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ON THE CONVERGENCE OF THE BACKWARD EULER ALGORITHM FOR THE MULTIDIMENSIONAL HEAT EQUATION

Abstract. The backward Euler algorithm for the multidimensional nonhomogeneous heat equation is analyzed, based on the finite element method. The existence and uniqueness of the numerical solution is investigated. Also, the convergence of the numerical solutions is studied.

1. Introduction. The multidimensional heat equation describes diffusion processes or heat transfer. Much progress has been made in developing more efficient finite difference and finite element algorithms. Some methods for numerical solving of the multidimensional heat equation are based on explicit finite differences [MG]. More efficient implicit methods such as alternating directions were studied in [N], [D]. Higher order split schemes in two or three dimensions were examined by Mitchell and Griffith [MG] and Gourlay [G].

Finite element splitting constructions were investigated by Fletcher [F] and Zienkiewicz, Taylor and Zhu [ZTZ]. Also, methods for determining whether or not the numerical solutions are indeed good approximations to the solutions were considered in [ZTZ].

Discretizations of the heat equation by θ-schemes in time and conforming finite elements in space were investigated in [V].

Methods based on combining the finite element method with backward Euler time discretization for the solution of diffusion problems on dynamically changing meshes were studied in [DK]. Error estimates for piecewise linear nonconforming finite element approximations of the heat equation

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in $\mathbb{R}^n$, $n = 2, 3$, using the backward Euler scheme were discussed in [NS]. In [LW] a weak Galerkin method was analyzed, based on totally discontinuous functions in approximation space. Error estimates in space and time were established.

In [T, Chapters 1, 2] the backward Euler algorithm was studied for numerical solution of the Dirichlet problem for the heat equation. An error estimate was obtained using the so-called elliptic or Ritz projection R_h onto the approximation space V_h.

In this paper, we investigate approximation by the finite element method of the Neumann–Dirichlet boundary value problem for the heat equation. We establish a convergence result based on the stability and approximation properties of the numerical scheme.

First, we state the classical problem for the multidimensional heat equation, then the variational problem is formulated. Further, we deal with the approximation problem. A theorem on existence and uniqueness of the numerical solution is presented. We continue with the numerical implicit scheme and study the convergence of the algorithm.

2. Statement of the problem. Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$, with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, and let $T > 0$ be fixed. The classical problem for the multidimensional heat equation is to find $u : [0, T] \times \bar{\Omega} \to \mathbb{R}$, $u \in C^1([0, T], C^2(\Omega))$, such that

$$\begin{aligned} &\frac{\partial u}{\partial t} - \alpha^2 \Delta u = f(t, x) \quad \text{ in } [0, T] \times \Omega, \\ &u|_{\Gamma_1} = g_1(t, x), \\ &\frac{\partial u}{\partial n}\bigg|_{\Gamma_2} = \operatorname{grad} u \cdot u|_{\Gamma_2} = g_2(t, x), \\ &u(0, x) = z(x) \quad \text{ in } \bar{\Omega} \end{aligned} \tag{2.1}$$

($n =$ the external normal to the boundary).

In the variational formulation of problem (2.1) suppose that

$$\begin{aligned} &f \in C([0, T], L^2(\Omega)), \\ &\tilde{g}_1 = \begin{cases} g_1 & \text{on } \Gamma_1 \\ 0 & \text{on } \Gamma_2 \end{cases} \quad \text{is in } C^1([0, T], H^{1/2}(\Gamma)), \\ &g_2 \in L^2([0, T], L^2(\Gamma_2)), \\ &z \in H^1(\Omega). \end{aligned}$$

This means there is a function $u_0 \in C^1([0, T], H^1(\Gamma))$ such that $u_0|_\Gamma = \tilde{g}_1$ (see [A]).

3. Main results. Consider the vector space $V=\{v\in H^1(\Omega):v|_{\Gamma_1}=0\}$, and let V^* be the dual space of V. Denote $\bar u=u-u_0$, where u is the solution of (2.1). It follows that

$$\bar u|_{\Gamma_1}=0,\qquad \bar u|_{\Gamma_2}=u|_{\Gamma_2}.$$

Integration on Ω and the Gauss formula lead to the variational problem: find $\bar u\in L^2([0,T],V)$ with $\bar u_t\in L^2([0,T),V^*)$ such that

$$\begin{aligned}(3.1)\quad &\int_\Omega \frac{\partial \bar u}{\partial t}v\,dx+\alpha^2\int_\Omega\sum_{i=1}^n\frac{\partial \bar u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx=\int_\Omega fv\,dx-\int_\Omega\frac{\partial u_0}{\partial t}v\,dx\\ &\quad-\alpha^2\int_\Omega\sum_{i=1}^n\frac{\partial u_0}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx+\alpha^2\int_{\Gamma_2}g_2v\,d\sigma,\qquad\text{a.e. } t\in(0,T),\ \forall v\in V.\end{aligned}$$

Problem (3.1) has a unique solution (see [DL]).

Further, we formulate the approximation problem by means of finite elements. Consider a triangulation T_h for Ω:

$$\Omega=\bigcup_{K\in T_h}K$$

where each K is an n-simplex or n-parallelepiped. Denote

$$W_h=\{u_h\in C(\bar\Omega):u_h|_K=\text{interpolation polynomial},\ \forall K\in T_h\},$$

which is included in $H^1(\Omega)$ ([C]).

It follows that

$$V_h:=\{v_h\in W_h:v_h|_{\Gamma_1}=0\}\subset V.$$

The norm in V_h and W_h will be the induced norm of $H^1(\Omega)$. Let $u_{h0}\in C^1([0,T],W_h)$ be defined by the values at the nodes of the triangulation:

$$u_{h0}(t,a_i)=\begin{cases}0 & \text{for } a_i\in\Omega\cup\Gamma_2,\\ g_1(t,a_i) & \text{for } a_i\in\Gamma_1.\end{cases}$$

Now we formulate the first approximation variational equation: find $\bar u_h\in C^1([0,T],V_h)$ such that

$$\begin{aligned}(3.2)\quad &\int_\Omega\frac{\partial\bar u_h}{\partial t}(t,x)v_h(x)\,dx+\alpha^2\int_\Omega\sum_{i=1}^n\frac{\partial\bar u_h}{\partial x_i}(t,x)\frac{\partial v_h}{\partial x_i}(x)\,dx\\ &=\int_\Omega f(t,x)v_h(x)\,dx-\int_\Omega\frac{\partial u_{h0}}{\partial t}(t,x)v_h(x)\,dx\\ &\quad-\alpha^2\int_\Omega\sum_{i=1}^n\frac{\partial u_{h0}}{\partial x_i}(t,x)\frac{\partial v_h}{\partial x_i}(x)\,dx+\alpha^2\int_{\Gamma_2}g_2(t,x)v_h(x)\,d\sigma\end{aligned}$$

for all $v_h \in V_h$ and $t \in (0,T)$. The initial condition is

$$\bar{u}_h(0,a_i) = \begin{cases} z(a_i) & \text{for } a_i \in \Omega \cup \Gamma_2, \\ 0 & \text{for } a_i \in \Gamma_1, \end{cases} \tag{3.3}$$

where a_i are the nodes of the triangulation.

THEOREM 3.1. *The variational problem* (3.2), (3.3) *has a unique solution.*

Proof. We denote by $\bar{a}_s$, $s = \overline{1,N}$, the nodes which are in $\Omega \cup \Gamma_2$.

The approximation space V_h is finite-dimensional; consider its basis $\{v_{hi}(x) : i = \overline{1,N}\}$ such that $v_{hi}(\bar{a}_s) = \delta_{is}$, $i,s = \overline{1,N}$. We deduce that

$$\bar{u}_h(t,x) = \sum_{i=1}^{N} \varphi_i(t) v_{hi}(x)$$

and $\bar{u}_h(t,\bar{a}_s) = \varphi_s(t)$, $s = \overline{1,N}$. It follows that $\varphi_s(0) = z(\bar{a}_s)$, $s = \overline{1,N}$. On the other hand, (3.2) yields

$$\begin{aligned} \sum_{i=1}^{N} \varphi_i'(t) \int_{\Omega} v_{hi}(x) v_{hj}(x)\,dx \\ + \alpha^2 \sum_{i=1}^{N} \varphi_i(t) \int_{\Omega} \sum_{k=1}^{n} \frac{\partial v_{hi}}{\partial x_k} \frac{\partial v_{hj}}{\partial x_k}\,dx = F_j(t), \qquad j = \overline{1,N}. \end{aligned} \tag{3.4}$$

Taking into consideration the inclusion $V_h \subset C(\bar{\Omega})$, we deduce that

$$\langle u,v \rangle := \int_{\Omega} u(x)v(x)\,dx$$

is a scalar product in V_h. Thus,

$$\det\left(\langle v_{hi}, v_{hj} \rangle\right)_{i,j=\overline{1,N}} \neq 0.$$

From (3.4) we infer that

$$A \begin{bmatrix} \varphi_1'(t) \\ \dots \\ \varphi_N'(t) \end{bmatrix} + \alpha^2 B \begin{bmatrix} \varphi_1(t) \\ \dots \\ \varphi_N(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ \dots \\ F_N(t) \end{bmatrix}. \tag{3.5}$$

Here A is a nonsingular matrix, so this provides a linear differential system of equations with initial conditions:

$$\begin{cases} \dfrac{d\varphi}{dt} = C\varphi(t) + \tilde{F}(t), & t \in [0,T], \\ \varphi_s(0) = z(\bar{a}_s), & s = \overline{1,N}. \end{cases} \tag{3.6}$$

This yields the existence and uniqueness of the solution of the variational problem (3.2), (3.3). ∎

Consider a partition of the interval $[0,T]$ with $t_m = mk$, where $k = T/M$, $M \in \mathbb{N}^*$. We proceed to the second variational problem: find $\bar{u}_{hk}^{(m+1)} \in V_h$ for $m = \overline{0, M-1}$ such that

$$\begin{aligned}(3.7)\quad & \int_\Omega \bar{u}_{hk}^{(m+1)}(x)v_h(x)\,dx + a^2k\int_\Omega \sum_{i=1}^n \frac{\bar{u}_{hk}^{(m+1)}}{\partial x_i}(x)\frac{\partial v_h}{\partial x_i}(x)\,dx \\ & = \int_\Omega \bar{u}_{hk}^{(m)}(x)v_h(x)\,dx + k\int_\Omega f(t_{m+1},x)v_h(x)\,dx \\ & \quad - k\int_\Omega \frac{\partial u_{h0}}{\partial t}(t_{m+1},x)v_h(x)\,dx - \alpha^2k\int_\Omega \sum_{i=1}^n \frac{\partial u_{h0}}{\partial x_i}(t_{m+1},x)\frac{\partial v_h}{\partial x_i}(x)\,dx \\ & \quad + \alpha^2k\int_{\Gamma_2} g_2(t_{m+1},x)v_h(x)\,d\sigma\end{aligned}$$

for all $v_h \in V_h$. The function $\bar{u}_{hk}^{(0)} \in V_h$ is defined by

$$\bar{u}_{hk}^{(0)}(a_i) = \begin{cases} z(a_i) & \text{for } a_i \in \Omega \cup \Gamma_2, \\ 0 & \text{for } a_i \in \Gamma_1, \end{cases}$$

where a_i are the nodes of the triangulation. It follows from the Lax–Milgram lemma that the variational equation (3.7) has a unique solution.

Further, we shall prove that the solution $\bar{u}_{hk}^{(m+1)} \in V_h$ of (3.7) is an approximation for $\bar{u}_h(t_{m+1})$, where $\bar{u}_h$ is the solution of the variational problem (3.2), (3.3). This will follow from the approximation and stability properties [M].

Consider a certain construction of the space V_h by means of interpolation polynomials:

$$\bar{u}_{hk}^{(m)}|_K = [DF_K(\bar{u}_{hk}^{(m)})]_{1\times s} \cdot [p(x)]_{s\times 1}$$

where

$$[DF_K(\bar{u}_{hk}^{(m)})]_{1\times s} = [\bar{u}_{hk}^{(m)}(a_1^K), \ldots, \bar{u}_{hk}^{(m)}(a_s^K)]$$

are the degrees of freedom of $\bar{u}_{hk}^{(m)}$ in K and

$$[p(x)]_{s\times 1} = [p_1(x), \ldots, p_s(x)]^t$$

is a basis in the local polynomial space P_K.

Inserting these in (3.7) we obtain

$$\begin{aligned}& \sum_{K\in T_h} [DF_K(v_h)]_{1\times s} \int_K [p(x)]_{s\times 1}[p(x)]_{1\times s}^t\,dx\,[DF_K(\bar{u}_{hk}^{(m+1)})]_{s\times 1}^t \\ & \quad + a^2k\sum_{K\in T_h} [DF_K(v_h)]_{1\times s} \int_K \sum_{i=1}^n \left[\frac{\partial p}{\partial x_i}\right]_{s\times 1} \left[\frac{\partial p}{\partial x_i}\right]_{1\times s}^t dx\,[DF_K(\bar{u}_{hk}^{(m+1)})]_{s\times 1}^t\end{aligned}$$

$$= \sum_{K\in T_h} [DF_K(v_h)]_{1\times s} \int_K [p(x)]_{s\times 1}[p(x)]^t_{1\times s}\, dx\, [DF_K(\bar u^{(m)}_{hk})]^t_{s\times 1}$$
$$+ k \sum_{K\in T_h} [DF_K(v_h)]_{1\times s} \int_K [p(x)]_{s\times 1} f(t_{m+1},x)\, dx$$
$$- k \sum_{K\in T_h} [DF_K(v_h)]_{1\times s} \int_K [p(x)]_{s\times 1}[p(x)]^t_{1\times s}\, dx\, [DF_K \frac{\partial u_{h0}}{\partial t}(t_{m+1})]^t_{s\times 1}$$
$$- \alpha^2 k \sum_{K\in T_h} [DF_K(v_h)]_{1\times s} \int_K \sum_{i=1}^n \left[\frac{\partial p}{\partial x_i}\right]_{s\times 1} \left[\frac{\partial p}{\partial x_i}\right]^t_{1\times s} dx\, [DF_K(u^{(m+1)}_{h0})]^t_{s\times 1}$$
$$+ \alpha^2 k \sum_{I\in T'_h} [DF_l(v_h)]_{1\times r} \cdot \int_I [\bar p(x)]_{r\times 1} g_2(t_{m+1},x)\, d\sigma.$$

Here

$$\bigcup_{I\in T'_h} I = \varGamma_2,$$

$$v_h(x)|_I = [v_h(a^I_1),\ldots,v(a^l_r)] \cdot \begin{bmatrix} \bar p_1(x) \\ \ldots \\ \bar p_r(r) \end{bmatrix} \qquad \text{for } v_h \in V_h,\ I \in T'_h.$$

The process of numerical integration and assemblage produces a linear algebraic system:

$$\begin{aligned}(3.8)\quad (A+\alpha^2 kB)[DF(\bar u^{(m+1)}_{hk})]^t \\ = A\Big([DF(\bar u^{(m)}_{hk})]^t - k\Big[DF\Big(\frac{\partial u_{h0}}{\partial t}(t_{m+1})\Big)\Big]^t\Big) \\ + k[F(t_{m+1})] + \alpha^2 k[G_2(t_{m+1})] - \alpha^2 kB[DF(u_{h0}(t_{m+1}))]^t,\end{aligned}$$

for all $m \in \mathbb{N}$.

The function $\bar u^{(0)}_{hk}$ was defined before. The matrix A (mass matrix) and B (stiffness matrix) in (3.8) are symmetric and positive definite, so that the matrix $R = A + \alpha^2 kB$ is also symmetric and positive definite. We have $[DF(\bar u^{(m)}_{hk}) = [\bar u^{(m)}_{hk}(\bar a_1),\ldots,\bar u^{(m)}_{hk}(\bar a_N)] \in \mathbb{R}^N$, where N is the number of the unknown degrees of freedom for $\bar u^{(m)}_{hk}$. Further, we define the space $U_{hk} = \{u_{hk} : u_{hk} : \{t_0,t_1,\ldots,t_M\} \to V_h\}$ and the operator $L_{hk} : U_{hk} \to \mathbb{R}^{(M+1)N}$ by

$$\begin{aligned}L_{hk}(u_{hk}) &= L_{hk}\big((u_{hk}(t_m))_{m=\overline{0,M}}\big) \\ &= \Big(\frac{1}{k}A\big([DF(u_{hk}(t_{m+1}))]^t - [DF(u_{hk}(t_m))]^t\big) \\ &\quad + \alpha^2 B[DF(u_{hk}(t_{m+1}))]^t|_{m=\overline{0,M-1}}, [DF(u_{hk}(t_0))]^t\Big).\end{aligned}$$

Consider the function $\bar{u}_{hk} \in U_{hk}$, $\bar{u}_{hk}(t_m) := \bar{u}_{hk}^{(m)}$ for $m = \overline{0, M}$, where $\bar{u}_{hk}^{(m)}$ is the unique solution of (3.7). We deduce that (3.8) can be rewritten as

$$L_{hk}\bar{u}_{hk} = \tilde{F}_{hk} \tag{3.9}$$

where

$$\tilde{F}_{hk} = \big((\tilde{F}^{(m+1)})_{m=\overline{0,M-1}},\ z(\bar{a}_1), \ldots, z(\bar{a}_N)\big),$$

$$\tilde{F}^{(m+1)} = -A\bigg[DF\bigg(\frac{\partial u_{h0}}{\partial t}(t_{m+1})\bigg)\bigg]^t + [F(t_{m+1})]^t + \alpha^2[G_2(t_{m+1})]^t - \alpha^2 B[DF(u_{h0}(t_{m+1}))]^t$$

for $m = \overline{0, M-1}$.

We consider the following norm over the space U_{hk}:

$$\|u_{hk}\| = \sup\{\|[DF(u_{hk}(t_m))]\| : m = \overline{0, M}\}$$

where $\|[DF(u_{hk}(t_m))]\|$ will be defined later so as to obtain stability.

THEOREM 3.2. *Let $\bar{u}_h \in C^2([0,T], V_h)$ be the solution of the variational problem* (3.2), (3.3). *Then $L_{hk}\bar{u}_h = \tilde{F}_{hk} + \delta_k$ with $\lim_{k\to 0}\|\delta_k\| = 0$.*

Proof. By Taylor's formula in integral form we infer that

$$\frac{\partial \bar{u}_h}{\partial t}(t_{m+1}, x) = \frac{\bar{u}_h(t_{m+1}, x) - \bar{u}_h(t_m, x)}{k} + k\int_0^1 \zeta \frac{\partial^2 \bar{u}_h}{\partial t^2}(t_m + \zeta k, x)\, d\zeta. \tag{3.10}$$

From (3.2) and (3.10) it follows that

$$\begin{aligned}
&\int_\Omega \frac{\bar{u}_h(t_{m+1}, x) - \bar{u}_h(t_m, x)}{k} v_h(x)\, dx \\
&\qquad + \alpha^2 \int_\Omega \sum_{i=1}^n \frac{\partial \bar{u}_h}{\partial x_i}(t_{m+1}, x) \frac{\partial v_h}{\partial x_i}(x)\, dx \\
&\quad = \int_\Omega f(t_{m+1}, x) v_h(x)\, dx - \int_\Omega \frac{\partial u_{h0}}{\partial t}(t_{m+1}, x) v_h(x)\, dx \\
&\qquad - \alpha^2 \int_\Omega \sum_{i=1}^n \frac{\partial u_{h0}}{\partial x_i}(t_{m+1}, x)\frac{\partial v_h}{\partial x_i}(x)\, dx + \alpha^2 \int_{\Gamma_2} g_2(t_{m+1}, x) v_h(x)\, d\sigma \\
&\qquad - k\int_\Omega \int_0^1 \zeta \frac{\partial^2 \bar{u}_h}{\partial t^2}(t_m + \zeta k, x) v_h(x)\, d\zeta\, dx
\end{aligned} \tag{3.11}$$

for $m = \overline{0, M-1}$. Equation (3.11) yields

$$\begin{aligned}
&\frac{1}{k}A([DF(\bar{u}_h(t_{m+1}))]^t - [DF(\bar{u}_h(t_m))]^t) + \alpha^2 B[DF(\bar{u}_h(t_{m+1}))]^t \\
&\qquad = \tilde{F}^{(m+1)} - kA\int_0^1 \zeta\bigg[DF\bigg(\frac{\partial^2 \bar{u}_h}{\partial t^2}(t_m + \zeta k)\bigg)\bigg]^t d\zeta.
\end{aligned}$$

It follows that

$$L_{hk}\bar{u}_h = \tilde{F}_{hk} + \left(\left(-kA\int_0^1 \zeta\left[DF\left(\frac{\partial^2 \bar{u}_h}{\partial t^2}(t_m+\zeta k)\right)\right]^t d\zeta\right)_{m=\overline{0,M-1}}, 0_N\right).$$

Since $\partial^2\bar{u}_h/\partial t^2 \in C([0,T],V_h)$ with $V_h \in C(\bar{\Omega})$, we deduce that

$$\|\delta_k\| = \sup\left\{\left\|-kA\int_0^1 \zeta\left[DF\left(\frac{\partial^2 \bar{u}_h}{\partial t^2}(t_m+\zeta k)\right)\right]^t d\zeta\right\| : m = \overline{0, M-1}\right\}$$
$$\leq c\|A\|k. \ \blacksquare$$

Equation (3.9) can be rewritten as

$$\begin{aligned} &[DF(\bar{u}_{hk}^{(m+1)})]^t = R^{-1}A[DF(\bar{u}_{hk}^{(m)})]^t + kR^{-1}\tilde{F}^{(m+1)}, \\ &[DF(\bar{u}_{hk}^{(0)})] = [z(\bar{a}_1), \ldots, z(\bar{a}_N)]. \end{aligned} \tag{3.12}$$

The stability of the approximation scheme arises from a proposition referring to symmetric, positive definite matrices which we shall prove later. For this purpose we apply a result due to Householder:

PROPOSITION 3.1. *Let A be a symmetric, nonsingular matrix, and $A = M - N$ with M a nonsingular matrix. Suppose the symmetric matrix $Q = M + M^t - A$ is positive definite. Then the following assertions are equivalent:*

(1) *the spectral radius $\rho(M^{-1}N) < 1$;*
(2) *A is a positive definite matrix.*

We now prove the following result:

PROPOSITION 3.2. *Let A, B be symmetric, positive definite matrices and $\alpha > 0$. Then there is a matrix norm such that*

$$\|(A+\alpha B)^{-1}A\| < 1.$$

Proof. We know that $A + \alpha B$ is a symmetric, positive definite, nonsingular matrix. Define now $M = A + \alpha B$, $N = A$ and $\tilde{A} = M - N = \alpha B$. We infer that $\tilde{A}$ is symmetric, positive definite, and nonsingular, M is nonsingular and $Q = M + M^t - \tilde{A} = 2A + \alpha B$ is symmetric and positive definite. Proposition 3.1 yields

$$\rho((A+\alpha B)^{-1}A) < 1. \tag{3.13}$$

From a known result in numerical analysis, (3.13) is equivalent to the existence of a norm such that $\|(A+\alpha B)^{-1}A\| < 1$. $\blacksquare$

THEOREM 3.3. *The numerical approximation scheme* (3.9) *is unconditionally stable.*

Proof. Consider a perturbation of the scheme (3.9),

$$H_k = \tilde{F}_k + \varepsilon_k = \left((\tilde{F}^{(m+1)} + \varepsilon_k^{(m+1)})_{m=\overline{0,M-1}}, z(\bar{a}_1) + \varepsilon_1, \ldots, z(\bar{a}_N) + \varepsilon_N\right)$$

with $\varepsilon_k^{(0)} = (\varepsilon_1, \dots, \varepsilon_N)$ and $\varepsilon_k^{(m+1)} \in \mathbb{R}^N$, $m = \overline{0, M-1}$. From the relation $L_{hk} w_k = H_k$ it follows that

$$\begin{aligned} &[DF(w_k^{(m+1)})]^t = R^{-1}A[DF(w_k^{(m)})]^t + kR^{-1}\tilde{F}^{(m+1)} + kR^{-1}\varepsilon_k^{(m+1)}, \\ &\hfill m = \overline{0, M-1}, \\ &[DF(w_k^{(0)})] = (z(\bar{a}_1) + \varepsilon_1, \dots, z(\bar{a}_N) + \varepsilon_N). \end{aligned} \tag{3.14}$$

Denote

$$\begin{aligned} v_k^{(m)} &= [DF(w_k^{(m)})]^t - [DF(\bar{u}_{hk}^{(m)})]^t \in \mathbb{R}^N \quad \text{for } m = \overline{0, M}, \\ L &= \|R^{-1}A\|. \end{aligned}$$

Considering the norm provided by Proposition 3.2, we deduce $L < 1$. Relations (3.12) and (3.14) yield

$$\begin{aligned} |v_k^{(m+1)}\| &= \|R^{-1}Av_k^{(m)} + kR^{-1}\varepsilon_k^{(m+1)}\| \\ &\le L\|v_k^{(m)}\| + k\|R^{-1}\|\,\|\varepsilon_k^{(m+1)}\| \\ &\le L(L\|v_k^{(m-1)}\| + k\|R^{-1}\|\,\|\varepsilon_k^{(m)}\|) + k\|R^{-1}\|\,\|\varepsilon_k\| \\ &\le L^2\|v_k^{(m-1)}\| + k\|R^{-1}\|(1+L)\|\varepsilon_k\| \le \dots \le \\ &\le L^{m+1}\|v_k^{(0)}\| + k\|R^{-1}\|(1 + L + \dots + L^m)\|\varepsilon_k\| \end{aligned} \tag{3.15}$$

for $m = \overline{0, M-1}$. We have

$$\|v_k^{(0)}\| = \|[DF(w_k^{(0)})] - [DF(\bar{u}_{hk}^{(0)})]\| = \|(\varepsilon_1, \dots, \varepsilon_N)\| \le \|\varepsilon_k\|. \tag{3.16}$$

Substituting (3.16) in (3.15) and taking into consideration that $L < 1$, we obtain

$$\|v_k^{(m+1)}\| \le [1 + \|R^{-1}\|k(m+1)]\|\varepsilon_k\| \le [1 + \|(A + \alpha^2 kB)^{-1}\|T]\|\varepsilon_k\| \tag{3.17}$$

for $m = \overline{0, M-1}$.

We have $\lim_{k\to 0} \|(A + \alpha^2 kB)^{-1}\| = \|A^{-1}\|$, which yields

$$\|(A + \alpha^2 kB)^{-1}\| \le c_1 \quad \text{for } k \le k_0. \tag{3.18}$$

Now (3.16)–(3.18) yield

$$\|\bar{u}_{hk} - w_k\| = \sup_{m=\overline{0,M}} \|v_k^{(m)}\| \le c_2\|\varepsilon_k\|$$

with $c_2 > 0$ a constant independent of k. ∎

Teorems 3.2 and 3.3 prove the convergence:

$$\|\bar{u}_{hk} - \bar{u}_h\| = \sup_{m=\overline{0,M}} \|[DF(\bar{u}_{hk}^{(m)})] - [DF(\bar{u}_h(t_m))]\| \le c_3 k \to 0.$$

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