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LOCAL CONVERGENCE OF TWO COMPETING THIRD ORDER METHODS IN BANACH SPACE

Abstract. We present a local convergence analysis for two popular third order methods of approximating a solution of a nonlinear equation in a Banach space setting. The convergence ball and error estimates are given for both methods under the same conditions. A comparison is given between the two methods, as well as numerical examples.

1. Introduction. In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought to a form like (1.1) using mathematical modelling [2]–[5], [11], [14], [15]. Solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study of convergence of iterative procedures is usually of two types: semilocal and local convergence analysis. The semilocal convergence analysis is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of numerical functional analysis for finding a solution x^* of equation (1.1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if

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the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev–Halley-type methods [1], [3], [5], [7]–[16] require the evaluation of the second Fréchet derivative, which is very expensive in general. However, there are integral equations where the second Fréchet derivative is block diagonal and inexpensive [10]–[13] or for quadratic equations the second Fréchet derivative is constant [4], [13]. Moreover, in some applications involving stiff systems [2], [5], [9], high order methods are useful. However, in general the use of the second Fréchet derivative restricts the use of these methods as their informational efficiency is less than or equal to unity.

That is why in the present study we study the local convergenve of two popular competing third order methods defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \left(\frac{F'(x_n) + F'(y_n)}{2}\right)^{-1} F(x_n) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - F'\left(\frac{x_n + y_n}{2}\right)^{-1} F(x_n). \end{aligned} \tag{1.3}$$

Method (1.3) is usually called the midpoint method. The two methods are obtained by modifications to the classic Newton method; at the cost of a twice higher cost of one iteration the third order of convergence rate is achieved.

The informational cost of one iteration of each of the two methods is equal to the cost of two consecutive iterations of the Newton method, which may be considered as one iteration of the two-step method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - F'(y_n)^{-1}F(y_n), \end{aligned} \tag{1.4}$$

with quadratic convergence. Obviously, the ball of convergence for this method is the same as in the case of the Newton method. Its radius has estimates greater than the radii of balls of convergence for the two methods considered in the paper. Nevertheless, their analysis is illustrative.

Notice, however, that methods (1.2) and (1.3) have been used by several authors in the semilocal convergence case, since their sufficient convergence criterion can be verified in cases where the corresponding ones for method (1.4) cannot be verified [5]–[11], [13]. Hence, these methods are practical alternatives to the Newton method in such cases. There is a plethora of semilocal convergence results for these methods under conditions $(\mathcal{C})$ [1]–[16]:

($\mathcal{C}_1$) $F : D \to Y$ is twice Fréchet-differentiable and $F'(x_0)^{-1} \in L(Y, X)$ for some $x_0 \in D$ such that
$$\|F'(x_0)^{-1}\| \leq \beta;$$
($\mathcal{C}_2$) $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$;
($\mathcal{C}_3$) $\|F'(x_0)^{-1}F''(x)\| \leq \beta_1$ for each $x \in D$;
($\mathcal{C}_4$) $\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq \beta_2\|x - y\|^p$ for each $x, y \in D$ and some $p \in (0, 1]$.

Conditions ($\mathcal{C}_3$) and ($\mathcal{C}_4$) restrict the applicability of these methods. As an academic example, define $f : [-1, 1] \to (-\infty, \infty)$ by $f(x) = x^2 \ln x^2 + x_1x^2 + c_2x + c_3$, $f(0) = c_3$, where c_1, c_2, c_3 are given real parameters. Then we have $\lim_{x\to 0} x^2 \ln x^2 = 0$, $\lim_{x\to 0} x \ln x^2 = 0$, $f'(x) = 2x \ln x^2 + 2(c_1 + 1)x + c_2$ and $f''(x) = 2(2 \ln x + 3 + c_1)$. Thus f does not satisfy condition ($\mathcal{C}_3$) or ($\mathcal{C}_4$) (for $p = 1$). Note that conditions ($\mathcal{C}_3$) and ($\mathcal{C}_4$) have also been used in the local convergence of the method (1.3) [2], [5] by simply replacing x_0 by x^*. In our study we assume for both methods (1.2) and (1.3) conditions ($\mathcal{A}$):

($\mathcal{A}_1$) $F : D \to Y$ is twice Fréchet-differentiable and there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$;
($\mathcal{A}_2$) $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|^p$ for each $x \in D$ and some $p \in (0, 1]$;
($\mathcal{A}_3$) $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|^p$ for each $x, y \in D$ and some $p \in (0, 1]$;
($\mathcal{A}_4$) $\|F'(x^*)^{-1}F'(x)\| \leq K$ for each $x \in D$.

The convergence ball for method (1.3) is shown to be smaller than the convergence ball of method (1.2) for $p \in (0, 1)$. Moreover, the error estimates on the distances $\|x_n - x^*\|$ for method (1.3) are shown to be smaller than the corresponding estimates for method (1.2) for $p \in (0, 1)$. The convergence balls and error estimates for these methods are the same for $p = 1$.

The paper is organized as follows: In Section 2 we present the local convergence of these methods as well as their comparison. Numerical examples are given in Section 3.

In the rest of this study, $U(w, q)$ and $\overline{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

2. Local convergence. In this section we present the local convergence of method (1.2) and method (1.3). It is convenient for the local convergence of method (1.2) to introduce some functions and parameters. Let $I = [0, L_0^{-1/p})$. Define
$$g : I^2 \to [0, \infty), \qquad f : I \to [0, \infty), \qquad h : \mathbb{R} \to \mathbb{R}$$

by

$$g(s,t) = t + \frac{1}{2}\frac{KLs(s+t)^p}{\big(1-\frac{L_0}{2}(s^p+t^p)\big)(1-L_0s^p)},$$
$$f(s) = \frac{Ls^p}{(1+p)(1-L_0s^p)} + \frac{2^{p-1}KLs^p}{(1-L_0s^p)^2},$$
$$h(s) = L_0(L+(1+p)L_0)s^2 - (L+(1+p)(2L_0+2^{p-1}KL))s + 1 + p.$$

The discriminant Δ of the quadratic polynomial h is given by

$$\begin{aligned}\Delta &= (L+2L_0(1+p)+2^{p-1}(1+p)KL)^2 - 4(1+p)L_0(L+(1+p)L_0)\\ &= L^2 + (1+p)^2K^2L^2 2^{2(p-1)} + (1+p)KL^2 2^p + L_0LK(1+p)^2 2^{p+1} > 0.\end{aligned}$$

It follows from $\Delta > 0$ that h has two roots ρ_1 and ρ_2 with $\rho_1 < \rho_2$. Moreover, by the Viète relations,

$$\rho_1\rho_2 = \frac{1+p}{L_0(L+(1+p)L_0)} > 0,$$
$$\rho_1+\rho_2 = \frac{L+(1+p)(2L_0+2^{p-1}KL)}{L_0(L+(1+p)L_0)} > 0,$$

we deduce that $0 < \rho_1 < \rho_2$. In particular,

$$\rho_1 = \frac{L+(1+p)(2L_0+2^{p-1}KL)-\sqrt{\Delta}}{2L_0(L+(1+p)L_0)} < \frac{1}{L_0}.$$

Set

$$r = \rho_1^{1/p}. \tag{2.1}$$

Notice that if for each $s, t \in (0, r] \subset I$ with $t \le s$ and

$$t \le \frac{Ls^{1+p}}{(1+p)(1-L_0s^p)}$$

we have

$$\frac{Ls^p}{(1+p)(1-L_0s^p)} \le 1 \tag{2.2}$$

and

$$g(s,t) \le f(s)s < s, \tag{2.3}$$

then we can show the following local convergence result for method (1.2) under conditions $(\mathcal{A})$:

THEOREM 2.1. *Suppose that conditions $(\mathcal{A})$ hold and $\overline{U}(x^*, r) \subseteq D$, where r is given by (2.1). Then the sequence $\{x_n\}$ generated by method (1.2) for some $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \dots$ and converges to x^*. Moreover, for each $n = 0, 1, 2, \dots,$*

$$\|y_n - x^*\| \le \frac{L\|x_n - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_n - x^*\|^p)} \le \|x_n - x^*\|, \tag{2.4}$$

$$\begin{aligned} \|x_{n+1} - x^*\| &\le g(\|x_n - x^*\|, \|y_n - x^*\|) \le f(\|x_n - x^*\|)\|x_n - x^*\| \\ &< \|x_n - x^*\|. \end{aligned} \tag{2.5}$$

Proof. We shall use induction to show that estimates (2.4), (2.5) hold and $y_n, x_{n+1} \in U(x^*, r)$ for each $n = 0, 1, 2, \ldots$. Using $(\mathcal{A}_2)$ and the hypothesis $x_0 \in U(x^*, r)$ we have

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\|^p < L_0 r^p < 1. \tag{2.6}$$

It follows from (2.6) and the Banach Lemma on invertible operators [2], [5], [14] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|^p} < \frac{1}{1 - L_0 r^p}. \tag{2.7}$$

Hence, y_0 is well defined. We shall show $y_0 \in U(x^*, r)$. It follows from the first substep in method (1.2) for $n = 0$ and $F(x^*) = 0$ that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -F'(x_0)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)] \\ &= -[F'(x_0)^{-1}F'(x^*)]F'(x^*)^{-1} \\ &\quad \cdot \int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]\, d\theta\, (x_0 - x^*). \end{aligned} \tag{2.8}$$

Then by (2.2), $(\mathcal{A}_3)$, (2.7) and (2.8) we get

$$\begin{aligned} &\|y_0 - x^*\| \\ &\quad \le \|F'(x_0)^{-1}F'(x^*)\| \\ &\qquad \cdot \left\|F'(x^*)^{-1}\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]d\theta\right\| \|x_0 - x^*\| \\ &\quad \le \frac{1}{1 - L_0\|x_0 - x^*\|^p}\, \frac{L_0\|x_0 - x^*\|^p}{1+p} \|x_0 - x^*\| \\ &\quad \le \frac{L\|x_0 - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_0 - x^*\|^p)} \le \|x_0 - x^*\| < r, \end{aligned} \tag{2.9}$$

which shows (2.4) for $n = 0$ and $y_0 \in U(x^*, r)$.

Next, we shall show that $\left(\frac{F'(x_0) + F'(y_0)}{2}\right)^{-1} \in L(Y, X)$. As in (2.6), we obtain

$$\begin{aligned}(2.10)\quad &\left\|F'(x^*)^{-1}\left[\frac{F'(x_0)+F'(y_0)}{2}-F'(x^*)\right]\right\| \\ &\le \frac{1}{2}[\|F'(x^*)^{-1}(F'(x_0)-F'(x^*)]\| + \|F'(x^*)^{-1}(F'(y_0)-F'(x^*))\|] \\ &\le \frac{L_0}{2}(\|x_0-x^*\|^p + \|y_0-x^*\|^p) < \frac{L_0}{2}(r^p+r^p) = Lr^p < 1.\end{aligned}$$

It follows from (2.10) and the Banach Lemma on invertible operators that $\left(\frac{F'(x_0)+F'(y_0)}{2}\right)^{-1} \in L(Y,X)$ and

$$\begin{aligned}(2.11)\quad &\left\|\left(\frac{F'(x_0)+F'(y_0)}{2}\right)^{-1}F'(x^*)\right\| \\ &\le \frac{1}{1-\frac{L_0}{2}(\|x_0-x^*\|^p+\|y_0-x^*\|^p)} < \frac{1}{1-L_0r^p}.\end{aligned}$$

Then, it follows from the second substep in method (1.2) for $n=0$ that x_1 is well defined. We shall show that (2.5) holds for $n=0$ and that $x_1 \in U(x^*,r)$. By subtracting the first from the second substep in method (1.2) for $n=0$, we get

$$\begin{aligned}(2.12)\quad x_1 &= y_0 + \left[F'(x_0)^{-1} - \left(\frac{F'(x_0)+F'(y_0)}{2}\right)^{-1}\right]F'(x_0) \\ &= y_0 - \frac{1}{2}\left[\left(\frac{F'(x_0)+F'(y_0)}{2}\right)^{-1}F'(x^*)\right] \\ &\quad \cdot [F'(x^*)^{-1}(F'(x_0)-F'(y_0))][F'(x_0)^{-1}F'(x^*)] \\ &\quad \cdot \left[F'(x^*)^{-1}\int_0^1 F'(x^*+\theta(x_0-x^*))(x_0-x^*)\,d\theta\right].\end{aligned}$$

Using, $(\mathcal{A}_3)$, $(\mathcal{A}_4)$, (2.7), (2.3), (2.11) and (2.12) we get

$$\begin{aligned}(2.13)\quad \|x_1-x^*\| &\le \|y_0-x^*\| + \frac{1}{2}\frac{L\|x_0-y_0\|^p}{1-\frac{L_0}{2}\|x_0-x^*\|}\frac{K}{1-L_0\|x_0-x^*\|^p} \\ &\le g(\|x_0-x^*\|,\|y_0-x^*\|) \le f(\|x_0-x^*\|)\|x_0-x^*\| \\ &< \|x_0-x^*\| < r,\end{aligned}$$

which shows (2.5) for $n=0$ and that $x_1 \in U(x^*,r)$. Then, by replacing in the preceding estimates x_0, x_1, y_0 by x_k, x_{k+1}, y_k, respectively, we complete the induction. Finally, from the estimate $\|x_{k+1}-x^*\| < \|x_k-x^*\|$ we deduce that $\lim_{k\to\infty} x_k = x^*$. ∎

Similarly, for method (1.3) we define corresponding functions and parameters by

$$g_1(s,t) = t + \frac{KLs(s+t)^p}{2^p\big(1-\frac{L_0}{2^p}(s+t)^p\big)(1-L_0s^p)},$$
$$f_1(s) = \frac{Ls^p}{(1+p)(1-L_0s^p)} + \frac{KLs^p}{(1-L_0s^p)^2},$$
$$h_1(s) = L_0(L+(1+p)L_0)s^2 - (L+(1+p)(2L_0+KL))s + 1 + p.$$

Then

$$\varDelta_1 = (L+(1+p)(2L_0+KL))^2 - 4(1+p)L_0(L+(1+p)L_0) > 0$$

and $0 < \bar{\rho_1} < \bar{\rho_2}$ with

$$\bar{\rho_1} = \frac{L+(1+p)(2L_0+KL) - \sqrt{\varDelta_1}}{2L_0(L+(1+p)L_0)} < \frac{1}{L_0}.$$

Set

$$\bar{r} = \bar{\rho}_1^{1/p}. \tag{2.14}$$

Then, as in the proof of Theorem 2.1 but using method (1.3) in the form

$$y_n = x_n - F'(x_n)^{-1}F(x_n),$$
$$x_{n+1} = y_n - F'\left(\frac{x_n+y_n}{2}\right)^{-1}\left[F'(x_n) - F'\left(\frac{x_n+y_n}{2}\right)\right]F'(x_n)^{-1}F(x_n)$$

and the estimate

$$\left\|F'(x^*)^{-1}\left[F'(x^*) - F'\left(\frac{x_0+y_0}{2}\right)\right]\right\| \le L_0\left\|\frac{x_0+y_0}{2} - x^*\right\|^p$$
$$\le \frac{L_0}{2^p}(\|x_0-x^*\| + \|y_0-x^*\|)^p < \frac{L_0}{2^p}(\bar{r}^p + \bar{r}^p) = L_0\bar{r}^p < 1,$$

instead of (2.11) we arrive at the following local result for method (1.3) under conditions $(\mathcal{A})$.

THEOREM 2.2. *Suppose that conditions* $(\mathcal{A})$ *hold and* $\overline{U}(x^*, r) \subseteq D$, *where* $\bar{r}$ *is given by* (2.14). *Then the sequence* $\{x_n\}$ *generated by method* (1.3) *for some* $x_0 \in U(x^*, r)$ *is well defined, remains in* $U(x^*, r)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to* x^*. *Moreover, for each* $n = 0, 1, 2, \ldots,$

$$\|y_n - x^*\| \le \frac{L\|x_n-x^*\|^{1+p}}{(1+p)(1-L_0\|x_n-x^*\|^p)} \le \|x_n - x^*\|,$$
$$\|x_{n+1}-x^*\| \le g_1(\|x_n-x^*\|, \|y_n-x^*\|) \le f_1(\|x_n-x^*\|)\|x_n-x^*\|$$
$$< \|x_n - x^*\|.$$

REMARK 2.3. (a) Condition $(\mathcal{A}_2)$ can be dropped, since it follows from $(\mathcal{A}_3)$. Notice, however, that in general

$$L_0 \le L \tag{2.15}$$

and L/L_0 can be arbitrarily large [2]–[6].

(b) In view of condition $(\mathcal{A}_2)$ and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}[F'(x)-F'(x^*)]+I\| \\ &\le 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\| \\ &\le 1+L_0\|x-x^*\|^p,\end{aligned}$$

condition $(\mathcal{A}_4)$ can be dropped and K can be replaced by

$$K(r)=1+L_0r^p. \tag{2.16}$$

(c) It is worth noticing that it follows from (2.2) and (2.4) that r (or $\bar{r}$) is such that

$$r<r_A=\left(\frac{1+p}{L+(1+p)L_0}\right)^{1/p}. \tag{2.17}$$

The convergence ball of radius r_A was given by us in [2], [3], [5] for Newton's method under conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. Estimate (2.17) shows that the convergence balls of cubically convergent methods (1.2) and (1.3) are smaller than the convergence balls of the quadratically convergent Newton's method.

(d) If $p\in(0,1)$, then $2^{p-1}\le 1$. Hence, it follows from (2.1) and (2.14) that

$$\bar{r}<r.$$

Moreover it follows that

$$g(s,t)<g_1(s,t)$$

and

$$f(s)<f_1(s)$$

for all $s,t\in(0,\bar{r})$. Furthermore, equality holds in the three preceding inequalities if $p=1$.

(e) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2]–[5], [14], [15].

(f) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2]–[5], [14], [15]:

$$F'(x)=T(F(x)),$$

where T is a known continuous operator. Since $F'(x^*)=T(F(x^*))=T(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x)=e^x-1$. Then we can choose $T(x)=x+1$ and $x^*=0$.

3. Numerical examples. We present numerical examples where we compute the radii of the convergence balls.

EXAMPLE 3.1. Let $X = Y = \mathbb{R}$. Define a function F on $D = [1, 3]$ by

$$F(x) = \frac{2}{3}x^{2/3} - x. \tag{3.1}$$

Then $x^* = 9/4 = 2.25$, $F'(x^*)^{-1} = 2$, $L_0 = 1 < L = 2$, $p = 0.5$ and $K = 2(\sqrt{3} - 1)$, $r = 0.0411$, $\bar{r} = 0.0291$ and $r_A = 0.1837$.

EXAMPLE 3.2. Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0, 1)$. Define F on D for $v = (x, y, z)$ by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right). \tag{3.2}$$

Then the Fréchet derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0, 0, 0)$, $F'(x^*) = F'(x^*)^{-1} = \operatorname{diag}\{1, 1, 1\}$, $L_0 = e - 1 < L = K = e$, $p = 1$, $r = 0.0852$, $\bar{r} = 0.0852$ and $r_A = 0.3249$.

EXAMPLE 3.3. Let $X = Y = C[0, 1]$, the space of continuous functions defined on $[0, 1]$, be equipped with the max norm. Let $D = \overline{U}(0, 1)$. Define a function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3\, d\theta. \tag{3.3}$$

We have

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)\, d\theta \quad \text{for each } \xi \in D.$$

Moreover, $x^* = 0$, $L_0 = 7.5$, $L = 15$, $p = 1$ and $K = K(r) = 1 + 7.5r$.

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