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STRONG UNIFORM CONSISTENCY RATES OF SOME CHARACTERISTICS OF THE CONDITIONAL DISTRIBUTION ESTIMATOR IN THE FUNCTIONAL SINGLE-INDEX MODEL

Abstract. The aim of this paper is to establish a nonparametric estimate of some characteristics of the conditional distribution. Kernel type estimators for the conditional cumulative distribution function and for the successive derivatives of the conditional density of a scalar response variable Y given a Hilbertian random variable X are introduced when the observations are linked with a single-index structure. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimator of this model. Asymptotic properties are stated for each of these estimators, and they are applied to the estimation of the conditional mode and conditional quantiles.

1. Introduction. Single-index models are becoming increasingly popular, and have received considerable attention recently because of their importance in several areas of science, including econometrics, biostatistics, medicine and financial econometrics. The single-index model, a special case of projection pursuit regression, has proven to be an efficient way of coping with high dimensional problems in nonparametric regression. In the present work we study single-index modeling in the case of a functional explanatory variable. More precisely, we consider the problem of estimating some

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characteristics of the conditional distribution of a real variable Y given a functional variable X when the explanation of Y given X is done through its projection on one functional direction. The conditional density plays an important role in nonparametric prediction, because several prediction tools in nonparametric statistic, such as the conditional mode, conditional median or conditional quantiles, are based on the preliminary estimate of this functional parameter.

Nonparametric estimation of the conditional density has been widely studied when the data is real. The first related result in nonparametric functional statistic was obtained by Ferraty et al. [15]: the almost complete consistency in the independent and identically distributed (i.i.d.) random variables of the kernel estimator of the conditional probability density was established. The asymptotic normality of this kernel estimator has been studied for dependent data by Ezzahrioui and Ould Saïd [10]. The single-index approach is widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. Such kind of modeling is intensively studied in the multivariate case. We cite for example Härdle et al. [18] and Hristache et al. [19]. Based on the regression function, Delecroix et al. [7] studied the estimation of the single-index model and established some asymptotic properties. The literature is rather limited in the case where the explanatory variable is functional (that is, a curve). The first asymptotic properties in the fixed functional single-index model were obtained by Ferraty et al. [12], who established the almost complete convergence, in the i.i.d. case, of the link regression function of this model. Their results were extended to the dependent case by Aït Saidi et al. [1]. Aït Saidi et al. [2] studied the case where the functional single-index is unknown. They proposed an estimator of this parameter, based on the cross-validation procedure.

The goal of this paper is to establish a nonparametric estimate of some characteristics of the conditional distribution where kernel type estimators for the conditional cumulative distribution function and the successive derivatives of the conditional density in the functional single-index model are introduced. We establish the pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimator of this model. Asymptotic properties are stated for each of these estimators, and they are applied to the estimation of the conditional mode and conditional quantiles.

Now, let us outline the paper. In Section 2, we present general notation and some conditions necessary for our study. In Sections 3 we propose estimators of the conditional cumulative distribution function and of the conditional density derivatives, and we prove their pointwise almost complete convergence (with rate). In Section 4, we study the uniform almost complete convergence of the conditional cumulative distribution function (resp. the conditional density derivatives) estimator given in Section 3. Section 5 is devoted to some applications; we first consider the problem of estimation of the conditional mode in a functional single-index model, then we investigate the asymptotic properties of the conditional quantile function of a scalar response and functional covariate when the observations are in a functional single-index model and data are i.i.d. We finish our paper by giving the technical proofs of the lemmas and the the corollary (Appendix).

2. General notation and conditions. All along the paper, when no confusion is possible, we will denote by C, C' or/and $C_{\theta,x}$ some generic constants in $\mathbb{R}_+^*$; any real function with an integer in brackets as exponent denotes its derivative of the corresponding order.

Let $(X_i, Y_i)_{1\le i\le n}$ be n random variables, identically distributed as the random pair (X, Y) with values in $\mathcal{H}\times\mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$. We consider the semimetric d_θ associated to a single index $\theta\in\mathcal{H}$, defined by $d_\theta(x_1, x_2) := |\langle x_1 - x_2, \theta\rangle|$ for $x_1, x_2 \in \mathcal{H}$; our aim is to build nonparametric estimates of several functions related to the conditional cumulative distribution function (cond-cdf) of Y given $\langle X, \theta\rangle = \langle x, \theta\rangle$ for some $x \in \mathcal{H}$:

$$\forall y \in \mathbb{R}, \qquad F(\theta, y, x) = \mathbb{P}(Y \le y \,|\, \langle X, \theta\rangle = \langle x, \theta\rangle),$$

which also shows the relationship between X and Y but is often unknown.

If this distribution is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, then we will denote by $f(\theta, \cdot, x)$ (resp. $f^{(j)}(\theta, \cdot, x)$) the conditional density of Y given $\langle X, \theta\rangle = \langle x, \theta\rangle$ (resp. its jth order derivative). In Sections 3 and 4, we will give almost complete convergence ([1]) results (with rates of convergence ([2])) for nonparametric estimates of both $F(\theta, \cdot, x)$ and $f^{(j)}(\theta, \cdot, x)$.

In the following, for any $x \in \mathcal{H}$ and $y \in \mathbb{R}$, let $\mathcal{N}_x$ be a fixed neighborhood of x in $\mathcal{H}$, $\mathcal{S}_\mathbb{R}$ will be a fixed compact subset of $\mathbb{R}$, and we will use the notation $B_\theta(x, h) = \{X \in \mathcal{H} : 0 < |\langle x - X, \theta\rangle| < h\}$. Our nonparametric models will be quite general in the sense that we will just need the following simple assumption for the marginal distribution of $\langle\theta, X\rangle$:

([1]) Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to *converge almost completely* to some variable T if $\sum_n \mathbb{P}(|T_n - T| > \epsilon) < \infty$ for any $\epsilon > 0$. This mode of convergence implies both almost sure convergence and convergence in probability (see for instance Bosq and Lecoutre [5]).

([2]) Recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of random variables is said to be of *order of almost complete convergence* u_n if there exists some $\epsilon > 0$ for which $\sum_n \mathbb{P}(|T_n| > \epsilon u_n) < \infty$. This is denoted by $T_n = \mathcal{O}(u_n)$, a.co. (or $T_n = \mathcal{O}_{\text{a.co.}}(u_n)$).

(H1) $\mathbb{P}\left(X \in B_\theta(x,h)\right) =: \phi_{\theta,x}(h) > 0$, $\phi_{\theta,x}(h) \to 0$, as $h \to 0$

together with some usual smoothness conditions on the function to be estimated. According to the type of estimation problem to be considered, we will assume that for some $b_1, b_2 > 0$ either

(H2) $\forall (y_1,y_2) \in \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}$, $\forall (x_1,x_2) \in \mathcal{N}_x \times \mathcal{N}_x$,
$$|F(\theta,y_1,x_1) - F(\theta,y_2,x_2)| \le C_{\theta,x}(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}),$$

or

(H3) $\forall (y_1,y_2) \in \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}$, $\forall (x_1,x_2) \in \mathcal{N}_x \times \mathcal{N}_x$,
$$|f^{(j)}(\theta,y_1,x_1) - f^{(j)}(\theta,y_2,x_2)| \le C_{\theta,x}(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}).$$

3. Pointwise almost complete estimation. In this section we give the pointwise almost complete estimate (with rate) of the conditional cumulative distribution and of the successive derivatives of the conditional density.

3.1. Conditional cumulative distribution estimation. The purpose of this section is to estimate the cond-cdf $F^x(\theta,\cdot,x)$. We introduce a kernel type estimator $\widehat{F}^x(\theta,\cdot,x)$ of $F^x(\theta,\cdot,x)$ as follows:

$$\widehat{F}(\theta,y,x) = \frac{\sum_{i=1}^n K(h_K^{-1}(\langle x - X_i, \theta\rangle))H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1}(\langle x - X_i, \theta\rangle))}, \tag{3.1}$$

where K is a kernel, H is a cumulative distribution function (cdf) and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) is a sequence of positive real numbers which tends to zero as n tends to infinity, and with the convention $0/0 = 0$. Note that a similar estimator was already introduced by Ferraty et al. [11] in the case where X is valued in some semimetric space which can be of infinite dimension. In our functional single-index context, we need the following conditions for our estimate:

(H4) $|H(y_1) - H(y_2)| \le C|y_1 - y_2|$ for all $(y_1,y_2) \in \mathbb{R}^2$ and $\int |t|^{b_2} H^{(1)}(t)\,dt < \infty$,

(H5) K is a positive bounded function with support $[-1,1]$,

(H6) $\lim_{n\to\infty} h_K = 0$ with $\lim_{n\to\infty} \dfrac{\log n}{n\phi_{\theta,x}(h_K)} = 0$,

(H7) $\lim_{n\to\infty} h_H = 0$ with $\lim_{n\to\infty} n^\alpha h_H = \infty$ for some $\alpha > 0$.

Comments on the assumptions. Our assumptions are rather standard for this kind of model. Assumptions (H1) and (H5) are the same as those given in Ferraty et al. [12]. Assumptions (H2) and (H3) are regularity conditions which characterize the functional space of our model and are needed to evaluate the bias term of our asymptotic results. Assumptions (H4) and

(H6)–(H7) are technical conditions and are also similar to those in Ferraty et al. [15].

THEOREM 3.1. *Under Assumptions* (H1), (H2) *and* (H4)–(H7), *for any fixed* y,

$$|\widehat{F}(\theta, y, x) - F(\theta, y, x)| = \mathcal{O}(h_K^{b_1} + h_H^{b_2}) + \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right), \tag{3.2}$$

Proof. For $i = 1, \dots, n$, we consider the quantities

$$K_i(\theta, x) := K(h_K^{-1}(\langle x - X_i, \theta\rangle))$$

and, for all $y \in \mathbb{R}$,

$$H_i(y) = H(h_H^{-1}(y - Y_i)),$$

and define

$$\widehat{F}_N(\theta, y, x) = \frac{1}{n\,\mathbb{E}\,K_1(\theta, x)} \sum_{i=1}^{n} K_i(\theta, x) H_i(y),$$

$$\widehat{F}_D(\theta, x) = \frac{1}{n\,\mathbb{E}\,K_1(\theta, x)} \sum_{i=1}^{n} K_i(\theta, x).$$

The proof is based on the decomposition

$$\begin{aligned} \widehat{F}(\theta, y, x) - F(\theta, y, x) = {} & \frac{1}{\widehat{F}_D(\theta, x)}(\widehat{F}_N(\theta, y, x) - \mathbb{E}\,\widehat{F}_N(\theta, y, x)) \\ & - \frac{1}{\widehat{F}_D(\theta, x)}(F(\theta, y, x) - \mathbb{E}\,\widehat{F}_N(\theta, y, x)) \\ & + \frac{F(\theta, y, x)}{\widehat{F}_D(\theta, x)}(1 - \widehat{F}_D(\theta, x)) \end{aligned} \tag{3.3}$$

and on the following intermediate results.

LEMMA 3.2 ([1]). *Under Assumptions* (H1) *and* (H5)–(H6),

$$|\widehat{F}_D(\theta, x) - 1| = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\right). \tag{3.4}$$

COROLLARY 3.3. *Under the hypotheses of Lemma* 3.2,

$$\sum_{n=1}^{\infty} \mathbb{P}\big(|\widehat{F}_D(\theta, x)| \le 1/2\big) < \infty. \tag{3.5}$$

LEMMA 3.4. *Under Assumptions* (H1), (H2) *and* (H4)–(H6),

$$|F(\theta, y, x) - \mathbb{E}\,\widehat{F}_N(\theta, y, x)| = \mathcal{O}(h_K^{b_1} + h_H^{b_2}). \tag{3.6}$$

Lemma 3.5. *Under Assumptions* (H1), (H2) *and* (H4)–(H7),

$$(3.7)\qquad |\widehat{F}_N(\theta,y,x)-\mathbb{E}\,\widehat{F}_N(\theta,y,x)| = \mathcal{O}_{\text{a.co.}}\bigg(\sqrt{\frac{\log n}{n\phi_{\theta,x}(h_K)}}\bigg),$$

3.2. Estimating successive derivatives of the conditional density. The main objective of this part is the estimation of the successive derivatives of the conditional density of Y given $\langle X,\theta\rangle=\langle x,\theta\rangle$, denoted by $f(\theta,\cdot,x)$. It is well known that, in nonparametric statistics, this provides an alternative approach to studying the links between Y and X and it can also be used, in single-index modeling, to estimate the functional index θ if it is unknown.

So, we propose an estimator $\widehat{f}^{(j)}(\theta,y,x)$ of $f^{(j)}(\theta,y,x)$ as follows:

$$(3.8)\quad \widehat{f}^{(j)}(\theta,y,x)=\frac{\sum_{i=1}^{n}K(h_K^{-1}(\langle x-X_i,\theta\rangle))H^{(j+1)}(h_H^{-1}(y-Y_i))}{h_H^{j+1}\sum_{i=1}^{n}K(h_K^{-1}(\langle x-X_i,\theta\rangle))},\ y\in\mathbb{R}.$$

A similar estimator was already introduced by Ferraty et al. [11] in the case where X is valued in some semimetric space which can be of infinite dimension, and then widely studied (see for instance Attaoui et al. [3] for several asymptotic results and references). In addition to the conditions introduced along the previous section, we need the following ones, which are technical conditions and are also similar to those given in Ferraty et al. [15]:

$$\text{(H8)}\ \begin{cases}\forall(y_1,y_2)\in\mathbb{R}^2,\ |H^{(j+1)}(y_1)-H^{(j+1)}(y_2)|\le C_{\theta,x}|y_1-y_2|,\\ \exists\nu>0,\ \forall j'\le j+1,\ \lim_{y\to\infty}|y|^{1+\nu}|H^{(j'+1)}(y)|=0.\end{cases}$$

$$\text{(H9)}\ \lim_{n\to\infty}h_K=0 \text{ with } \lim_{n\to\infty}\frac{\log n}{nh_H^{2j+1}\phi_{\theta,x}(h_K)}=0.$$

The next result concerns the asymptotic behavior of the kernel functional estimator $\widehat{f}^{(j)}(\theta,\cdot,x)$ of the jth order derivative of the conditional density function.

Theorem 3.6. *Under Assumptions* (H1), (H3)–(H5), *and* (H7)–(H9), *and for any fixed* y, *we have, as* $n\to\infty$,

$$(3.9)\qquad |\widehat{f}^{(j)}(\theta,y,x)-f^{(j)}(\theta,y,x)| = \mathcal{O}(h_K^{b_1}+h_H^{b_2})+\mathcal{O}_{\text{a.co.}}\bigg(\sqrt{\frac{\log n}{nh_H^{2j+1}\phi_{\theta,x}(h_K)}}\bigg).$$

Proof. This result is based on the same kind of decomposition as (3.3).

Indeed, we can write

$$\widehat{f}^{(j)}(\theta,y,x)-f^{(j)}(\theta,y,x) = \frac{1}{\widehat{F}_D(\theta,x)}(\widehat{f}_N^{(j)}(\theta,y,x) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)) - \frac{1}{\widehat{F}_D(\theta,x)}(f^{(j)}(\theta,y,x) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)) + \frac{f^{(j)}(\theta,y,x)}{\widehat{F}_D(\theta,x)}(1-\widehat{F}_D(\theta,x)) \tag{3.10}$$

where

$$\widehat{f}_N^{(j)}(\theta,y,x) = \frac{1}{n\,h_H^{j+1}\,\mathbb{E}\,K_1(\theta,x)}\sum_{i=1}^{n} K_i(\theta,x)H_i^{(j+1)}(y).$$

Then Theorem 3.6 can be deduced from the two lemmas below, together with Lemma 3.2 and Corollary 3.3.

Lemma 3.7. *Under Assumptions* (H1)–(H3), (H5) *and* (H6),

$$|f^{(j)}(\theta,y,x) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)| = \mathcal{O}(h_K^{b_1}+h_H^{b_2}).$$

Lemma 3.8. *Under Assumptions* (H1)–(H7),

$$|\widehat{f}_N^{(j)}(\theta,y,x) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)| = \mathcal{O}_{\text{a.co.}}\bigg(\sqrt{\frac{\log n}{nh_H^{2j+1}\phi_{\theta,x}(h_K)}}\bigg).$$

The proofs of the above lemmas and corollary are done in the same manner as in [11], since Lemmas 3.4, 3.5, 3.7 and 3.8, and Corollary 3.3, are special cases of Lemmas 2.3.2, 2.3.3, 2.3.4 and 2.3.5 and Corollary 2.3.1 of [12]. It suffices to replace $\widehat{f}^{(j)}(y,x)$ (resp. $f^{(j)}(y,x)$) by $\widehat{f}^{(j)}(\theta,y,x)$ (resp. $f^{(j)}(\theta,y,x)$), and $\widehat{F}_D(x)$ (resp. $F_D(x)$) by $\widehat{F}_D(\theta,x)$ (resp. $F_D(\theta,x)$) with $d(x_1,x_2) = \langle x_1-x_2,\theta\rangle$. ■

4. Uniform almost complete convergence. In this section we derive the uniform versions of Theorems 3.1 and 3.6. The study of uniform consistency is motivated by the fact that the latter is an indispensable tool for studying the asymptotic properties of all estimates of the functional index if it is unknown. In the multivariate case, uniform consistency is a standard extension of pointwise consistency: however, in our functional case, it requires some additional tools and topological conditions (see Ferraty and Vieu [15] for more discussion on uniform convergence in nonparametric functional statistics). Thus, in addition to the conditions introduced previously, we need the following ones. Firstly, consider

$$\mathcal{S}_{\mathcal{H}} \subset \bigcup_{k=1}^{d_n^{\mathcal{S}_{\mathcal{H}}}} B(x_k,r_n) \quad \text{and} \quad \Theta_{\mathcal{H}} \subset \bigcup_{j=1}^{d_n^{\Theta_{\mathcal{H}}}} B(t_j,r_n) \tag{4.1}$$

where x_k, $t_j \in \mathcal{H}$ and $r_n, d_n^{\mathcal{S}_\mathcal{H}}, d_n^{\Theta_\mathcal{H}}$ are sequences of positive real numbers which tend to infinity as $n \to \infty$.

4.1. Conditional cumulative distribution estimation. In this section we propose to study the uniform almost complete convergence of our estimator (3.1). For this, we need the following assumptions:

(A1) There exists a differentiable function $\phi(\cdot)$ such that for all $x \in \mathcal{S}_\mathcal{H}$ and for all $\theta \in \Theta_\mathcal{H}$,

$$0 < C\phi(h) \le \phi_{\theta,x}(h) \le C'\phi(h) < \infty$$

and there exists $\eta_0 > 0$ such that $\phi'(\eta) < C$ for all $\eta < \eta_0$.

(A2) $\forall (y_1, y_2) \in \mathcal{S}_\mathbb{R} \times \mathcal{S}_\mathbb{R}, \forall (x_1, x_2) \in \mathcal{S}_\mathcal{H} \times \mathcal{S}_\mathcal{H}, \forall \theta \in \Theta_\mathcal{H}$,

$$|F(\theta, y_1, x_1) - F(\theta, y_2, x_2)| \le C_\theta(\|x_1 - x_2\|^{b_1} + |y_1 - y_2|^{b_2}).$$

(A3) The kernel K satisfies (H3) and the Lipschitz condition

$$|K(x) - K(y)| \le C\|x - y\|.$$

(A4) For $r_n = \mathcal{O}((\log n)/n)$ the sequences $d_n^{\mathcal{S}_\mathcal{H}}$ and $d_n^{\Theta_\mathcal{H}}$ satisfy

$$\frac{(\log n)^2}{n\phi(h_K)} < \log d_n^{\mathcal{S}_\mathcal{H}} + \log d_n^{\Theta_\mathcal{H}} < \frac{n\phi(h_K)}{\log n},$$

$$\sum_{n=1}^{\infty} n^{1/2b_2} (d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}})^{1-\beta} < \infty \quad \text{for some } \beta > 1.$$

Remark 4.1. Note that Assumptions (A1) and (A2) are, respectively, the uniform versions of (H1) and (H2). Assumptions (A1) and (A4) are linked with the topological structure of the functional variable (see Ferraty et al. [13]).

Theorem 4.2. *Under Assumptions* (A1)–(A4) *and* (H4), *as* $n \to \infty$,

$$\sup_{\theta \in \Theta_\mathcal{H}} \sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{y \in \mathcal{S}_\mathbb{R}} |\widehat{F}(\theta, y, x) - F(\theta, y, x)| = \mathcal{O}(h_K^{b_1} + h_H^{b_2}) + \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_\mathcal{H}} d_n^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right). \tag{4.2}$$

In the particular case where the functional single-index is fixed we get the following result.

Corollary 4.3. *Under Assumptions* (A1)–(A4) *and* (H4), *as* $n \to \infty$,

$$\sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{y \in \mathcal{S}_\mathbb{R}} |\widehat{F}(\theta, y, x) - F(\theta, y, x)| = \mathcal{O}(h_K^{b_1} + h_H^{b_2}) + \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log d_n^{\mathcal{S}_\mathcal{H}}}{n\phi(h_K)}}\right). \tag{4.3}$$

Clearly, Theorem 3.5 and Corollary 4.3 can be deduced from the following intermediate results which are just uniform versions of Lemmas 3.2–3.5 and Corollary 3.3.

LEMMA 4.4. *Under Assumptions* (A1), (A3) *and* (A4), *as* $n \to \infty$,

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}|\widehat{F}_D(\theta,x)-1| = \mathcal{O}_{\text{a.co.}}\Bigg(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg).$$

COROLLARY 4.5. *Under the assumptions of Lemma* 4.4,

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in\mathcal{S}_{\mathcal{H}}}\widehat{F}_D(\theta,x) < 1/2\Big) < \infty.$$

LEMMA 4.6. *Under Assumptions* (A1), (A2) *and* (H4), *as* $n \to \infty$,

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|F(\theta,y,x)-\mathbb{E}\,\widehat{F}_N(\theta,y,x)| = \mathcal{O}(h_K^{b_1}+h_H^{b_2}). \tag{4.4}$$

LEMMA 4.7. *Under the assumptions of Theorem* 3.5, *as* $n \to \infty$,

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{F}_N(\theta,y,x)-\mathbb{E}\,\widehat{F}_N(\theta,y,x)| = \mathcal{O}_{\text{a.co.}}\Bigg(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg).$$

4.2. Estimating successive derivatives of the conditional density. In this part we focus on the study of uniform almost complete convergence of our estimator (3.8). In addition to the conditions introduced in Section 4, we need the following ones:

(A5) $\forall(y_1,y_2)\in\mathcal{S}_{\mathbb{R}}\times\mathcal{S}_{\mathbb{R}},\ \forall(x_1,x_2)\in\mathcal{S}_{\mathcal{H}}\times\mathcal{S}_{\mathcal{H}},\ \forall\theta\in\Theta_{\mathcal{H}}$,
$$|f^{(j)}(\theta,y_1,x_1)-f^{(j)}(\theta,y_2,x_2)| \le C_\theta(\|x_1-x_2\|^{b_1}+|y_1-y_2|^{b_2}).$$

(A6) For some $\gamma\in(0,1)$, $\lim_{n\to\infty} n^\gamma h_H=\infty$, and for $r_n=\mathcal{O}((\log n)/n)$ the sequences $d_n^{\mathcal{S}_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy
$$\frac{(\log n)^2}{nh_H^{2j+1}\phi(h_K)} < \log d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \frac{nh_H^{2j+1}\phi(h_K)}{\log n},$$
$$\sum_{n=1}^{\infty} n^{(3\gamma+1)/2}(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})^{1-\beta} < \infty \quad \text{for some } \beta>1.$$

THEOREM 4.8. *Under Assumptions* (A1), (A3), (A5)–(A6) *and* (H8), *as* $n \to \infty$,

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{f}^{(j)}(\theta,y,x)-f^{(j)}(\theta,y,x)| = \mathcal{O}(h_K^{b_1}+h_H^{b_2})+\mathcal{O}_{\text{a.co.}}\Bigg(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\Bigg). \tag{4.5}$$

Proof. This result is based on the same kind of decomposition as in (3.10), so it can be deduced from the two lemmas below, together with Lemma 4.4 and Corollary 4.5.

LEMMA 4.9. *Under Assumptions* (A1), (A5) *and* (H8), *as* $n \to \infty$,

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|f^{(j)}(\theta,y,x)-\mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)|=\mathcal{O}(h_K^{b_1}+h_H^{b_2}).$$

LEMMA 4.10. *Under the assumptions of Theorem* 4.8, *as* $n \to \infty$,

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\widehat{f}_N^{(j)}(\theta,y,x)-\mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)|=\mathcal{O}_{\text{a.co.}}\biggl(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\biggr).$$

5. Applications

5.1. The conditional mode in a functional single-index model. In this section we will consider the problem of estimation of the conditional mode in a functional single-index model. Our main aim here is to establish the a.co. convergence of the kernel estimator of the conditional mode of Y given $\langle X,\theta\rangle=\langle x,\theta\rangle$, denoted by $M_\theta(x)$, uniformly on a fixed subset $\mathcal{S}_{\mathcal{H}}$ of $\mathcal{H}$. For this, we assume that $M_\theta(x)$ satisfies on $\mathcal{S}_{\mathcal{H}}$ the following uniform uniqueness property (see Ould-Saïd and Cai [24] for the multivariate case):

(A7) $\forall\varepsilon_0>0$, $\exists\eta>0$, $\forall\varphi:\mathcal{S}_{\mathcal{H}}\to\mathcal{S}_{\mathbb{R}}$,

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}}|M_\theta(x)-\varphi(x)|\ge\varepsilon_0\ \Rightarrow\ \sup_{x\in\mathcal{S}_{\mathcal{H}}}|f(\theta,\varphi(x),x)-f(\theta,M_\theta(x),x)|\ge\eta.$$

We estimate the conditional mode $\widehat{M_\theta}(x)$ with a random variable M_θ such that

$$\widehat{M_\theta}(x)=\arg\sup_{y\in\mathcal{S}_{\mathbb{R}}}\widehat{f}(\theta,y,x). \tag{5.1}$$

Note that the estimate $\widehat{M_\theta}$ is not necessarily unique, and if this is the case, all what follows will concern any value $\widehat{M_\theta}$ satisfying (5.1). The difficulty of the problem is naturally linked with the flatness of the function $f(\theta,y,x)$ around the mode M_θ. This flatness can be controlled by the number of vanishing derivatives at M_θ, and this parameter will also have a great influence on the asymptotic rates of our estimates. More precisely, we introduce the following additional smoothness condition:

$$\text{(A8)}\quad\begin{cases} f^{(l)}(\theta,M_\theta(x),x)=0 \text{ if } 1\le l<j,\\ f^{(j)}(\theta,\cdot,x) \text{ is uniformly continuous on } \mathcal{S}_{\mathbb{R}},\\ |f^{(j)}(\theta,\cdot,x)|>C>0.\end{cases}$$

THEOREM 5.1. *Under the assumptions of Theorem* 4.8 *together with* (A7)–(A8),

$$\sup_{x\in\mathcal{S}_{\mathcal{H}}}|\widehat{M_\theta}(x)-M_\theta(x)|=\mathcal{O}(h_K^{b_1/j}+h_H^{b_2/j})+\mathcal{O}_{\text{a.co.}}\biggl(\biggl(\frac{\log d_n^{\mathcal{S}_{\mathcal{H}}}}{n^{1-\gamma}\phi(h_K)}\biggr)^{1/2j}\biggr).$$

From the above theorem, we obtain the following result.

COROLLARY 5.2. *Under the assumptions of Theorem* 5.1, *as* $n \to \infty$,

$$\widehat{M_\theta}(x) - M_\theta(x) \to 0, \qquad a.co.$$

5.2. Conditional quantile in a functional single-index model. In this part of the paper we investigate the asymptotic properties of the conditional quantile function of a scalar response and functional covariate when the observations are from a functional single-index model and data are i.i.d.

Let $(Y_i, X_i)_{1\le i\le n}$ be n random variables, identically distributed as the random pair (Y, X) with values in $\mathbb{R}\times\mathcal{H}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$. We consider the semimetric d_θ associated to a single index $\theta \in \mathcal{H}$, defined by $d_\theta(x_1, x_2) := |\langle x_1 - x_2, \theta\rangle|$ for $x_1, x_2 \in \mathcal{H}$. Under such topological structure and for the fixed functional θ, we suppose that the conditional distribution function of Y given $X = x$, denoted by $F^x(\cdot)$, exists and is given by

$$\forall y \in \mathbb{R}, \qquad F^x_\theta(y) := F(y \,|\, \langle x, \theta\rangle) = F(\theta, y, x).$$

Clearly, the identifiability of the model is ensured, and we have for all $x \in \mathcal{H}$,

$$F_1(\cdot \,|\, \langle x, \theta_1\rangle) = F_2(\cdot \,|\, \langle x, \theta_2\rangle) \;\Rightarrow\; F_1 \equiv F_2 \text{ and } \theta_1 = \theta_2;$$

for more details see Aït Saidi et al. [2].

We will consider the problem of estimation of conditional quantiles. Here, we are implicitly assuming the existence of a regular version for the conditional distribution of Y given $\langle X, \theta\rangle$. Now, let $t_\theta(\alpha)$ be the α-order quantile of the distribution of Y given $\langle X, \theta\rangle = \langle x, \theta\rangle$. From the cond-cdf $F(\theta, \cdot, x)$, it is easy to give the general definition of the α-order quantile:

$$t_\theta(\alpha) = \inf\{t \in \mathbb{R} : F(\theta, t, x) \ge \alpha\}, \qquad \forall \alpha \in (0, 1).$$

In order to simplify our framework and to focus on the main point of interest of this part (the functional feature of $\langle X, \theta\rangle$), we assume that $F(\theta, \cdot, x)$ is strictly increasing and continuous in a neighborhood of $t_\theta(\alpha)$. This ensures uniqueness of the conditional quantile $t_\theta(\alpha)$ which is defined by

$$t_\theta(\alpha) = F^{-1}(\theta, \alpha, x). \tag{5.2}$$

In what remains, we wish to stay in a free distribution framework. This will lead us to make only smoothness restrictions on the cond-cdf $F(\theta, \cdot, x)$ through nonparametric modeling (see Section 2).

As a by-product of (5.2) and (3.1), it is easy to derive an estimator $\widehat{t}_\theta(\alpha)$ of $t_\theta(\alpha)$:

$$\widehat{t}_\theta(\alpha) = \widehat{F}^{-1}(\theta, \alpha, x). \tag{5.3}$$

As we will see later on, such an estimator is unique as soon as H is an increasing continuous function.

Naturally, we will estimate this quantile by means of the conditional distribution estimator studied in the previous sections. Here also, the literature on (conditional and/or unconditional) quantile estimation is quite vast when the explanatory variable X is real (see for instance Samanta [26] for previous results and Berlinet et al. [4] for recent advances and references). In the functional case, the conditional quantiles for scalar response and a scalar/multivariate covariate have received considerable interest in the statistical literature. For completely observed data, several nonparametric approaches have been proposed; for instance, Gannoun et al. [16] introduced a smoothed estimator based on double kernel and local constant kernel methods. Under random censoring, Gannoun et al. [17] introduced a local linear (LL) regression (see Koenker and Bassett [20] for the definition), and El Ghouch and Van Keilegom [8] studied the same LL estimator. Ould-Saïd [23] constructed a kernel estimator of the conditional quantile under i.i.d. censorship model and established its strong uniform convergence rate. Liang and De Uña-Álvarez [22] established the strong uniform convergence (with rate) of the conditional quantile function under the α-mixing assumption.

Recently, many authors are interested in the estimation of conditional quantiles for a scalar response and functional covariate. Ferraty et al. [14] introduced a nonparametric estimator of the conditional quantile defined as the inverse of the conditional cumulative distribution function when the sample is considered as an α-mixing sequence. They stated its rate of almost complete consistency and used it to forecast the well-known El Niño time series and to build confidence prediction bands. Ezzahrioui and Ould-Saïd [9] established the asymptotic normality of the kernel conditional quantile estimator under the α-mixing assumption. Recently, and within the same framework, Dabo-Niang and Laksaci [6] proved the consistency in L^p norm of the conditional quantile estimator for functional dependent data.

In this work we propose to estimate $t_\theta(\alpha)$ by $\widehat{t}_\theta(\alpha)$ defined in (5.3) or in

$$\widehat{F}(\theta, \widehat{t}_\theta(\alpha), x) = \alpha. \tag{5.4}$$

To ensure existence and uniqueness of this quantile, we will assume that

(A9) $F(\theta, \cdot, x)$ is strictly increasing.

Note that because H is a cdf satisfying (H4), such a value $\widehat{t}_\theta(\alpha)$ always exists. It may not be unique, but if this happens all what follows will concern any one of the values $\widehat{t}_\theta(\alpha)$ satisfying (5.4).

In order to ensure uniqueness of $\widehat{t}_\theta(\alpha)$ we will make the following, quite unrestrictive, assumption:

(A10) H is is strictly increasing.

As for the mode estimation problem discussed before, the difficulty in estimating the conditional quantile $t_\theta(\alpha)$ is linked with the flatness of the curve of the conditional distribution $F(\theta,\cdot,x)$ around $t_\theta(\alpha)$. More precisely, we will suppose that there exists some integer $j>0$ such that:

$$\text{(A11)}\quad \begin{cases} F^{(l)}(\theta,t_\theta(\alpha),x)=0 \text{ if } 1\le l<j,\\ F^{(j)}(\theta,\cdot,x) \text{ is uniformly continuous on } \mathcal{S}_{\mathbb{R}},\\ |F^{(j)}(\theta,t_\theta(\alpha),x)|>C>0. \end{cases}$$

THEOREM 5.3. *If the conditions of Theorem* 4.8 *hold, together with* (A9)–(A11), *then*

$$\sup_{x\in\mathcal{S}_\mathcal{H}}|\widehat{t}_\theta(\alpha)-t_\theta(\alpha)|=\mathcal{O}(h_K^{b_1/j}+h_H^{b_2/j})+\mathcal{O}_{\text{a.co.}}\Big(\Big(\frac{\log d_n^{\mathcal{S}_\mathcal{H}}}{n\phi(h_K)}\Big)^{1/2j}\Big). \tag{5.5}$$

Proof. Let us write the Taylor expansion of the function $\widehat{F}(\theta,\cdot,x)$:

$$\begin{aligned}\widehat{F}(\theta,t_\theta(\alpha),x)-\widehat{F}(\theta,\widehat{t}_\theta(\alpha),x)&=\sum_{l=1}^{j-1}\frac{(t_\theta(\alpha)-\widehat{t}_\theta(\alpha))^l}{l!}\widehat{F}^{(l)}(\theta,t_\theta(\alpha),x)\\&\quad+\frac{(t_\theta(\alpha)-\widehat{t}_\theta(\alpha))^j}{j!}\widehat{F}^{(j)}(\theta,t^*,x),\end{aligned}$$

where t^* is some point between $t_\theta(\alpha)$ and $\widehat{t}_\theta(\alpha)$. We use the first part of condition (A11) to rewrite this expression as

$$\begin{aligned}&\widehat{F}(\theta,t_\theta(\alpha),x)-\widehat{F}(\theta,\widehat{t}_\theta(\alpha),x)\\&\qquad=\sum_{l=1}^{j-1}\frac{(t_\theta(\alpha)-\widehat{t}_\theta(\alpha))^l}{l!}\big(\widehat{f}_\theta^{x,(l-1)}(t_\theta(\alpha))-f_\theta^{x,\,(l-1)}(t_\theta(\alpha))\big)\\&\qquad\quad+\frac{(t_\theta(\alpha)-\widehat{t}_\theta(\alpha))^j}{j!}\widehat{f}^{(j-1)}(\theta,t^*,x),\end{aligned}$$

where $\widehat{f}_\theta^{x,(l-1)}(t_\theta(\alpha))=\widehat{f}^{(l-1)}(\theta,t_\theta(\alpha),x)$.

If we could establish that

$$\exists\tau>0,\qquad \sum_{n=1}^{\infty}\mathbb{P}(f^{(j-1)}(\theta,t^*,x)<\tau)<\infty, \tag{5.6}$$

we would have

$$\begin{aligned}&(t_\theta(\alpha)-\widehat{t}_\theta(\alpha))^j\\&\quad=\mathcal{O}_{\text{a.co.}}\Big(\sum_{l=1}^{j-1}(t_\theta(\alpha)-\widehat{t}_\theta(\alpha))^l\big(\widehat{f}^{(l-1)}(\theta,t_\theta(\alpha),x)-f^{(l-1)}(\theta,t_\theta(\alpha),x)\big)\Big)\\&\qquad+\mathcal{O}\big(\widehat{F}_\theta^x(t_\theta(\alpha))-F_\theta^x(t_\theta(\alpha))\big).\end{aligned} \tag{5.7}$$

By comparing the rates of convergence given in Theorems 3.5 and 4.8, we see that the leading term on the right hand side of (5.7) is the first one. So we have

$$(t_\theta(\alpha) - \widehat{t}_\theta(\alpha))^j = \mathcal{O}_{\text{a.co.}}\big(\widehat{F}(\theta, t_\theta(\alpha), x) - F(\theta, t_\theta(\alpha), x)\big).$$

Because of Theorem 4.8, this is enough to get the claimed result, and so (5.6) is the only result that remains to be checked. This will be done directly by using the uniform continuity of the function $f^{(j-1)}(\theta, \cdot, x)$ given by the second part of (A11) together with the third part of (A8) and with the following lemma.

LEMMA 5.4. *If the conditions of Theorem* 3.5 *hold, together with* (A9) *and* (A10), *then*

$$\widehat{t}_\theta(\alpha) - t_\theta(\alpha) \to 0, \qquad a.co. \tag{5.8}$$

6. Appendix

Proof of Lemma 4.4. For all $x \in \mathcal{S}_\mathcal{H}$ and $\theta \in \Theta_\mathcal{H}$, we set

$$k(x) = \arg \min_{k \in \{1,\dots,r_n\}} \|x - x_k\| \quad \text{and} \quad j(\theta) = \arg \min_{j \in \{1,\dots,l_n\}} \|\theta - t_j\|.$$

Consider the decomposition

$$\sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{\theta \in \Theta_\mathcal{H}} |\widehat{F}_D(\theta, x) - \mathbb{E}\,\widehat{F}_D(\theta, x)| \le \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5,$$

where

$$\Pi_1 = \sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{\theta \in \Theta_\mathcal{H}} |\widehat{F}_D(\theta, x) - \widehat{F}_D(\theta, x_{k(x)})|,$$

$$\Pi_2 = \sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{\theta \in \Theta_\mathcal{H}} |\widehat{F}_D(\theta, x_{k(x)}) - \widehat{F}_D(t_{j(\theta)}, x_{k(x)})|,$$

$$\Pi_3 = \sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{\theta \in \Theta_\mathcal{H}} |\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\,\widehat{F}_D(t_{j(\theta)}, x_{k(x)})|,$$

$$\Pi_4 = \sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{\theta \in \Theta_\mathcal{H}} |\mathbb{E}\,\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\,\widehat{F}_D(\theta, x_{k(x)})|,$$

$$\Pi_5 = \sup_{x \in \mathcal{S}_\mathcal{H}} \sup_{\theta \in \Theta_\mathcal{H}} |\mathbb{E}\,\widehat{F}_D(\theta, x_{k(x)}) - \mathbb{E}\,\widehat{F}_D(\theta, x)|.$$

For Π_1 and Π_2, we employ the Hölder continuity of K, Cauchy–Schwarz's and Bernstein's inequalities to get

$$\Pi_1 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right), \qquad \Pi_2 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right). \tag{6.1}$$

Then, since $\Pi_4 \le \Pi_1$ and $\Pi_5 \le \Pi_2$, we get, for $n \to \infty$,

$$\Pi_4 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right), \qquad \Pi_5 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{S_\mathcal{H}} d_n^{\Theta_\mathcal{H}})}{n\phi(h_K)}}\right). \tag{6.2}$$

Now, we deal with Π_3. For all $\eta > 0$, we have

$$\mathbb{P}\Big(\Pi_3 > \eta\Big(\sqrt{\frac{\log(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Big)\Big)$$
$$\le d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}} \max_{k\in\{1,\dots,d_n^{S_{\mathcal{H}}}\}} \max_{j\in\{1,\dots,d_n^{\Theta_{\mathcal{H}}}\}} \mathbb{P}\Big(\Pi_3' > \eta\Big(\sqrt{\frac{\log(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Big)\Big),$$

where $\Pi_3' = |\widehat{F}_D(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\,\widehat{F}_D(t_{j(\theta)}, x_{k(x)})|$.

Applying Bernstein's exponential inequality to

$$\frac{1}{\phi(h_K)}(K_i(t_{j(\theta)}, x_{k(x)}) - \mathbb{E}\,K_i(t_{j(\theta)}, x_{k(x)})),$$

under (A8) we get

$$\Pi_3 = \mathcal{O}_{\text{a.co.}}\Big(\sqrt{\frac{\log(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Big).$$

The conclusion can be easily deduced from the latter together with (6.1) and (6.2). ∎

Proof of Corollary 4.5. It is easy to see that

$$\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{F}_D(\theta,x)| \le 1/2 \;\Rightarrow\; \exists x\in\mathcal{S}_{\mathcal{H}},\ \exists\theta\in\Theta_{\mathcal{H}},\ 1-\widehat{F}_D(\theta,x)\ge 1/2$$
$$\Rightarrow \sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}} |1-\widehat{F}_D(\theta,x)| \ge 1/2.$$

We deduce from Lemma 4.4 that

$$\mathbb{P}\Big(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{F}_D(\theta,x)| \le 1/2\Big) \le \mathbb{P}\Big(\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}} |1-\widehat{F}_D(\theta,x)| \le 1/2\Big).$$

Consequently,

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\inf_{\theta\in\Theta_{\mathcal{H}}}\inf_{x\in\mathcal{S}_{\mathcal{H}}} \widehat{F}_D(\theta,x) < 1/2\Big) < \infty. \;∎$$

Proof of Lemma 4.6. One has

$$\mathbb{E}\,\widehat{F}_N(\theta,y,x) - F_\theta^x(y) = \frac{1}{\mathbb{E}\,K_1(x,\theta)}\,\mathbb{E}\Big[\sum_{i=1}^n K_i(x,\theta)H_i(y)\Big] - F_\theta^x(y) \tag{6.3}$$
$$= \frac{1}{\mathbb{E}\,K_1(x,\theta)}\,\mathbb{E}\big(K_1(x,\theta)[\mathbb{E}(H_1(y)\,|\,\langle X_1,\theta\rangle) - F_\theta^x(y)]\big),$$

where $F_\theta^{(j),\xi}(y) = F^{(j)}(\theta,y,\xi)$, for $j=0,1$.

Moreover, we have

$$\mathbb{E}(H_1(y)\,|\,\langle X_1,\theta\rangle) = \int_{\mathbb{R}} H(h_H^{-1}(y-z))f(\theta,z,X_1)\,dz.$$

Now, integrating by parts and using the fact that H is a cdf, we obtain

$$\mathbb{E}(H_1(y)\,|\,\langle X_1,\theta\rangle) = \int_{\mathbb{R}} H^{(1)}(t)F(\theta, y-h_H t, X_1)\,dt.$$

Thus,

$$|\mathbb{E}(H_1(y)\,|\,\langle X_1,\theta\rangle) - F_\theta^x(y)| \le \int_{\mathbb{R}} H^{(1)}(t)|F_\theta^{X_1}(y-h_H t) - F_\theta^x(y)|\,dt.$$

Finally, the use of (A2) implies that

$$|\mathbb{E}(H_1(y)\,|\,\langle X_1,\theta\rangle) - F_\theta^x(y)| \le C_\theta \int_{\mathbb{R}} H^{(1)}(t)(h_K^{b_1} + |t|^{b_2}h_H^{b_2})\,dt. \tag{6.4}$$

Because this inequality is uniform in $(\theta, y, x) \in \Theta_{\mathcal{H}} \times \mathcal{S}_{\mathcal{H}} \times \mathcal{S}_{\mathbb{R}}$ and because of (H4), (4.4) is a direct consequence of (6.3), (6.4) and of Corollary 4.5. ∎

Proof of Lemma 4.7. We keep the notation of Lemma 4.4. Since $\mathcal{S}_{\mathbb{R}}$ is compact, we have $\mathcal{S}_{\mathbb{R}} \subset \bigcup_{m=1}^{z_n}(y_m - l_n, y_m + l_n)$ for some $t_1,\ldots,t_{z_n} \in \mathcal{S}_{\mathbb{R}}$ with $l_n = n^{-1/2b_2}$ and $z_n \le Cn^{-1/2b_2}$. Take

$$m(y) = \arg\min_{m\in\{1,\ldots,z_n\}} |y - t_m|.$$

Thus, we have the decomposition

$$|\widehat{F}_N(\theta,y,x) - \mathbb{E}\,\widehat{F}_N(\theta,y,x)| = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5,$$

where

$$\begin{aligned}
\Gamma_1 &= |\widehat{F}_N(\theta,y,x) - \widehat{F}_N(\theta,y,x_{k(x)})|,\\
\Gamma_2 &= |\widehat{F}_N(\theta,y,x_{k(x)}) - \mathbb{E}\,\widehat{F}_N(\theta,y,x_{k(x)})|,\\
\Gamma_3 &= 2|\widehat{F}_N(t_{j(\theta)},y,x_{k(x)}) - \widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})|,\\
\Gamma_4 &= 2|\mathbb{E}\,\widehat{F}_N(t_{j(\theta)},y,x_{k(x)}) - \mathbb{E}\,\widehat{F}_N(t_{j(\theta)},y_{m(y)},x_{k(x)})|,\\
\Gamma_5 &= |\mathbb{E}\,\widehat{F}_N(\theta,y,x_{k(x)}) - \mathbb{E}\,\widehat{F}_N(\theta,y,x)|.
\end{aligned}$$

Concerning Γ_1 we have

$$\begin{aligned}
&|\widehat{F}_N(\theta,y,x) - \widehat{F}_N(\theta,y,x_{k(x)})|\\
&\quad\le \frac{1}{n}\sum_{i=1}^n |H_i(y)| \left| \frac{1}{\mathbb{E}\,K_1(\theta,x)}K_i(\theta,x) - \frac{1}{\mathbb{E}\,K_1(\theta,x_{k(x)})}K_i(\theta,x_{k(x)})\right|.
\end{aligned}$$

We use the Hölder continuity of K, the Cauchy–Schwarz inequality, the Bernstein inequality and the boundedness of H (Assumption (H4)) to get

$$\begin{aligned}
&|\widehat{F}_N(\theta,y,x) - \widehat{F}_N(\theta,y,x_{k(x)})|\\
&\qquad\le \frac{C}{\phi(h_K)}\,\frac{1}{n}\sum_{i=1}^n |H_i(y)|\,|K_i(\theta,x) - K_i(\theta,x_{k(x)})| \le \frac{C'r_n}{\phi(h_K)}.
\end{aligned}$$

Concerning Γ_2, the monotonicity of $\mathbb{E}\,\widehat{F}_N(\theta,\cdot,x)$ and $\widehat{F}_N(\theta,\cdot,x)$ permits us to write, for all $m \le z_n$, $x \in \mathcal{S}_{\mathcal{H}}$, and $\theta \in \Theta_{\mathcal{H}}$,

$$\begin{aligned}\mathbb{E}\,\widehat{F}_N(\theta, y_{m(y)}-l_n, x_{k(x)}) &\le \sup_{y\in(y_{m(y)}-l_n, y_{m(y)}+l_n)} \mathbb{E}\,\widehat{F}_N(\theta,y,x)\\ &\le \mathbb{E}\,\widehat{F}_N(\theta, y_{m(y)}+l_n, x_{k(x)}),\\ \widehat{F}_N(\theta, y_{m(y)}-l_n, x_{k(x)}) &\le \sup_{y\in(y_{m(y)}-l_n, y_{m(y)}+l_n)} \widehat{F}_N(\theta,y,x)\\ &\le \widehat{F}_N(\theta, y_{m(y)}+l_n, x_{k(x)}).\end{aligned}$$

Next, we use the Hölder condition on $F(\theta,y,x)$ to show that, for any $y_1, y_2 \in \mathcal{S}_{\mathbb{R}}$ and for all $x \in \mathcal{S}_{\mathcal{H}}$, $\theta \in \Theta_{\mathcal{H}}$,

$$\begin{aligned}(6.5)\qquad &|\mathbb{E}\,\widehat{F}_N(\theta,y_1,x) - \mathbb{E}\,\widehat{F}_N(\theta,y_2,x)|\\ &\quad= \frac{1}{\mathbb{E}\,K_1(x,\theta)}|\mathbb{E}(K_1(x,\theta)F_\theta^{X_1}(y_1)) - \mathbb{E}(K_1(x,\theta)F_\theta^{X_1}(y_2))|\\ &\quad\le C|y_1-y_2|^{b_2}.\end{aligned}$$

Now, we have, for all $\eta > 0$,

$$\begin{aligned}&\mathbb{P}\Bigg(\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in\mathcal{S}_{\mathcal{H}}}\sup_{y\in\mathcal{S}_{\mathbb{R}}}|\Xi_n - \mathbb{E}\,\Xi_n| > \eta\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg)\\ &= \mathbb{P}\Bigg(\max_{j\in\{1,\dots,d_n^{\Theta_{\mathcal{H}}}\}}\max_{k\in\{1,\dots,d_n^{\mathcal{S}_{\mathcal{H}}}\}}\max_{m(y)\in\{1,\dots,z_n\}}|\Xi_n - \mathbb{E}\,\Xi_n| > \eta\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg)\\ &\le z_n d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\\ &\quad\times \max_{j\in\{1,\dots,d_n^{\Theta_{\mathcal{H}}}\}}\max_{k\in\{1,\dots,d_n^{\mathcal{S}_{\mathcal{H}}}\}}\max_{m(y)\in\{1,\dots,z_n\}}\mathbb{P}\Bigg(|\Xi_n - \mathbb{E}\,\Xi_n| > \eta\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg)\\ &\le 2z_n d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}\exp(-C\eta^2\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}))\end{aligned}$$

where $\Xi_n = \widehat{F}_N(\theta,y,x_{k(x)})$. Choosing $z_n = \mathcal{O}(l_n^{-1}) = \mathcal{O}(n^{1/2b_2})$, we get

$$\mathbb{E}\Bigg(|\Xi_n - \mathbb{E}\,\Xi_n| > \eta\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg) \le C' z_n (d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})^{1-C\eta^2}.$$

Setting $C\eta^2 = \beta$ and using (A4), we get

$$\Gamma_2 = \mathcal{O}_{\mathrm{a.co.}}\Bigg(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\Bigg).$$

Concerning the terms $\varGamma_3$ and $\varGamma_4$, using the Lipschitz condition on H, one can write

$$\begin{aligned}|\varUpsilon_N| &\le C\frac{1}{n\phi(h_K)}\sum_{i=1}^{n} K_i(t_{j(\theta)}, x_{k(x)})|H_i(y) - H_i(y_{m(y)})| \\ &\le \frac{Cl_n}{nh_H\phi(h_K)}\sum_{i=1}^{n} K_i(t_{j(\theta)}, x_{k(x)}).\end{aligned}$$

where $\varUpsilon_N = \widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)})$.

Once again a standard exponential inequality for a sum of bounded variables allows us to write

$$\begin{aligned}&\widehat{F}_N(t_{j(\theta)}, y, x_{k(x)}) - \widehat{F}_N(t_{j(\theta)}, y_{m(y)}, x_{k(x)}) \\ &\qquad = \mathcal{O}\left(\frac{l_n}{h_H}\right) + \mathcal{O}_{\text{a.co.}}\left(\frac{l_n}{h_H}\sqrt{\frac{\log n}{n\phi_x(h_K)}}\right).\end{aligned}$$

Now, the facts that $\lim_{n\to\infty} n^{\gamma} h_H = \infty$ and $l_n = n^{-1/2b_2}$ imply that

$$\frac{l_n}{h_H\phi(h_K)} = o\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right),$$

and so

$$\varGamma_3 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right).$$

Hence, for n large enough, we have

$$\varGamma_3 \le \varGamma_4 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right).$$

Concerning $\varGamma_5$, we have

$$\mathbb{E}\,\widehat{F}_N(\theta, y, x_{k(x)}) - \mathbb{E}\,\widehat{F}_N(\theta, y, x) \le \sup_{x\in\mathcal{S}_{\mathcal{H}}} |\widehat{F}_N(\theta, y, x) - \widehat{F}_N(\theta, y, x_{k(x)})|.$$

Then following the proof used for $\varGamma_1$ and using the same idea as for $\mathbb{E}\,\widehat{F}_D(\theta, x_{k(x)}) - \mathbb{E}\,\widehat{F}_D(\theta, x)$ we get, for $n \to \infty$,

$$\varGamma_5 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n\phi(h_K)}}\right). \ \blacksquare$$

Proof of Lemma 4.9. Let $H_i^{(j+1)}(y) = H^{(j+1)}(h_H^{-1}(y - Y_i))$, and note that

$$\begin{aligned}(6.6)\quad &\varPsi_n(\theta, y, x) \\ &\quad = \frac{h_H^{-j-1}}{\mathbb{E}\,K_1(x,\theta)}\,\mathbb{E}\big(K_1(x,\theta)[\mathbb{E}(H_1^{(j+1)}(y)\mid\langle X,\theta\rangle) - h_H^{j+1} f^{(j)}(\theta, y, x)]\big),\end{aligned}$$

where $\varPsi_n(\theta, y, x) = \mathbb{E}\,\widehat{f}_N^{(j)}(\theta, y, x) - f^{(j)}(\theta, y, x)$.

Moreover,

$$\mathbb{E}(H_1^{(j+1)}(y)\,|\,\langle X,\theta\rangle) = \int_{\mathbb{R}} H^{(j+1)}(h_H^{-1}(y-z))f(\theta,z,X)\,dz \tag{6.7}$$
$$= -\sum_{l=1}^{j} h_H^l \big[H^{(j-l+1)}(h_H^{-1}(y-z))f^{(l-1)}(\theta,z,X)\big]_{-\infty}^{\infty} + h_H^j \int_{\mathbb{R}} H^{(1)}(h_H^{-1}(y-z))f^{(j)}(\theta,z,X)\,dz.$$

Condition (H8) allows us to cancel the first term on the right side of (6.7) and we can write

$$|\mathbb{E}(H_1^{(j+1)}(y)\,|\,\langle X,\theta\rangle) - h_H^{j+1} f^{(j)}(\theta,y,x)| \le h_H^{j+1}\int_{\mathbb{R}} H^{(1)}(t)|f^{(j)}(\theta,y-h_H t,X) - f^{(j)}(\theta,y,x)|\,dt.$$

Finally, (A5) allows us to write

$$|\mathbb{E}(H_1^{(j+1)}(y)\,|\,\langle X,\theta\rangle) - h_H^{j+1} f^{(j)}(\theta,y,x)| \le C_{\theta,x} h_H^{j+1}\int_{\mathbb{R}} H'(t)(h_K^{b_1} + (|t|h_H)^{b_2})\,dt.$$

This inequality is uniform in $(\theta,y,x)\in\Theta_{\mathcal{H}}\times\mathcal{S}_{\mathcal{H}}\times\mathcal{S}_{\mathbb{R}}$; now to finish the proof it is sufficient to use (H4). ■

Proof of Lemma 4.10. Let $l_n = n^{-3\gamma/2-1/2}$ and $z_n \le Cn^{-3\gamma/2-1/2}$. Consider the decomposition

$$|\widehat{f}_N^{(j)}(\theta,y,x) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)| = \Delta_1+\Delta_2+\Delta_3+\Delta_4+\Delta_5,$$

where

$$\begin{aligned}
\Delta_1 &= |\widehat{f}_N^{(j)}(\theta,y,x) - \widehat{f}_N^{(j)}(\theta,y,x_{k(x)})|,\\
\Delta_2 &= |\widehat{f}_N^{(j)}(\theta,y,x_{k(x)}) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x_{k(x)})|,\\
\Delta_3 &= 2|\widehat{f}_N^{(j)}(t_{j(\theta)},y,x_{k(x)}) - \widehat{f}_N^{(j)}(t_{j(\theta)},y_{m(y)},x_{k(x)})|,\\
\Delta_4 &= 2|\mathbb{E}\,\widehat{f}_N^{(j)}(t_{j(\theta)},y,x_{k(x)}) - \mathbb{E}\,\widehat{f}_N^{(j)}(t_{j(\theta)},y_{m(y)},x_{k(x)})|,\\
\Delta_5 &= |\mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x_{k(x)}) - \mathbb{E}\,\widehat{f}_N^{(j)}(\theta,y,x)|.
\end{aligned}$$

Concerning Δ_1, we have

$$|\widehat{f}_N^{(j)}(\theta,y,x) - \widehat{f}_N^{(j)}(\theta,y,x_{k(x)})| \le \frac{h_H^{-1-j}}{n}\sum_{i=1}^{n}|H_i^{(j+1)}(y)|\bigg|\frac{1}{\mathbb{E}\,K_1(\theta,x)}K_i(\theta,x) - \frac{1}{\mathbb{E}\,K_1(\theta,x_{k(x)})}K_i(\theta,x_{k(x)})\bigg|.$$

We use the Hölder continuity of K, the Cauchy–Schwarz inequality, the Bernstein inequality and the boundedness of $H^{(j+1)}$ (Assumption (H8)) to get

$$\begin{aligned}|\Psi_n^1| &\leq \frac{Ch_H^{-j-1}}{\phi(h_K)}\frac{1}{n}\sum_{i=1}^{n}|H_i^{(j+1)}(y)|\,|K_i(\theta,x)-K_i(\theta,x_{k(x)})| \\ &\leq \frac{Ch_H^{-j-1}}{\phi(h_K)}\frac{1}{n}\sum_{i=1}^{n}|H_i^{(j+1)}(y)|\,|K_i(\theta,x)-K_i(\theta,x_{k(x)})| \\ &\leq \frac{C'r_n}{h_H^{j+2}\phi(h_K)},\end{aligned}$$

where $\Psi_n^1 = \widehat{f}_N^{(j)}(\theta,y,x) - \widehat{f}_N^{(j)}(\theta,y,x_{k(x)})$. From (A6), for n large enough, we have

$$\Delta_1 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\right).$$

Then using the fact that $\Delta_5 \leq \Delta_1$, we obtain

$$\Delta_5 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\right). \tag{6.8}$$

For Δ_2, we follow the same idea as for Γ_2 to get

$$\Delta_2 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\right).$$

Concerning Δ_3 and Δ_4, using the Lipschitz condition on H, we get

$$|\widehat{f}_N^{(j)}(t_{j(\theta)},y,x_{k(x)}) - \widehat{f}_N^{(j)}(t_{j(\theta)},y_{m(y)},x_{k(x)})| \leq \frac{l_n}{h_H^{j+2}\phi(h_k)}.$$

Using the fact that $\lim_{n\to\infty} n^{\gamma}h_H = \infty$ and choosing $l_n = n^{-3\gamma/2-1/2}$ implies

$$\frac{l_n}{h_H^{j+2}\phi(h_k)} = o\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\right).$$

So, for n large enough, we have

$$\Delta_3 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}})}{nh_H^{2j+1}\phi(h_K)}}\right),$$

and as $\Delta_4 \leq \Delta_3$, we also obtain

$$\Delta_4 = \mathcal{O}_{\text{a.co.}}\left(\sqrt{\frac{\log(d_n^{\mathcal{S}_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}})}{n h_H^{2j+1}\phi(h_K)}}\right). \tag{6.9}$$

Finally, the lemma can be easily deduced from (6.8) and (6.9). ∎

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References

[1] A. Aït Saidi, F. Ferraty and R. Kassa, *Single functional index model for a time series*, Rev. Roumaine Math. Pures Appl. 50 (2005), 321–330.

[2] A. Aït Saidi, F. Ferraty, R. Kassa and Ph. Vieu, *Cross-validated estimation in the single-functional index model*, Statistics 42 (2008), 475–494.

[3] S. Attaoui, A. Laksaci and E. Ould-Saïd, *A note on the conditional density estimate in the single functional index model*, Statist. Probab. Lett. 81 (2011), 45–53.

[4] A. Berlinet, B. Cadre and A. Gannoun, *On the conditional L_1-median and its estimation*, J. Nonparametr. Statist. 13 (2001), 631–645.

[5] D. Bosq and J.-P. Lecoutre, *Théorie de l'estimation fonctionnelle*, Economica, Paris, 1987.

[6] S. Dabo-Niang and A. Laksaci, *Nonparametric quantile regression estimation for functional dependent data*, Comm. Statist. Theory Methods 41 (2012), 1254–1268.

[7] M. Delecroix, W. Härdle and M. Hristache, *Efficient estimation in conditional single-index regression*, J. Multivariate Anal. 86 (2003), 213–226.

[8] A. El Ghouch and I. Van Keilegom, *Local linear quantile regression with dependent censored data*, Statist. Sinica 19 (2009), 1621–1640.

[9] M. Ezzahrioui and E. Ould-Saïd, *Asymptotic results of a non-parametric conditional quantile estimator for functional time-series*, Comm. Statist. Theory Methods 37 (2008), 2735–2759.

[10] M. Ezzahrioui and E. Ould Saïd, *Some asymptotic results of a non-parametric conditional mode estimator for functional time series data*, Statist. Neerlandica 64 (2010), 171–201.

[11] F. Ferraty, A. Laksaci and Ph. Vieu, *Estimating some characteristics of the conditional distribution in non-parametric functional models*, Statist. Inference Stoch. Process. 9 (2006), 47–76.

[12] F. Ferraty, A. Peuch et Ph. Vieu, *Modèle à indice fonctionnel simple*, C. R. Math. Acad. Sci. Paris 336 (2003), 1025–1028.

[13] F. Ferraty, A. Laksaci, A. Tadj and Ph. Vieu, *Rate of uniform consistency for nonparametric estimates with functional variables*, J. Statist. Plann. Inference 140 (2010), 335–352.

[14] F. Ferraty, A. Rabhi and Ph. Vieu, *Conditional quantiles for functionally dependent data with application to the climatic El Niño phenomenon*, Sankhyā 67 (2005), 378–498.

[15] F. Ferraty and Ph. Vieu, *Nonparametric Functional Data Analysis: Theory and Practice*, Springer Series in Statistics, Springer, New York, 2006.

[16] A. Gannoun, J. Saracco and K. Yu, *Nonparametric prediction by conditional median and quantiles*, J. Statist. Plann. Inference 117 (2003), 207–223.
[17] A. Gannoun, J. Saracco, A. Yuan and G. E. Bonney, *Non-parametric quantile regression with censored data*, Scand. J. Statist. 32 (2005), 527–550.
[18] W. Härdle, P. Hall and H. Ichumira, *Optimal smoothing in single-index models*, Ann. Statist. 21 (1993), 157–178.
[19] M. Hristache, A. Juditsky and V. Spokoiny, *Direct estimation of the index coefficient in the single-index model*, Ann. Statist. 29 (2001), 595–623.
[20] R. Koenker and J. G. Bassett, *Regression quantiles*, Econometrica 46 (1978), 33–50.
[21] A. Laksaci, M. Lemdani and E. Ould Saïd, *A generalized L^1-approach for a kernel estimator of conditional quantile with functional regressors: Consistency and asymptotic normality*, Statist. Probab. Lett. 79 (2009), 1065–1073.
[22] H. Y. Liang and J. de Uña-Álvarez, *Asymptotic properties of conditional quantile estimator for censored dependent observations*, Ann. Inst. Statist. Math. 63 (2011), 267–289.
[23] E. Ould-Saïd, *A strong uniform convergence rate of kernel conditional quantile estimator under random censorship*, Statist. Probab. Lett. 76 (2006), 579–586.
[24] E. Ould-Saïd and Z. Cai, *Strong uniform consistency of nonparametric estimation of the censored conditional mode function*, Nonparametric Statist. 17 (2005), 797–806.
[25] M. Rosenblatt, *Conditional probability density and regression estimators*, in: Multivariate Analysis II, P. R. Krishnaiah (ed.), Academic Press, New York, 1969.
[26] M. Samanta, *Nonparametric estimation of conditional quantiles*, Statist. Probab. Lett. 7 (1989), 407–412.

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