefore we can observe some irregularities in Figure 2 for $k \geq 50$.

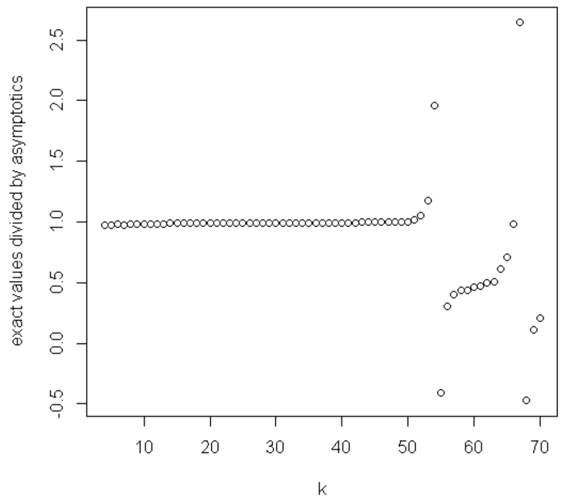

Fig. 2. Values of i_k for $a = 1$ and $b(t) = t$

5. Estimates for IBNR losses; decomposition of the process S. It is obvious that the first payment does not have to be executed by time $v \geq 1$ for each claim. Therefore we introduce a decomposition $S(t) = S^{\circ}(t) + S^{*}(t)$ $(t \geq v)$ into two processes of the total amount of claims paid by time t, with the first payment before v or not respectively.

Recall that the original claim arriving process N is a non-homogeneous Poisson process with intensity function $a(t)$ on $[0,1]$. We now decompose $N = N^{\circ} \cup N^{*}$, where N^{*} consists of points of N such that the first payment is before v (process N°) or not (process N^{*}). Recall that a generic payment process M is a non-homogeneous Poisson process with intensity function $b(u)$ with consecutive points $0 < D_1 < D_2 < \cdots$. The distribution F_D of D_1 has p.d.f.

$$f_D(u) = b(u)e^{-\int_0^u b(w)\,dw}, \qquad u \geq 0.$$

We can represent

$$M(t) = \mathbb{1}(D_1 \le t) + \Pi(\Lambda_{D_1}(t)), \tag{5.1}$$

where $\Pi(t)$ is the standard homogeneous Poisson process with rate 1 and

$$\Lambda_u(t) = \begin{cases} 0 & \text{for } t < u, \\ \int_u^t b(w)\,dw & \text{for } t \ge u. \end{cases}$$

From the theory of Poisson processes we know that N^{o}, N^* are independent, with intensity functions $a^{\mathrm{o}}(t) = a(t)\,\mathbb{P}(D_1 \le v - t)$ and $a^*(t) = a(t)\,\mathbb{P}(D_1 \ge v - t)$ $(t \le 1)$ respectively. Thus the independent r.v.s $N^*(1)$, $N^{\mathrm{o}}(1)$ are Poissonian with respective means

$$\bar{a}^{\mathrm{o}} = \int_0^1 a(t)\big(1 - e^{-\int_0^{v-t} b(w)\,dw}\big)\,dt,$$
$$\bar{a}^* = \int_0^1 a(t)\big(e^{-\int_0^{v-t} b(w)\,dw}\big)\,dt.$$

Let $M_1, M_2, \ldots$ be i.i.d. copies of M independent of N. Consecutive points of M_i are $0 < D_{i1} < D_{i2} < \cdots$. For a fixed $v > 1$ and $s > 0$ we define

$$S^{\mathrm{o}}(v+s) = \sum_{j=1}^{L^{\mathrm{o}}(v+s)} C_j \quad \text{and} \quad S^*(v+s) = \sum_{j=1}^{L^*(v+s)} C_j,$$

where (C_j) are i.i.d. with distribution of generic C and L^{o}, L^* are the total numbers of payments for claims with the first payment before or after time v respectively. Using representation (5.1) we can write, for $v > 1$ and $s > 0$,

$$\begin{aligned} L^{\mathrm{o}}(v+s) &= \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} \le v)\big(\mathbb{1}(D_{i1} \le v + s - T_i) + \Pi_i(\Lambda_{D_{i1}}(v+s-T_i))\big) \\ &= \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} \le v)\big(1 + \Pi_i(\Lambda_{D_{i1}}(v+s-T_i))\big) \end{aligned}$$

and

$$L^*(v+s) = \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} \ge v)\big(\mathbb{1}(D_{i1} \le v + s - T_i) + \Pi_i(\Lambda_{D_{i1}}(v+s-T_i))\big),$$

where Π_i $(i = 1, 2, \ldots)$ are i.i.d. standard Poisson processes independent of the other random elements. Hence

$$S^{\mathrm{o}}(v+s) = \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} \le v)\Big(C_{i1} + \sum_{j=2}^{\Pi_i(\Lambda_{D_{i1}}(v+s-T_i))} C_{ij}\Big)$$

and similarly

$$S^{*}(v+s) = \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} > v)\Big(\mathbb{1}(D_{i1} \le v+s-T_i)C_{i1} + \sum_{j=2}^{\Pi_i(\Lambda_{D_{i1}}(v+s-T_i))} C_{ij}\Big).$$

Since for each i we have

$$(5.2)\qquad \Pi_i(\Lambda_{D_{i1}}(v+s-T_i)) - \Pi_i(\Lambda_{D_{i1}}(v-T_i)) =_{\mathrm{d}} \tilde{\Pi}_i(\Lambda_{D_{i1}}(v-T_i, v+s-T_i]),$$

where Π_i and $\tilde{\Pi}_i$ are independent, standard Poisson processes, careful examination yields

$$\begin{aligned} L^{\mathrm{o}}(v, v+s] &= L^{\mathrm{o}}(v+s) - L^{\mathrm{o}}(v) \\ &=_{\mathrm{d}} \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} \le v)\tilde{\Pi}_i(\Lambda_{D_{i1}}(v-T_i, v+s-T_i]) \end{aligned}$$

and

$$\begin{aligned} S^{\mathrm{o}}(v, v+s] &= S^{\mathrm{o}}(v+s) - S^{\mathrm{o}}(v) \\ &=_{\mathrm{d}} \sum_{i=1}^{N(1)} \mathbb{1}(T_i + D_{i1} \le v) \sum_{j=1}^{\tilde{\Pi}_i(\Lambda_{D_{i1}}(v-T_i, v+s-T_i])} C_{ij}. \end{aligned}$$

On the other hand, we have

$$L^{*}(v, v+s] =_{\mathrm{d}} L^{*}(v+s), \qquad S^{*}(v, v+s] =_{\mathrm{d}} S^{*}(v+s).$$

In this section, we present formulas for IBNR reserve, but for this purpose we need another representation for our processes. As the reader may have noticed, the new processes we have defined in this section are constructed by thinning a Poisson measure. For further details consider now a Poisson measure $\mathcal{M}$ on $[0,1] \times \mathbb{R}_+$ with points

$$\mathcal{M} = (T_i, D_{i1}), \qquad i = 1, \ldots, N(1).$$

This process has intensity measure $\mu(dx, dy) = a(x)dx\, F_D(dy)$ on $[0,1] \times \mathbb{R}_+$, where F_D is the distribution of D_{i1}. We next perform thinning of points of the process leaving only points such that $T_i + D_{i1} \le v$, that is, we obtain a Poisson measure $\mathcal{M}^{\mathrm{o}}$ with points in

$$\Delta_v^{\mathrm{o}} = \{(x,y) \in [0,1] \times \mathbb{R}_+ : 0 \le x \le 1,\, x \le x + y \le v\}$$

with intensity measure μ^{o}. Furthermore we define $\mathcal{M}^*$ leaving only points of $\mathcal{M}$ such that $T_i + D_{i1} > v$, that is, we obtain a Poisson measure with points in

$$\Delta_v^* = \{(x,y) \in [0,1] \times \mathbb{R}_+ : 0 \le x \le 1,\, x + y > v\}$$

with intensity measure μ^*. We denote points in the process $\mathcal{M}^{\mathrm{o}}$ by $(T_i^{\mathrm{o}}, D_i^{\mathrm{o}})$, $i = 1, \ldots, N^{\mathrm{o}}(1)$, and those in $\mathcal{M}^*$ by $(T_i^*, D_i^*), i = 1, \ldots, N^*(1)$.

Note that points in $\mathcal{M}^{\mathrm{o}}$ can be obtained by drawing first $N^{\mathrm{o}}(1)$, which is Poissonian with mean $\bar{a}^{\mathrm{o}}$, and then drawing independently points $(T_i^{\mathrm{o}}, D_i^{\mathrm{o}})$ with p.d.f.

$$f^{\mathrm{o}}(dx, dy) = \begin{cases} \dfrac{a(x)\, \mathbb{P}(x + D_1 \in dy)\, dx}{\int_{\Delta_v^{\mathrm{o}}} a(x)\, \mathbb{P}(x + D_1 \in dy)\, dx}, & (x,y) \in \Delta_v^{\mathrm{o}}, \\ 0, & (x,y) \notin \Delta_v^{\mathrm{o}}, \end{cases}$$
$$= \begin{cases} \dfrac{a(x) F_D(dy - x)\, dx}{\bar{a}^{\mathrm{o}}}, & (x,y) \in \Delta_v^{\mathrm{o}}, \\ 0, & (x,y) \notin \Delta_v^{\mathrm{o}}. \end{cases}$$

Thus e.g.

$$T_i^{\mathrm{o}} \sim \frac{a(t) F_D(v - t)}{\int_0^1 a(w) F_D(v - w)\, dw} = \frac{a(t) H(v - t)}{\bar{a}^{\mathrm{o}}}.$$

Similarly points in $\mathcal{M}^*$ can be obtained by drawing first $N^*(1)$, which is Poissonian with mean $\bar{a}^*$, and then drawing independently points (T_i^*, D_i^*) with p.d.f.

$$f^*(dx, dy) = \begin{cases} \dfrac{a(x)\, \mathbb{P}(x + D_1 \in dy, D_1 > v - x)\, dx}{\int_{\Delta_v^*} a(x)\, \mathbb{P}(x + D_1 \in dy)\, dx}, & (x,y) \in \Delta_v^*, \\ 0, & (x,y) \notin \Delta_v^*, \end{cases}$$
$$= \begin{cases} \dfrac{a(x)(F_D(dy - x) - F_D(v - x))\, dx}{\bar{a}^*}, & (x,y) \in \Delta_v^{\mathrm{o}} d, \\ 0, & (x,y) \notin \Delta_v^*. \end{cases}$$

We are now ready to derive another representation for the processes S^{o} and S^*. We denote by Π_i independent copies, $\Pi_i =_{\mathrm{d}} \Pi$. Then

$$S^{\mathrm{o}}(v + s) =_{\mathrm{d}} \sum_{i=1}^{N^{\mathrm{o}}(1)} \Big(\tilde{C}_{i1} + \sum_{j=1}^{\Pi_i(\Lambda_{D_i^{\mathrm{o}}}(v+s-T_i^{\mathrm{o}}))} C_{ij} \Big),$$

$$S^{\mathrm{o}}(v, v + s] =_{\mathrm{d}} \sum_{i=1}^{N^{\mathrm{o}}(1)} \sum_{j=1}^{\Pi_i(\Lambda_{D_{i1}^{\mathrm{o}}}(v-T_i^{\mathrm{o}}, v+s-T_i^{\mathrm{o}}])} C_{ij},$$

where $C_{ij}, \tilde{C}_{ij}$ are i.i.d. with distribution of C. Furthermore

$$S^*(v, v+s] = S^*(v+s) =_{\mathrm{d}} \sum_{i=1}^{N^*(1)} \Big(\tilde{C}_{i1} \mathbb{1}(T_i^* + D_i^* \le v+s) + \sum_{j=1}^{\Pi_i(\Lambda_{D_i^*}(v+s-T_i^*))} \tilde{\tilde{C}}_{ij}\Big), \tag{5.3}$$

where $\tilde{C}_{ij}, \tilde{\tilde{C}}_{ij}$ are i.i.d. with distribution of C.

Using the representation given in (5.3) we can evaluate the IBNR reserve. For this note that $S^*(v, v+s]$ is independent of $S(v) = S^{\mathrm{o}}(v)$. The following proposition gives the form of the IBNR reserve.

Proposition 4.

$$\mathbb{E}[S^*(v, v+s] \,|\, S(v) = k] = \mathbb{E}\, C\Big(\int_0^1 a(x)(F_D(v+s-x) - F_D(v-x))\,dx + \int_0^1 a(x) \int_{v-x}^{v-x+s} \int_y^{v-x+s} b(v)\,dv\,dF_D(y)\,dx\Big).$$

Proof. It is clear that

$$\mathbb{E}[S^*(v, v+s] \,|\, S(v) = k] = \mathbb{E}[S^*(v, v+s]].$$

Now

$$\begin{aligned}
&\mathbb{E}[S^*(v, v+s]] \\
&= \mathbb{E}\Big[\sum_{i=1}^{N^*(1)} \Big(\tilde{C}_{i1} \mathbb{1}(T_i^* + D_i^* \le v+s) + \sum_{j=1}^{\Pi_i(\Lambda_{D_i^*}(v+s-T_i^*))} \tilde{\tilde{C}}_{ij}\Big)\Big] \\
&= \mathbb{E}\, C \cdot \mathbb{E}\Big[\sum_{i=1}^{N^*(1)} \mathbb{1}(T_i^* + D_i^* \le v+s) + \sum_{i=1}^{N^*(1)} \Pi_i(\Lambda_{D_i^*}(v+s-T_i^*))\Big] \\
&= \mathbb{E}\, C \cdot \mathbb{E}\, N^*(1) \cdot \big(\mathbb{P}(T^* + D^* \le v+s) + \mathbb{E}[\Lambda_{D_i^*}(v+s-T_i^*)]\big) \\
&= \mathbb{E}\, C\, \bar{a}^* \Big(\frac{1}{\bar{a}^*} \int_0^1 a(x)(F_D(v+s-x) - F_D(v-x))\,dx. \\
&\qquad + \frac{1}{\bar{a}^*} \int_0^1 dx\, a(x) \int_{v-x}^{\infty} (\Lambda_y(v+s-x))\, \mathbb{P}(D > v-x, D \in dy)\Big) \\
&= \mathbb{E}\, C \Big(\int_0^1 a(x)(F_D(v+s-x) - F_D(v-x))\,dx \\
&\qquad + \int_0^1 dx\, a(x) \int_{v-x}^{v-x+s} \int_y^{v-x+s} b(v)\,dv\,dF_D(y)\Big). \qquad \blacksquare
\end{aligned}$$

For claims that have been reported before time v we get an analogous formula to that in Corollary 1. We define

$$p_k^{\mathrm{o}}(n,t_1,d_1,\ldots,t_n,d_n) = \mathbb{P}\Big(\sum_{i=1}^{n}\Big(C_{i1} + \sum_{j=1}^{\Pi_i(\Lambda_{d_i}(v-t_i))} \tilde{C}_{ij}\Big)\Big),$$

where $C_{ij}, \tilde{C}_{ij}$ are i.i.d. with distribution of C.

Proposition 5.

$$\begin{aligned}&\mathbb{E}[S^{\mathrm{o}}(v,v+s] \mid S^{\mathrm{o}}(v)=k] \\ &\quad = \mathbb{E}\,C_{ij}\cdot\frac{\mathbb{E}[N^{\mathrm{o}}(1)\ g(T_1^{\mathrm{o}})\ p_k^{\mathrm{o}}(N^{\mathrm{o}}(1),T_1^{\mathrm{o}},D_1^{\mathrm{o}}\ldots,T_{N^{\mathrm{o}}(1)}^{\mathrm{o}},D_{N^{\mathrm{o}}(1)}^{\mathrm{o}})]}{\mathbb{E}[p_k^{\mathrm{o}}(N^{\mathrm{o}}(1),T_1^{\mathrm{o}},D_1^{\mathrm{o}}\ldots,T_{N^{\mathrm{o}}(1)}^{\mathrm{o}},D_{N^{\mathrm{o}}(1)}^{\mathrm{o}})]}.\end{aligned}$$

Proof. First notice that

$$\Lambda_{D^{\mathrm{o}}}(v-T^{\mathrm{o}},v+s-T^{\mathrm{o}}] = \Lambda_{D^{\mathrm{o}}}(v+s-T^{\mathrm{o}}) - \Lambda_{D^{\mathrm{o}}}(v-T^{\mathrm{o}}) = g(T_i^{\mathrm{o}}),$$

where $g(x)$ is defined in (3.1). Now for the numerator,

$$\begin{aligned}&\mathbb{E}[S^{\mathrm{o}}(v,v+s];\ S^{\mathrm{o}}(v)=k] \\ &= \mathbb{E}\,C_{11}\cdot\sum_{n=1}^{\infty}\mathbb{P}(N^{\mathrm{o}}(1)=n)\,\mathbb{E}\Big[\sum_{i=1}^{n}\tilde{\Pi}_i(g(T_i^{\mathrm{o}}));\ \sum_{i=1}^{n}\Big(C_{i1}+\sum_{j=1}^{\Pi_i(\Lambda_{D_i^{\mathrm{o}}}(v-T_i^{\mathrm{o}}))}\tilde{C}_{ij}\Big)\Big] \\ &= \mathbb{E}\,C_{11}\cdot\sum_{n=1}^{\infty}\mathbb{P}(N^{\mathrm{o}}(1)=n)\sum_{i=1}^{n}\mathbb{E}\Big[\tilde{\Pi}_i(g(T_i^{\mathrm{o}}));\ \sum_{i=1}^{n}\Big(C_{i1}+\sum_{j=1}^{\Pi_i(\Lambda_{D_i^{\mathrm{o}}}(v-T_i^{\mathrm{o}}))}\tilde{C}_{ij}\Big)\Big] \\ &= \mathbb{E}\,C_{11}\cdot\sum_{n=1}^{\infty}\mathbb{P}(N^{\mathrm{o}}(1)=n)n\,\mathbb{E}\Big[\tilde{\Pi}_i(g(T_i^{\mathrm{o}}));\ \sum_{i=1}^{n}\Big(C_{i1}+\sum_{j=1}^{\Pi_i(\Lambda_{D_i^{\mathrm{o}}}(v-T_i^{\mathrm{o}}))}\tilde{C}_{ij}\Big)\Big] \\ &= \mathbb{E}\,C_{11} \\ &\quad\times\sum_{n=1}^{\infty}\mathbb{P}(N^{\mathrm{o}}(1)=n)n\int_0^1 g(t_1)p_k^{\mathrm{o}}(n,t_1,D_1^{\mathrm{o}}\ldots,T_{N^{\mathrm{o}}(1)}^{\mathrm{o}},D_{N^{\mathrm{o}}(1)}^{\mathrm{o}})f_{T^{\mathrm{o}}}(t_1)\,dt_1 \\ &= \mathbb{E}\,C_{11}\,\mathbb{E}[N^{\mathrm{o}}(1)\ g(T_1^{\mathrm{o}})\ p_k^{\mathrm{o}}(N^{\mathrm{o}}(1),T_1^{\mathrm{o}},D_1^{\mathrm{o}}\ldots,T_{N^{\mathrm{o}}(1)}^{\mathrm{o}},D_{N^{\mathrm{o}}(1)}^{\mathrm{o}})].\end{aligned}$$

Setting $g(x)\equiv 1$ and $N^{\mathrm{o}}(1)\equiv 1$ gives the formula for the denominator, which completes the proof. ∎

Appendix. We recall here the basic facts about Poisson random measures (see e.g. [2, Chapter VI] or [3]). Let $(E,\mathcal{E},\mu)$ be a measure space with finite measure μ. Then N is a *Poisson random measure* on E if

(1) for $B \in \mathcal{E}$, the r.v. $N(B)$ is Poisson distributed with mean $\mu(B)$,
(2) for disjoint $B_1, \ldots, B_n \in \mathcal{E}$, the r.v.s $N(B_1), \ldots, N(B_n)$ are independent.

Then N has the following representation:

$$N = \sum_{j=1}^{\Pi} \delta_{T_j},$$

where $\Pi, T_1, T_2, \ldots$ are independent, Π is Poisson with mean $\mu(E)$ and $T_1, T_2, \ldots$ are i.i.d. with distribution $\mu(dx)/\mu(E)$.

We now consider an approach to Poisson measures via marked Poisson processes. Suppose that N is a Poisson process on (a, b) with intensity measure $A(dx)$ and suppose $A(a, b) < \infty$. Thus

$$N = \sum_{j=1}^{\Pi} \delta_{T_j}.$$

The space $\mathbb{K}$ of marks is endowed with a σ-field $\mathcal{K}$. To each point T_i of N one attaches a mark $K_i \in \mathcal{K}$. It is a random element with distribution $\nu_t(dk)$ depending on the position of the point $T = t$ only. Hence the marked point process or random measure

$$M = \sum_{j=1}^{\Pi} \delta_{(T_j, K_i)}$$

is a Poisson random measure on $E = (a, b) \times \mathbb{K}$ with intensity measure $\mu(dt \times dk) = A(dt)\nu_t(dk)$. Conversely, if M is a Poisson random measure on $E = (a, b) \times \mathbb{K}$ with intensity measure represented by $\mu(dt \times dk) = A(dt)\nu_t(dk)$, where $\nu_t(K)$ is a stochastic kernel, then M can be thought of as a marked point process.

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