APPLICATIONES MATHEMATICAE
41,4 (2014), pp. 361–384

Yuji Liu (Guangdong)
Pinghua Yang (Shijiazhuang)

IVPS FOR SINGULAR MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE BASE POINTS AND APPLICATIONS

Abstract. The purpose of this paper is to study global existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations. By constructing a special Banach space and employing fixed-point theorems, some sufficient conditions are obtained for the global existence and uniqueness of solutions of this kind of equations involving Caputo fractional derivatives and multiple base points. We apply the results to solve the forced logistic model with multi-term fractional derivatives.

1. Introduction. Fractional differential equations (FDEs for short) are generalizations of ordinary differential equations to arbitrary noninteger orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [12]. Fractional differential equations therefore find numerous applications in different branches of physics, chemistry and biological sciences, including visco-elasticity, feedback amplifiers, electrical circuits, electro-analytical chemistry, fractional multipoles and neuron modelling [15]. Many excellent books and monographs on this field are available [5, 6, 7, 9, 14, 11, 10, 16, 17].

In the literature, ${}^{c}D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0$ is known as a *single term equation*. In certain cases, we find equations containing more than one dif-

2010 *Mathematics Subject Classification*: 92D25, 34A37, 34K15.
Key words and phrases: initial value problem, multi-term fractional differential equation, single base point FDE, multiple base points FDE, Caputo fractional derivative, fractional order logistic model, fixed-point theorem.

DOI: 10.4064/am41-4-7

ferential term. These are called *multi-term equations*. A classical example is the so-called *Basset equation*

$$AD^1_{0+}y(x) + bD^n_{0+}y(x) + cy(x) = f(x), \quad y(0) = y_0,$$

where $0 < n < 1$. This equation is most frequently, but not exclusively, used with $n = 1/2$. It describes the forces that occur when a spherical object sinks in a (relatively dense) incompressible viscous fluid (see [1, 10]).

In the left and right fractional derivatives $D^\alpha_{a^+}x$ and $D^\alpha_{b^-}x$, a is called a *left base point* and b a *right base point*. Both a and b are called base points of fractional derivatives. An FDE containing more than one base point is called a *multiple base point FDE*. An FDE containing only one base point is called a *single base point FDE*.

In [2], the authors studied the initial value problem (IVP for short) for the fractional functional differential equation with one base point

$$\begin{cases} {}^c\!D^\alpha_{0+}x(t) = f(t, x_t), & t \in (t_0, \infty),\ t_0 \geq 0,\ 0 < \alpha < 1, \\ x(t) = \phi(t), & t \in [t_0 - \tau, t_0], \end{cases}$$

where ${}^c\!D^\alpha_{0+}$ is the standard Caputo fractional derivative at the base point $t = 0$, $f : J \times C^0([-\tau, 0], \mathbb{R}) \times \mathbb{R}$ with $J = (t_0, \infty)$ is a given function, $\tau > 0$ and $\phi \in C^0([t_0 - \tau, t_0], \mathbb{R})$; if $x \in C^0([t_0 - \tau, \infty), \mathbb{R})$, then for any $t \in [t_0, \infty)$, we define $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0]$.

In [8], the authors studied the global existence of solutions of the initial value problem for the fractional functional differential equation with one base point

$$\begin{cases} D^\alpha_{0+}x(t) = f(t, x(t)), & t \in (0, \infty),\ 0 < \alpha \leq 1, \\ \lim\limits_{t \to 0} t^{1-\alpha}x(t) = u_0, \end{cases}$$

where D^α_{0+} is the standard Riemann–Liouville fractional derivative at the base point $t = 0$, and $f : J \times \mathbb{R} \to \mathbb{R}$ with $J = (0, \infty)$ is a given function.

In this paper, we study the following IVP for the nonlinear multi-term fractional differential equation on the half line:

$$(1) \qquad \begin{cases} {}^c\!D^\alpha_* x(t) = q(t)f(t, x(t), {}^c\!D^p_* x(t)), & t \in (0, \infty), \\ x(0) = x_0, \end{cases}$$

where $x_0 \in \mathbb{R}$, $\alpha \in (0, 1]$, $0 < p < \alpha$, $q : (0, \infty) \to \mathbb{R}$ has the property that there exists $l > -\alpha$ such that $|q(t)| \leq t^l$ for all $t \in (0, \infty)$, q may be singular at $t = 0$, ${}^c\!D_*$ is the standard Caputo fractional derivative at the base points $t = t_k$ $(k = 1, 2, \ldots)$, $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_{k\to\infty} t_k = \infty$ and $\lambda_0 = \inf_{k=0,1,2,\ldots}[t_{k+1} - t_k] > 0$, i.e., ${}^c\!D^\alpha_*|_{(t_k, t_{k+1}]}u(t) = {}^c\!D^\alpha_{t_k^+}u(t)$ for all $t \in (t_k, t_{k+1}]$, and $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function.

Malthusian Geometrical Law is expressed as $N'(t) = rN(t)$, where $N(t)$ is the population at time t and r is the proportionality constant. When one considers the case where the growth of the population in any environment may be stopped due to the density of the population, this model is modified to the nonlinear logistic model $N'(t) = rN(t)(1 - N(t)/\pi)$, where π is the maximum that a given amount of food can support. A generalization of the nonlinear logistic model is represented by $N'(t) = rN(t)[1 - (N(t)/\pi)^\alpha]/\alpha$; for $\alpha \to 0$ this model is known as the Gompertz model and can be found in the actuarial literature and in the mortality analysis of elderly persons [3].

In [4], Das, Gupta and Vishal presented the following fractional-order logistic model (*Das model*):

$$D_{0^+}^{\beta} N(t) = \frac{r}{\alpha} N(t)\left[1 - \left(\frac{N(t)}{\pi}\right)^{\alpha}\right], \qquad 0 < \beta \le 1. \tag{2}$$

One purpose of this paper is to establish sufficient conditions for the existence and uniqueness of solutions of (1) (the definitions of positive solutions can be found in Section 2). Another purpose is to establish sufficient conditions for the existence of solutions of the following forced logistic models with multi-term fractional derivatives:

$$\begin{cases} {}^{c}D_{*}^{\alpha} x(t) = q(t)[r(t) + a(t)x(t) - b(t)(x(t))^{\delta} + c(t)x(t)(D_{*}^{p} x(t))^{\mu}], \\ \hfill t \in (0,\infty), \\ x(0) = x_0, \end{cases} \tag{3}$$

where $\alpha \in (0,1]$ and $p \in (0,\alpha)$, $0 < t_1 < t_2 < \cdots$ with $\lim_{k\to\infty} t_k = \infty$ and $\lambda_0 = \inf_{k=0,1,2,\dots}[t_{k+1} - t_k] > 0$, $\delta > 1, \mu > 0$, $a, b, c, r : (0,\infty) \to \mathbb{R}$ are continuous functions and r is called a *forced term*, $x_0 \in \mathbb{R}$.

The remainder of this paper is organized as follows: Preliminary results are given in Section 2, the main results are presented in Sections 3 and 4, and an application is shown in Section 5.

2. Preliminary results. For the convenience of the reader, we present here the necessary definitions from fixed point theory and fractional calculus theory. They can be found in [13, 14, 16]. Denote the Gamma function and Beta function respectively by

$$\Gamma(\alpha_1) = \int_0^\infty s^{\alpha_1 - 1} e^{-s}\, ds, \qquad \alpha_1 > 0,$$

$$\mathbf{B}(\alpha_2, \beta_2) = \int_0^1 (1-x)^{\alpha_2 - 1} x^{\beta_2 - 1}\, dx, \qquad \alpha_2, \beta_2 > 0.$$

DEFINITION 2.1 ([14]). Let $c \ge 0$. The *Riemann–Liouville fractional integral* of order $\alpha > 0$ of a function $f : (c,\infty) \to \mathbb{R}$ is given by

$$I_{c^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right-hand side exists.

Definition 2.2 ([14]). Let $c \geq 0$. The *Caputo derivative* of order α for a function $f : (c, \infty) \to \mathbb{R}$ is defined as

$$ {}^{c}D_{c^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \tag{4} $$

for $n-1 \leq \alpha < n$, $n \in \mathbb{N}$. If $0 < \alpha \leq 1$, then

$$ D_{c^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t (t-s)^{-\alpha} f^{(1)}(s)\, ds. \tag{5} $$

Obviously, the Caputo derivative of a constant is equal to zero.

Lemma 2.1 ([14]). *For $\alpha > 0$, the general solution of the fractional differential equation ${}^{c}D_{c^+}^{\alpha} x(t) = 0$ is given by $x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, $n-1 < \alpha \leq n$.*

Definition 2.3. A continuous function $x : [0, \infty) \to \mathbb{R}$ is said to be a *solution* of the IVP (1) if x satisfies the differential equation ${}^{c}D_{t_k^+}^{\alpha} x(t) = q(t) f(t, x(t), {}^{c}D_{t_k^+}^{p} x(t))$ on $(t_k, t_{k+1}]$ and $x(0) = x_0$.

Choose $\sigma > \alpha + l$ and $\mu > \sigma$. Let

$$ X = \left\{ x \in C^0([0,\infty), \mathbb{R}) : \begin{array}{l} {}^{c}D_*^{p} x|_{(t_k, t_{k+1}]} \in C^0((t_k, t_{k+1}]),\ k = 0, 1, 2, \ldots, \\ \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} x(t) \text{ is bounded on } (0, \infty), \\ \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} {}^{c}D_*^{p} x(t) \text{ is bounded on } (0, \infty). \end{array} \right\} $$

For $x \in X$, define

$$ \|x\| = \max \left\{ \sup_{t \in (0,\infty)} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} |x(t)|,\ \sup_{t \in (0,\infty)} \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} |{}^{c}D_*^{p} x(t)| \right\}. $$

It is easy to show that X is a real Banach space.

Definition 2.4. $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is called a *Carathéodory function* if it satisfies the following assumptions:

(i) $(t, x, y) \to f\big(t, \frac{(1+t)(1+t^{\mu})}{t^{\sigma-\alpha-l}} x, \frac{1+t^{\mu}}{t^{p+\sigma-\alpha-l}} y\big)$ is continuous on $[0, \infty) \times \mathbb{R}^2$;

(ii) for each $r > 0$ there exists a constant $M_r > 0$ such that $|x|, |y| \leq r$ imply

$$ \left| f\left(t, \frac{(1+t)(1+t^{\mu})}{t^{\sigma-\alpha-l}} x, \frac{1+t^{\mu}}{t^{p+\sigma-\alpha-l}} y\right) \right| \leq M_r, \qquad t \in [0, \infty). $$

If $b > a > 0$, then we have

$$ \sup_{t \in (0,\infty)} \frac{t^a}{1+t^b} = \frac{1}{b} a^{a/b} (b-a)^{(b-a)/b} =: M_{a,b}. \tag{6} $$

LEMMA 2.2. *Suppose that f is a Carathéodory function, and let $x \in X$. Then $y \in X$ is a solution of*

$$(7)\qquad \begin{cases} {}^cD_*^{\alpha} y(t) = q(s) f(t, x(t), {}^cD_*^p x(t)), & t \in (0,\infty), \\ y(0) = x_0, \end{cases}$$

if and only if $y \in X$ is a solution of the fractional integral equation

$$(8)\qquad y(t) = \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD_{t_k^+}^p x(s))\, ds + x_0 + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD_{t_{j-1}^+}^p x(s))\, ds, \quad t \in (t_k, t_{k+1}],\ k = 0,1,2,\ldots.$$

Proof. As $x \in X$, there exists $r > 0$ such that

$$\max\left\{ \sup_{t\in(0,\infty)} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} |x(t)|, \sup_{n=0,1,2,\ldots} \sup_{t\in(t_k,t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} |{}^cD_{t_k^+}^p x(t)| \right\} = r.$$

Since f is a Carathéodory function, there exists $M_r \geq 0$ such that

$$\begin{aligned} &|f(t, x(t), {}^cD_{t_k^+}^p x(t))| \\ &\quad = \left| f\left(t, \frac{(1+t)(1+t^{\mu})}{t^{\sigma-\alpha-l}} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} x(t), \frac{1+t^{\mu}}{t^{p+\sigma-\alpha-l}} \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} {}^cD_{t_k^+}^p x(t)\right)\right| \\ &\quad \leq M_r, \qquad t \in [0,\infty). \end{aligned}$$

Assume y satisfies (7). Then

$${}^cD_*^{\alpha} y(t) = q(t) f(t, x(t), {}^cD_*^p x(t)), y(0) = x_0.$$

By using Lemma 2.1, we can write the solution of (7) as

$$y(t) = \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD_{t_k^+}^p x(s))\, ds + c_k$$

for $t \in (t_k, t_{k+1}]$, $k = 0,1,2,\ldots$. We see that

$$\begin{aligned} &\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD_{t_k^+}^p x(s))\, ds \right| \\ &\qquad \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^l M_r\, ds = M_r t^{\alpha+l} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw \to 0, \qquad t \to 0. \end{aligned}$$

From $y(0) = x_0$, we get $c_0 = x_0$. Since

$$\left|\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_k^+}x(s))\,ds\right|\le M_r\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^l\,ds$$
$$=M_rt^{\alpha+l}\int_{t_k/t}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\to 0\quad\text{as } t\to t_k^+,\ k=1,2,\dots,$$

we get

$$c_k-\left(\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_{k-1}^+}x(s))\,ds+c_{k-1}\right)=0.$$

It follows that

$$c_k=c_{k-1}+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_{k-1}^+}x(s))\,ds$$
$$=x_0+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_{j-1}^+}x(s))\,ds,\qquad k=0,1,2,\dots.$$

We have the following form of the solution:

$$y(t)=\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_k^+}x(s))\,ds+x_0$$
$$+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_{j-1}^+}x(s))\,ds,$$
$$t\in(t_k,t_{k+1}],\ k=0,1,2,\dots.$$

Hence y satisfies (8). We need to prove that $y\in X$. In fact, we have

$${}^cD^p_{t_k^+}y(t)=\int_{t_k}^{t}\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}q(s)f(s,x(s),{}^cD^p_{t_k^+}x(s))\,ds.$$

It is easy to see that

$$y\in C^0([0,\infty)),\qquad {}^cD^p_{t_k^+}y|_{(t_k,t_{k+1}]}\in C^0(t_k,t_{k+1}],\qquad k=0,1,2,\dots.$$

Furthermore, for $t\in(t_k,t_{k+1}]$ we have

$$\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|y(t)|$$
$$=\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}\left|\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_k^+}x(s))\,ds+x_0\right.$$
$$\left.+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^cD^p_{t_{j-1}^+}x(s))\,ds\right|$$

$$
\begin{aligned}
&\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_r s^l\,ds + M_{\sigma-\alpha-l,\mu}|x_0|\\
&\quad+\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}M_r s^l\,ds\\
&\le M_r\frac{t^{\sigma}}{(1+t)(1+t^{\mu})}\int_{t_k/t}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw + M_{\sigma-\alpha-l,\mu}|x_0|\\
&\quad+\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}M_r\sum_{j=1}^{k}t_j^{\alpha+l}\int_{t_{j-1}/t_j}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\\
&\le M_r M_{\sigma,\mu}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw + M_{\sigma-\alpha-l,\mu}|x_0|\\
&\quad+M_r\sum_{j=1}^{k}\frac{t^{\sigma-\alpha-l}t_j^{\alpha+l}}{(1+t)(1+t^{\mu})}\int_{t_{j-1}/t_j}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\\
&\le M_r M_{\sigma,\mu}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} + M_{\sigma-\alpha-l,\mu}|x_0|\\
&\quad+M_r\sum_{j=1}^{k}\frac{t^{\sigma-\alpha-l}t_j^{\alpha+l}}{t^{\mu+1}}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\\
&\le M_r M_{\sigma,\mu}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} + M_{\sigma-\alpha-l,\mu}|x_0| + M_r\sum_{j=1}^{k}\frac{t_j^{\alpha+l}}{t_j^{\mu+1-\sigma+\alpha+l}}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}\\
&\le M_r M_{\sigma,\mu}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} + M_{\sigma-\alpha-l,\mu}|x_0| + M_r\sum_{j=1}^{k}\frac{1}{t_j^{\mu+1-\sigma}}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

Since $t_j - t_{j-1} \ge \lambda_0$ and $t_0 = 0$, we get $t_j \ge j\lambda_0$ for all $j = 0, 1, 2, \ldots$. Then

$$
\begin{aligned}
\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|y(t)| &\le M_r M_{\sigma,\mu}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} + M_{\sigma-\alpha-l,\mu}|x_0|\\
&\quad+M_r\frac{1}{\lambda_0^{\mu+1-\sigma}}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}\sum_{j=1}^{\infty}\frac{1}{j^{\mu+1-\sigma}},\, t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots.
\end{aligned}
$$

So

$$
\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|y(t)| \text{ is bounded on } (0,\infty).
$$

For $t \in (t_k, t_{k+1}]$ we have

$$\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^{c}D^{p}_{t_k^+}y(t)| = \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\Big|\int_{t_k}^{t}\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds\Big|$$
$$\le \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\int_{t_k}^{t}\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}M_r s^l\,ds = \frac{M_r t^{\sigma}}{1+t^{\mu}}\int_{t_k/t}^{1}\frac{(1-w)^{\alpha-p-1}}{\Gamma(\alpha-p)}w^l\,dw$$
$$\le M_r M_{\sigma,\mu}\frac{\mathbf{B}(\alpha-p,l+1)}{\Gamma(\alpha-p)} < \infty, \quad t\in(t_k,t_{k+1}], k=0,1,2,\ldots.$$

So

$$\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^{c}D^{p}_{*}y(t)| \text{ is bounded on } (0,\infty).$$

It follows that $y \in X$.

Conversely, assume that y satisfies (8). By a direct computation, it follows that $y \in X$ satisfies the system (7). This completes the proof of the lemma. ∎

3. Main theorems. We are now in a position to prove the existence and uniqueness of solutions of (1). Let us define an operator T on X by

$$(9)\qquad (Tx)(t) = \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds + x_0$$
$$+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^{c}D^{p}_{t_{j-1}^+}x(s))\,ds,$$
$$t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots.$$

LEMMA 3.1. *Suppose that f is a Carathéodory function and*

$$\lambda_0 := \inf_{k=1,2,\ldots}(t_k - t_{k-1}) > 0.$$

Then

(i) *$T: X\to X$ is well defined;*
(ii) *the fixed points of the operator T coincide with the solutions of* (1)*;*
(iii) *$T: X\to X$ is completely continuous.*

Proof. (i) For $x\in X$, we get

$$r = \max\Big\{\sup_{t\in(0,\infty)}\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|x(t)|,$$
$$\sup_{k=0,1,2,\ldots}\ \sup_{t\in(t_k,t_{k+1}]}\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^{c}D^{p}_{t_k^+}x(t)|\Big\} < \infty.$$

Since f is a Carathéodory function, there exists $M_r > 0$ such that

$$|f(t, x(t), {}^cD^p_{t_k^+}x(t))| \le M_r, \quad t \in [0, \infty).$$

It is easy to show that

$$Tx \in C^0([0,\infty)), \quad {}^cD^p_{t_k^+}Tx|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}], \quad k = 0, 1, 2, \dots.$$

As in Lemma 2.2, we can prove that

$$\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)}(Tx)(t) \text{ is bounded}$$

and

$$\left\{\sup_{t\in(t_k,t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu}\, {}^cD^p_{t_k^+}(Tx)(t)\right\}_{k=0}^{\infty} \text{ is bounded.}$$

Hence $Tx \in X$. Thus $T : X \to X$ is well defined.

(ii) It follows from Lemma 2.2 that the fixed points of T coincide with the solutions of (1).

(iii) To prove that T is completely continuous, we must show that

- T is continuous,
- T maps bounded subsets of X to bounded sets,
- T maps bounded subsets of X to relatively compact sets.

The proof is divided into five steps.

STEP 1. We prove that T is continuous.

Let $y_n \in X$ with $y_n \to y_0$ as $n \to \infty$. We will prove that $Ty_n \to Ty_0$ as $n \to \infty$. It is easy to see that there exists $r > 0$ such that

$$\sup_{t\in(0,\infty)} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)}|y_n(t)|, \sup_{k=0,1,2,\dots} \sup_{t\in(t_k,t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu}|{}^cD^p_{t_k^+}y_n(t)| \le r < \infty,$$
$$n = 0, 1, 2, \dots,$$

and

$$(10)\quad \begin{aligned} &\sup_{t\in(0,\infty)} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)}|y_n(t) - y_0(t)| \to 0 \quad \text{as } n \to \infty, \\ &\sup_{k=0,1,2,\dots} \sup_{t\in(t_k,t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu}|{}^cD^p_{t_k^+}y_n(t) - {}^cD^p_{t_k^+}y_0(t)| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Since f is a Carathéodory function, there exists $M_r > 0$ such that

$$|f(t, y_n(t), {}^cD^p_{t_k^+}y_n(t))| \le M_r, \quad t \in [0, \infty),\ k = 0, 1, 2, \dots.$$

One sees that, for $t \in (t_k, t_{k+1}], k = 0, 1, 2, \dots,$

$$(11)\qquad (Ty_n)(t) = \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, y_n(s), {}^{c}D^{p}_{t_k^+} y_n(s))\, ds + x_0 + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, y_n(s), {}^{c}D^{p}_{t_{j-1}^+} y_n(s))\, ds,$$

and

$$(12)\qquad {}^{c}D^{p}_{t_k}(Ty_n)(t) = \int_{t_k}^{t} \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} q(s) f(s, y_n(s), {}^{c}D^{p}_{t_k^+} y_n(s))\, ds.$$

From $\lambda_0 = \inf_{k=1,2,\dots}(t_k - t_{k-1}) > 0$, we get $t_k > k\lambda_0$ for all $k = 0, 1, 2, \dots$. Since $\sum_{j=K+1}^{\infty} 1/j^{\mu+1-\sigma}$ is convergent, there is $K > 0$ such that

$$\sum_{j=K+1}^{\infty} \frac{1}{j^{\mu+1-\sigma}} < \epsilon.$$

Then

$$(13)\qquad \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \times |q(s) f(s, y_n(s), {}^{c}D^{p}_{t_k^+} y_n(s)) - q(s) f(s, y_0(s), {}^{c}D^{p}_{t_k^+} y_0(s))|\, ds$$
$$+ \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \sum_{j=K+1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} \times |q(s) f(s, y_n(s), {}^{c}D^{p}_{t_k^+} y_n(s)) - q(s) f(s, y_0(s), {}^{c}D^{p}_{t_k^+} y_0(s))|\, ds$$
$$\le 2M_r \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^l\, ds + 2M_r \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \sum_{j=K+1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} s^l\, ds$$
$$\le 2M_r \frac{1}{t^{\mu+1-\sigma}} \int_{t_k/t}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw + 2M_r \frac{1}{t^{\mu+1-\sigma+\alpha+l}} \sum_{j=K+1}^{k} t_j^{\alpha+l} \int_{t_{j-1}/t_j}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw$$

$$
\begin{aligned}
&\le 2M_r \frac{1}{t_k^{\mu+1-\sigma}} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw \\
&\quad + 2M_r \sum_{j=K+1}^{k} \frac{1}{t_j^{\mu+1-\sigma+\alpha+l}} t_j^{\alpha+l} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw \\
&\le 4M_r \sum_{j=K+1}^{k} \frac{1}{t_j^{\mu+1-\sigma}} \frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \le 4M_r \frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \sum_{j=K+1}^{k} \frac{1}{(j\lambda_0)^{\mu+1-\sigma}} \\
&= 4M_r \frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \frac{1}{\lambda_0^{\mu+1-\sigma}} \sum_{j=K+1}^{\infty} \frac{1}{j^{\mu+1-\sigma}} \le 4M_r \frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \frac{1}{\lambda_0^{\mu+1-\sigma}} \epsilon.
\end{aligned}
$$

Since f is a Carathéodory function, there exists $\delta_1 > 0$ such that

$$
\begin{aligned}
&\bigg| f\bigg(t, \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} u_1, \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} v_1\bigg) \\
&\qquad\qquad - f\bigg(t, \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} u_2, \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} v_2\bigg)\bigg| < \frac{\epsilon}{\sum_{j=1}^{K} 1/t_j^{\mu+1-\sigma}}
\end{aligned}
$$

for all $t \in [0, t_{K+1}]$ and $u_1, u_2 \in [-r, r]$ with $|u_1 - u_2| < \delta_1, |v_1 - v_2| < \delta_1$. From (37), there exists an integer $N_2 > 0$ such that

$$
\begin{aligned}
&\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} |y_n(t) - y_0(t)| < \delta_1, && t \in (0,\infty),\ n > N_2, \\
&\frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} |{}^cD_{t_k^+}^p y_n(t) - {}^cD_{t_k^+}^p y_0(t)| < \delta_1, && t \in (t_k, t_{k+1}],\ n > N_2.
\end{aligned}
$$

So for $t \in [t_k, t_{k+1}]$ $(k \le K)$ and $n > N_2$, we have

$$
\begin{aligned}
(14)\qquad &\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=1}^{K} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\qquad \times |q(s)f(s, y_n(s), {}^cD_{t_k^+}^p y_n(s)) - q(s)f(s, y_0(s), {}^cD_{t_k^+}^p y_0(s))|\, ds \\
&\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=1}^{K} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} s^l \frac{\epsilon}{\sum_{j=1}^{K} 1/t_j^{\mu+1-\sigma}}\, ds \\
&\le \frac{\epsilon}{\sum_{j=1}^{K} 1/t_j^{\mu+1-\sigma}} \frac{1}{t^{\mu+1-\sigma+\alpha+l}} \sum_{j=1}^{K} t_j^{\alpha+l} \int_{t_{j-1}/t_j}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw \\
&\le \frac{\epsilon}{\sum_{j=1}^{K} 1/t_j^{\mu+1-\sigma}} \sum_{j=1}^{K} \frac{1}{t_j^{\mu+1-\sigma}} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw \le \frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \epsilon.
\end{aligned}
$$

Using (13) and (14), we see that for $t \in (t_k, t_{k+1}]$ $(k = 0, 1, 2, \dots)$ and $n > N_2$,

$$\begin{aligned}
&\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|(Ty_n)(t) - (Ty_0)(t)| \\
&\quad \le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_n(s), {}^cD^p_{t_k^+} y_n(s)) \\
&\qquad\qquad - f(s, y_0(s), {}^cD^p_{t_k^+} y_0(s))|\, ds \\
&\quad + \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s)|\, |f(s, y_n(s), {}^cD^p_{t_{j-1}^+} y_n(s)) \\
&\qquad\qquad - f(s, y_0(s), {}^cD^p_{t_{j-1}^+} y_0(s))|\, ds \\
&\quad \le 4M_r \frac{\mathbf{B}(\alpha, l+1)}{\Gamma(\alpha)} \frac{1}{\lambda_0^{\mu+1-\sigma}} \epsilon + \frac{\mathbf{B}(\alpha, l+1)}{\Gamma(\alpha)} \epsilon.
\end{aligned}$$

It follows that

$$(15) \qquad \sup_{k=0,1,2,\dots} \sup_{t \in (t_k, t_{k+1}]} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} |(Ty_n)(t) - (Ty_0)(t)| \to 0 \text{ as } n \to \infty.$$

Similarly we can show that

$$(16) \qquad \sup_{k=0,1,2,\dots} \sup_{t \in (t_k, t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} |{}^cD_{t_k^+}(Ty_n)(t) - {}^cD_{t_k^+}(Ty_0)(t)| \to 0$$

as $n \to \infty$. From (15) and (16) we get

$$\lim_{n\to\infty} Ty_n = Ty_0.$$

Thus T is continuous.

Recall that $\Omega \subset X$ is relatively compact if

- (i) Ω is bounded,
- (ii) both $K := \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} \Omega$ and $L := \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} {}^cD^p_{0^+} \Omega$ are equi-continuous on any closed subinterval $[a, b]$ of $(t_k, t_{k+1}]$ $(k = 0, 1, 2, \dots)$,
- (iii) both K and L are equi-convergent at $t = t_k$ $(k = 0, 1, 2, \dots)$,
- (iv) both K and L are equi-convergent at $t = \infty$.

Let $W \subset X$ be a nonempty bounded set. To prove that T is completely continuous, we need to prove that $T(W)$ is bounded, both $\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})} T(\Omega)$ and $\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}} {}^cD^p_{0^+} T(\Omega)$ are equi-continuous on finite closed subintervals of $(t_k, t_{k+1}]$ $(k = 0, 1, 2 \dots)$, both are equi-convergent at $t = t_k$ $(k = 0, 1, 2, \dots)$, and both are equi-convergent at $t = \infty$. Since W is bounded, there exists

$r > 0$ such that

$$\sup_{t\in(0,\infty)} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|x(t)| \le r,$$
$$\sup_{k=0,1,2,\dots} \sup_{t\in(t_k,t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^{c}D^{p}_{t_k^+}x(t)| \le r, \qquad x \in W.$$

Since f is a Carathéodory function, there exists $M_r > 0$ such that

$$|f(t,x(t),{}^{c}D^{p}_{t_k^+}x(t))| \le M_r, \qquad t \in t \in (t_k,t_{k+1}],\ k = 0,1,2,\dots.$$

STEP 2. By similar methods to those used in the proof of Lemma 2.2, it is easy to see that TW is bounded. We omit the details.

STEP 3. We prove that both $\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}T(\Omega)$ and $\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}{}^{c}D^{p}_{0^+}T(\Omega)$ are equi-continuous on finite closed subintervals of $(t_k,t_{k+1}]$ $(k = 0,1,2,\dots)$.

For $[a,b] \subset (t_k,t_{k+1}]$ with $s_1, s_2 \in [a,b]$ with $s_1 < s_2$ and $x \in W$, we have

$$\begin{aligned}
&\bigg|\frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})}\int_{t_k}^{s_1}\frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds\\
&\qquad - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})}\int_{t_k}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds\bigg|\\
&\le \left|\frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})}\right|\\
&\quad\times\int_{t_k}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}|q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))|\,ds\\
&\quad+\frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})}\int_{s_1}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}|q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))|\,ds\\
&\quad+\frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})}\\
&\quad\times\int_{t_k}^{s_1}\frac{|(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))|\,ds\\
&\le \left|\frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})}\right|M_r\int_{t_k}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{l}\,ds\\
&\quad+\frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})}M_r\int_{s_1}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{l}\,ds
\end{aligned}$$

$$+ M_r \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} \int_{t_k}^{s_1} \frac{|(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}|}{\Gamma(\alpha)} s^l\,ds$$

$$= \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| s_2^{\alpha+l} M_r \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} M_r s_2^{\alpha+l} \int_{s_1/s_2}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ M_r \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} \int_{t_k}^{s_1} \frac{(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^l\,ds$$

$$\le \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| s_2^{\alpha+l} M_r \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ M_r \max\{a^{\alpha+l}, b^{\alpha+l}\} \int_{s_1/s_2}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ M_r \left[s_1^{\alpha+l} \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw - s_2^{\alpha+l} \int_0^{s_1/s_2} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw \right]$$

$$\le \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \max\{a^{\alpha+l}, b^{\alpha+l}\} M_r$$

$$\times \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ M_r \max\{a^{\alpha+l}, b^{\alpha+l}\} \int_{s_1/s_2}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ M_r |s_1^{\alpha+l} - s_2^{\alpha+l}| \int_0^1 \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw$$

$$+ M_r \max\{a^{\alpha+l}, b^{\alpha+l}\} \int_{s_1/s_2}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\,dw \to 0$$

uniformly as $s_1 \to s_2$ with $s_1, s_2 \in [a,b] \subset (t_k, t_{k+1}]$. So

$$\left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} (Tx)(s_1) - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} (Tx)(s_2) \right|$$

$$= \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} \int_{t_k}^{s_1} \frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^{c}D_{t_k^+}^{p} x(s))\,ds \right.$$

$$
\begin{aligned}
&- \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \int\limits_{t_k}^{s_2} \frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))\, ds \Bigg| \\
&+ |x_0| \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \\
&+ \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \\
&\times \sum_{j=1}^{k} \int\limits_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))|\, ds \\
\le &\Bigg| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} \int\limits_{0}^{s_1} \frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{0^+} x(s))\, ds \\
&- \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \int\limits_{0}^{s_2} \frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{0^+} x(s))\, ds \Bigg| \\
&+ |x_0| \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \\
&+ M_r \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \sum_{j=1}^{k} \int\limits_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} s^l\, ds \\
\le &\Bigg| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} \int\limits_{0}^{s_1} \frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{0^+} x(s))\, ds \\
&- \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \int\limits_{0}^{s_2} \frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{0^+} x(s))\, ds \Bigg| \\
&+ |x_0| \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \\
&+ M_r \left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} \right| \\
&\times \sum_{j=1}^{k} t_k^{\alpha+l} \int\limits_{0}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw \to 0
\end{aligned}
$$

uniformly as $s_1 \to s_2$ with $s_1, s_2 \in [a,b] \subset (t_k, t_{k+1}]$. It follows that

$$
\left| \frac{s_1^{\sigma-\alpha-l}}{(1+s_1)(1+s_1^{\mu})} (Tx)(s_1) - \frac{s_2^{\sigma-\alpha-l}}{(1+s_2)(1+s_2^{\mu})} (Tx)(s_2) \right| \to 0 \tag{17}
$$

uniformly as $s_1 \to s_2$ with $s_1, s_2 \in [a,b] \subset (t_k, t_{k+1}]$.

Furthermore, similarly we have

$$\begin{aligned}(18)\qquad &\left|\frac{s_1^{p+\sigma-\alpha-l}}{1+s_1^{\mu}}\,{}^{c}D^{p}_{t_k^+}(Tx)(s_1)-\frac{s_2^{p+\sigma-\alpha-l}}{1+s_2^{\mu}}\,{}^{c}D^{p}_{t_k^+}(Tx)(s_2)\right|\\ &=\left|\frac{s_1^{p+\sigma-\alpha-l}}{1+s_1^{\mu}}\int_{t_k}^{s_1}\frac{(s_1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds\right.\\ &\qquad\left.-\frac{s_2^{p+\sigma-\alpha-l}}{1+s_2^{\mu}}\int_{t_k}^{s_2}\frac{(s_2-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds\right|\to 0\end{aligned}$$

uniformly as $s_1\to s_2$ with $s_1,s_2\in[a,b]\subset(t_k,t_{k+1}]$. From (17) and (18), we see that TW is equi-continuous on finite closed subintervals of $(t_k,t_{k+1}]$.

STEP 4. We prove that both $\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}T(\Omega)$ and $\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}{}^{c}D^{p}_{0^+}T(\Omega)$ are equi-convergent as $t\to t_k^+$ $(k=0,1,2,\dots)$.

Since $\mu>\sigma>0$ we see that

$$\begin{aligned}&\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|(Tx)(t)-x_0|\\ &\qquad\le\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))\,ds\\ &\qquad\le\frac{M_r t^{\sigma}}{(1+t)(1+t^{\mu})}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{l}\,dw\to 0\end{aligned}$$

uniformly in W as $t\to 0$. It follows that

$$(19)\qquad \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}|(Tx)(t)-x_0|\to 0\quad\text{uniformly in } W \text{ as } t\to 0.$$

Furthermore, we have

$$\begin{aligned}&\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^{c}D^{p}_{0^+}(Tx)(t)|\\ &\quad\le\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\int_0^t\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|q(s)f(s,x(s),{}^{c}D^{p}_{t_k^+}x(s))|\,ds\\ &\quad\le\frac{M_r t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\int_0^t\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}s^{l}\,ds=\frac{M_r t^{\sigma}}{1+t^{\mu}}\int_0^1\frac{(1-w)^{\alpha-p-1}}{\Gamma(\alpha-p)}w^{l}\,dw\to 0\end{aligned}$$

uniformly in W as $t\to 0$. It follows that

$$(20)\qquad \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^{c}D^{p}_{0^+}(Tx)(t)|\to 0\quad\text{uniformly in } W \text{ as } t\to 0.$$

From (19) and (20), we see that TW is equi-convergent as $t\to 0^+$.

For $t \to t_k^+$, we have

$$\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \times \left|(Tx)(t) - \left(\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))\, ds + x_0\right)\right|$$
$$\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))|\, ds$$
$$\le M_r \frac{t^\sigma}{(1+t)(1+t^\mu)} \int_{t_k/t}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw$$

and

$$\frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} \left| {}^cD_{t_k^+}(Tx)(t) - \int_{t_k}^{t} \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))\, ds\right|$$
$$\le M_r \frac{t^\sigma}{1+t^\mu} \int_{t_k/t}^{1} \frac{(1-w)^{\alpha-p-1}}{\Gamma(\alpha-p)} w^l\, dw.$$

Hence we have shown that TW is equi-convergent as $t \to t_k^+$ $(k = 1, 2, \dots)$.

STEP 5. We prove that both $\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} T(\Omega)$ and $\frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} {}^cD^p_{0^+} T(\Omega)$ are equi-convergent as $t \to \infty$.

We get

$$\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \times \left|(Tx)(t) - \left(x_0 + \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))\, ds\right)\right|$$
$$\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))|\, ds$$
$$+ \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=k+1}^{\infty} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s) f(s, x(s), {}^cD^p_{t_k^+} x(s))|\, ds$$
$$\le M_r \frac{t^\sigma}{(1+t)(1+t^\mu)} \int_{t_k/t}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw$$
$$+ M_r \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=k+1}^{\infty} t_j^{\alpha+l} \int_{t_{j-1}/t_j}^{1} \frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)} w^l\, dw$$

$$\leq M_r M_{\sigma,\mu}\frac{1}{1+t}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} + M_r\frac{1}{\lambda_0^{\mu-\sigma+1}}\sum_{j=k+1}^{\infty}\frac{1}{j^{\mu-\sigma+1}}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \to 0$$

uniformly in W as $t\to\infty$ $(k\to\infty)$. Furthermore, we have

$$\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^cD^p_{t_k^+}(Tx)(t)| \leq \frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}t^{\alpha+l-p}\int_{t_k/t}^{1}\frac{(1-w)^{\alpha-p-1}}{\Gamma(\alpha)}M_r w^l\,dw$$

$$\leq \frac{t^{\sigma}}{1+t^{\mu}}\int_0^1\frac{(1-w)^{\alpha-p-1}}{\Gamma(\alpha)}M_r w^l\,dw \to 0$$

uniformly in W as $t\to\infty$. Hence TW is equi-convergent as $t\to\infty$.

From the above discussion, we see that T is completely continuous. ■

We now present the main assumptions:

(G) Suppose that $\delta_i=\delta_{1i}+\delta_{2i}>0$ $(i=1,\ldots,m)$ and $\delta_1\leq\cdots\leq\delta_m$, and f is a Carathéodory function such that there exist $A_i\geq 0$ $(i=1,\ldots,m)$ and a continuous bounded function $r:(0,1)\to\mathbb{R}$ such that

$$\left|f\left(t,\frac{(1+t)(1+t^{\mu})}{t^{\sigma-\alpha-l}}u_1,\frac{1+t^{\mu}}{t^{p+\sigma-\alpha-l}}u_2\right)-r(t)\right|\leq\sum_{i=1}^{m}A_i|u_1|^{\delta_{1i}}|u_2|^{\delta_{2i}}$$

for all $t\in(0,\infty)$ and $u_1,u_2\in\mathbb{R}$.

Let

$$\Psi(t)=\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)r(s)\,ds$$

$$+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)r(s)\,ds+x_0,\qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots,$$

$$N_1=\left(M_{\sigma,\mu}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}+\frac{1}{\lambda_0^{\mu-\sigma+1}}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}\sum_{j=1}^{\infty}\frac{1}{j^{\mu-\sigma+1}}\right)\sum_{i=1}^{m}A_i\|\Psi\|^{\delta_i-\delta_m},$$

$$N_2=M_{\sigma,\mu}\frac{\mathbf{B}(\alpha-p,l+1)}{\Gamma(\alpha-p)}\sum_{i=1}^{m}A_i,$$

$$N_0=\max\{N_1,N_2\}.$$

THEOREM 3.1. *Suppose that* (G) *holds. Then IVP* (1) *has a solution* $x\in X$ *if*

$$\delta_m<1,\ \text{or}\ \delta_m=1\ \text{with}\ N_0<1,\ \text{or}$$

(21)
$$\delta_m>1\ \text{with}\ \frac{\|\Psi\|^{1-\delta_m}(\delta_m-1)^{\delta_m-1}}{\delta_m^{\delta_m}}\geq N_0.$$

Proof. Let the Banach space X and its norm $\|\cdot\|$ be defined as in Section 2. Let $T : X \to X$ be defined by (9). Since by (G), f is a Carathéodory function, by Lemma 2.2 we seek solutions of (1) by looking for fixed points of T in X, and T is well defined and completely continuous.

It is easy to show that $\Psi \in X$. Let $r > 0$ and define

$$\overline{\Omega}_r = \{x \in X : \|x - \Psi\| \le r\}.$$

For $x \in \overline{\Omega}_r$, we have $\|x - \Psi\| \le r$. Then

$$\|x\| = \max\left\{ \sup_{t\in(0,\infty)} \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)}|x(t)|, \sup_{k=0,1,2,\dots} \sup_{t\in(t_k,t_{k+1}]} \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu}\left|{}^cD^p_{t_k^+}x(t)\right| \right\}$$
$$\le \|x-\Psi\| + \|\Psi\| \le r + \|\Psi\|.$$

Using (G), we find

$$\begin{aligned}
&\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)}|(Tx)(t) - \Psi(t)| \\
&\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s)|\, |f(s, x(s), {}^cD^p_{0^+}x(s)) - r(s)|\, ds \\
&\quad + \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} |q(s)|\, |f(s, x(s), {}^cD^p_{0^+}x(s)) - r(s)|\, ds \\
&\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^l \\
&\quad \times \left[\sum_{i=1}^{m} A_i \left| \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} x(s) \right|^{\delta_{1i}} \left| \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} {}^cD^p_{0^+}x(s) \right|^{\delta_{2i}} \right] ds \\
&\quad + \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} s^l \\
&\quad \times \left[\sum_{i=1}^{m} A_i \left| \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} x(s) \right|^{\delta_{1i}} \left| \frac{t^{p+\sigma-\alpha-l}}{1+t^\mu} {}^cD^p_{0^+}x(s) \right|^{\delta_{2i}} \right] ds \\
&\le \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^l\, ds \sum_{i=1}^{m} A_i \|x\|^{\delta_i} \\
&\quad + \frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^\mu)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} s^l\, ds \sum_{i=1}^{m} A_i \|x\|^{\delta_i}
\end{aligned}$$

$$\le \frac{t^{\sigma}}{(1+t)(1+t^{\mu})}\int_{t_k/t}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$+\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}\sum_{j=1}^{k}t_j^{\alpha+l}\int_{t_{j-1}/t_j}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$\le M_{\sigma,\mu}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$+\frac{t^{\sigma-\alpha-l}}{t^{\mu+1}}\sum_{j=1}^{k}t_j^{\alpha+l}\int_{t_{j-1}/t_j}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$\le M_{\sigma,\mu}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$+\sum_{j=1}^{k}\frac{1}{t_j^{\mu-\sigma+1}}\int_0^1\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^l\,dw\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$\le M_{\sigma,\mu}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}\sum_{i=1}^{m}A_i\|x\|^{\delta_i}+\sum_{j=1}^{\infty}\frac{1}{j^{\mu-\sigma+1}\lambda_0^{\mu-\sigma+1}}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$

$$=\left(M_{\sigma,\mu}\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}+\frac{1}{\lambda_0^{\mu-\sigma+1}}\,\frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)}\sum_{j=1}^{\infty}\frac{1}{j^{\mu-\sigma+1}}\right)\sum_{i=1}^{m}A_i$$

$$\le N_1[r+\|\Psi\|]^{\delta_m}.$$

Furthermore, we have

$$\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^cD_{t_k^+}^p(Tx)(t)-{}^cD_{t_k^+}^p\Psi(t)|$$

$$\le\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\int_{t_k}^{t}\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|q(s)|\,|f(s,x(s),{}^cD_{t_k^+}^p x(s))-C|\,ds$$

$$\le\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\int_{t_k}^{t}\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}s^l$$

$$\times\left[\sum_{i=1}^{m}A_i\left|\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu})}x(s)\right|^{\delta_{1i}}\left|\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}{}^cD_{t_k^+}^p x(s)\right|^{\delta_{2i}}\right]ds$$

$$\le\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}\int_0^{t}\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}s^l\sum_{i=1}^{m}A_i\|x\|^{\delta_i}\,ds$$

$$\leq M_{\sigma,\mu}\frac{\mathbf{B}(\alpha-p,l+1)}{\Gamma(\alpha-p)}\sum_{i=1}^{m}A_i\|x\|^{\delta_i} \leq M_{\sigma,\mu}\frac{\mathbf{B}(\alpha-p,l+1)}{\Gamma(\alpha-p)}\sum_{i=1}^{m}A_i\|x\|^{\delta_i}$$
$$\leq M_{\sigma,\mu}\frac{\mathbf{B}(\alpha-p,l+1)}{\Gamma(\alpha-p)}\sum_{i=1}^{m}A_i[r+\|\Psi\|]^{\delta_i} \leq [r+\|\Psi\|]^{\delta_m}N_2.$$

It follows that

$$\|Tx-\Psi\| \leq [r+\|\Psi\|]^{\delta_m}N_0.$$

(i) If $\delta_m < 1$, we can choose $r_0 > 0$ so large that $[r_0+\|\Psi\|]^{\delta_m}N_0 < r_0$. Let $\Omega_{r_0} = \{x \in Y : \|x-\Psi\| < r_0\}$. It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then the Schauder fixed-point theorem implies that T has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a bounded solution of (1).

(ii) If $\delta_m = 1$, we choose

$$r_0 \geq \frac{\|\Psi\|N_0}{1-N_0}.$$

Let $\Omega_{r_0} = \{x \in Y : \|x-\Psi\| < r_0\}$. It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then the Schauder fixed-point theorem implies that T has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a bounded solution of (1).

(iii) If $\delta_m > 1$, we choose $r = r_0 = \|\Psi\|/(\delta_m - 1)$. By assumption,

$$\frac{r_0}{(r_0+\|\Psi\|)^{\delta_m}} = \frac{\|\Psi\|^{1-\delta_m}(\delta_m-1)^{\delta_m-1}}{\delta_m^{\delta_m}} \geq N_0.$$

Let $\Omega_{r_0} = \{x \in Y : \|x-\Psi\| < r_0\}$. It is easy to see that $T\overline{\Omega}_{r_0} \subset \overline{\Omega}_{r_0}$. Then the Schauder fixed-point theorem implies that F has a fixed point $x \in \overline{\Omega}_{r_0}$, which is a solution of (1). ∎

THEOREM 3.2 *Suppose that* (G) *holds with* $\delta_m = 1$, *and* $N_0 < 1$. *Then* (1) *has a unique solution* x.

Proof. By (G) and Theorem 3.1, (1) has at least one solution. If it has two different solutions x_1 and x_2, then $\|x_1 - x_2\| > 0$, $Tx_1 = x_1$ and $Tx_2 = x_2$. So the methods used in the proof of Theorem 3.1 imply that

$$\frac{t^{\sigma-\alpha-l}}{(1+t)(1+t^{\mu}}|(Tx_1)(t)-(Tx_2)(t)| \leq N_1\|x_1-x_2\|$$

and

$$\frac{t^{p+\sigma-\alpha-l}}{1+t^{\mu}}|{}^cD_{0+}^{p}(Tx_1)(t)-{}^cD_{0+}^{p}(Tx_2)(t)| \leq N_2\|x_1-x_2\|.$$

It follows that

$$\|Tx_1 - Tx_2\| \leq N_0\|x_1-x_2\|.$$

We get

$$0 < \|x_1 - x_2\| = \|Tx_1 - Tx_2\| \le N_0 \|x_1 - x_2\| < \|x_1 - x_2\|,$$

a contradiction. ■

4. Application. In this section, we apply the main theorem to solve the fractional order logistic model (3).

THEOREM 4.1. *Suppose that* $\lambda_0 = \inf_{k=0,1,2,\ldots}[t_{k+1} - t_k] > 0$, *$r$ is continuous and bounded and there exists $l > -\alpha$ such that $|q(t)| \le t^l$ for all $t \in (0,\infty)$ and there exist $\sigma > \alpha + l$ and $\mu > \sigma$ such that*

$$(22)\qquad \begin{aligned} &\frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}}|a(t)| \le a_0, && t \in (0,\infty),\\ &\left(\frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}}\right)^\delta |b(t)| \le b_0, && t \in (0,\infty),\\ &\frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}}\left(\frac{1+t^\mu}{t^{p+\sigma-\alpha-l}}\right)^\mu |c(t)| \le c_0, && t \in (0,\infty). \end{aligned}$$

Then (3) *has a solution if*

$$(23)\qquad \frac{\|\Psi\|^{1-\max\{\delta,1+\mu\}}(\max\{\delta,1+\mu\}-1)^{\max\{\delta,1+\mu\}-1}}{\max\{\delta,1+\mu\}^{\max\{\delta,1+\mu\}}} \ge N_0,$$

where

$$\begin{aligned} \Psi(t) = {}& \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)r(s)\,ds \\ &+ \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{(t_j-s)^{\alpha-1}}{\Gamma(\alpha)} q(s)r(s)\,ds + x_0, t \in (t_k, t_{k+1}], k = 0,1,2,\ldots, \end{aligned}$$

$$\begin{aligned} N_1 &= \left(M_{\sigma,\mu} \frac{\mathbf{B}(\alpha, l+1)}{\Gamma(\alpha)} + \frac{1}{\lambda_0^{\mu-\sigma+1}} \frac{\mathbf{B}(\alpha,l+1)}{\Gamma(\alpha)} \sum_{j=1}^{\infty} \frac{1}{j^{\mu-\sigma+1}} \right) \\ &\quad \times (a_0\|\Psi\|^{1-\max\{\delta,1+\mu\}} + b_0\|\Psi\|^{\delta-\max\{\delta,1+\mu\}} + c_0\|\Psi\|^{1+\mu-\max\{\delta,1+\mu\}}),\\ N_2 &= M_{\sigma,\mu}\frac{\mathbf{B}(\alpha-p, l+1)}{\Gamma(\alpha-p)}(a_0+b_0+c_0),\\ N_0 &= \max\{N_1, N_2\}. \end{aligned}$$

Proof. Let $f(t,u,v) = r(t) + a(t)u - b(t)u^\delta + c(t)v^\mu$. Then

$$\left| f\left(t, \frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}}u, \frac{1+t^\mu}{t^{p+\sigma-\alpha-l}}\right) - r(t)\right|$$

$$= \frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}} a(t)|u| + b(t)\left(\frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}}\right)^\delta |u|^\delta$$
$$+ c(t)\frac{(1+t)(1+t^\mu)}{t^{\sigma-\alpha-l}}\left(\frac{1+t^\mu}{t^{p+\sigma-\alpha-l}}\right)^\mu uv^\mu$$
$$\le a_0|u| + b_0|u|^\delta + c_0|u|\,|v|^\mu.$$

It is easy to see that f is a Carathéodory function. Choose $\delta_{11} = 1, \delta_{21} = 0$, $\delta_{12} = \delta, \delta_{22} = 0$ and $\delta_{13} = 1, \delta_{23} = \mu$. Then $\delta_1 = 1, \delta_2 = \delta, \delta_3 = 1 + \mu$ with $\max\{\delta_1, \delta_2, \delta_3\} = \max\{\delta, 1 + \mu\} > 1$.

Corresponding to Theorem 3.1, we choose $A_1 = a_0$, $A_2 = b_0$, $A_3 = c_0$. Then (G) holds. By Theorem 3.1, (3) has a solution. ∎

Acknowledgements. This research was supported by Guangdong University major research projects and achievements cultivating program.

The authors thank the editors for their excellent work.

References

[1] A. B. Basset, *On the descent of a sphere in a viscous liquid*, Quart. J. Pure Appl. Math. 41 (1910), 369–381.

[2] F. Chen and Y. Zhou, *Attractivity of fractional functional differential equations*, Comput. Math. Appl. 62 (2011) 1359–1369.

[3] J. S. Cramer, *The origins and development of the logit model*, Chapter 9 in: J. S. Cromer, Logit Models from Economics and Other Fields, Cambridge Univ. Press, 2003, 149–157.

[4] S. Das, P. K. Gupta and K. Vishal, *Approximate approach to the Das model of fractional logistic population growth*, Appl. Appl. Math. 5 (2010), 1702–1708.

[5] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Sci., River Edge, NJ, 2000.

[6] Z. Hu, W. Liu and W. Rui, *Periodic boundary value problem for fractional differential equation*, Int. J. Math. (2012) (online).

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujjilo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud. 204, North-Holland, Amsterdam, 2006.

[8] C. Kou, H. Zhou and Y. Yan, *Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis*, Nonlinear Anal. 74 (2011), 5975–5986.

[9] Y. Liu, *Existence and uniqueness of solutions for initial value problems of multi-order fractional differential equations on the half lines*, Sci. Sin. Math. 42 (2012) 735–756 (in Chinese).

[10] H. Maagli, *Existence of positive solutions for a nonlinear fractional differential equation*, Electron. J. Differential Equations 2013, no. 29, 5 pp.

[11] L. Mahto, S. Abbas and A. Favini, *Analysis of Caputo impulsive fractional order differential equations with applications*, Int. J. Differential Equations 2013, art. ID 704547, 11 pp.

[12] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, in: Fractals and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi (eds.), Springer, New York, 1997, 291–348.
[13] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Reg. Conf. Ser. Math. 40, Amer. Math. Soc., Providence, RI, 1979.
[14] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[15] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal. 5 (2002), 367–386.
[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, London, 1999.
[17] S. Zhang, *Monotone iterative methods for initial value problem involving Riemann–Liouville fractional derivatives*, Nonlinear Anal. 71 (2009), 2087–2093.

Yuji Liu
Department of Mathematics
Guangdong University of Finance
and Economics
Guangzhou 510320, P.R. China
E-mail: yuji_liu@126.com
liuyuji888@sohu.com

Pinghua Yang
Department of Basic Courses
Shijiazhuang Mechanical Engineering College
Shijiazhuang 050000, P.R. China

Received on 26.4.2014;
revised version on 2.9.2014 (2214)