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K. Kropielnicka (Gdańsk)

IMPLICIT DIFFERENCE METHODS FOR QUASILINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL PROBLEMS OF THE DIRICHLET TYPE

Abstract. Classical solutions of quasilinear functional differential equations are approximated with solutions of implicit difference schemes. Proofs of convergence of the difference methods are based on a comparison technique. Nonlinear estimates of the Perron type with respect to the functional variable for given functions are used. Numerical examples are given.

1. Introduction. For any two metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions defined on X and taking values in Y. Let $M_{n\times n}$ denote the set of all $n\times n$ real matrices. We will use vectorial inequalities, understanding that the same inequalities hold between the corresponding components. Let

$$E=[0,a]\times(-b,b),\qquad D=[-d_0,0]\times[-d,d],$$

where $a>0$, $b=(b_1,\ldots,b_n)$, $b_i>0$ for $1\le i\le n$, $d_0\in\mathbb{R}_+$, $d=(d_1,\ldots,d_n)\in\mathbb{R}_+^n$ and $\mathbb{R}_+=[0,\infty)$. We put $c=(c_1,\ldots,c_n)=b+d$ and

$$\partial_0E=[0,a]\times([-c,c]\setminus(-b,b)),\quad E_0=[-d_0,0]\times[-c,c],\quad \Omega=E\cup E_0\cup\partial_0E.$$

For a function $z:\Omega\to\mathbb{R}$ and a point $(t,x)\in[0,a]\times[-b,b]$ we define a function $z_{(t,x)}:D\to\mathbb{R}$ as follows:

$$z_{(t,x)}(\xi,y)=z(t+\xi,x+y)\quad\text{ for }(\xi,y)\in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t-d_0,t]\times[x-d,x+d]$ and this restriction is shifted to the set D. Elements of the space $C(D,\mathbb{R})$ will be denoted by w, $\overline{w}$ and so on. Write $\Sigma=E\times C(D,\mathbb{R})$ and suppose

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that the functions

$$f : \Sigma \to M_{n\times n}, \qquad f = [f_{ij}]_{i,j=1,\dots,n},$$
$$g : \Sigma \to \mathbb{R}^n, \qquad g = (g_1,\dots,g_n), \qquad G : \Sigma \to \mathbb{R}$$

and

$$\varphi : \partial_0 E \cup E_0 \to \mathbb{R}, \qquad \alpha : E \to \mathbb{R}^{1+n}, \qquad \alpha = (\alpha_0,\alpha'),\ \alpha' = (\alpha_1,\dots,\alpha_n),$$

are given. We assume that $\alpha(t,x) \in \overline{E}$ and $\alpha_0(t,x) \le t$ for $(t,x) \in E$, where $\overline{E}$ is the closure of E. We consider the problem consisting of the quasilinear differential functional equation

$$\begin{aligned}(1)\qquad \partial_t z(t,x) = &\sum_{i,j=1}^{n} f_{ij}(t,x,z_{\alpha(t,x)})\partial_{x_i x_j} z(t,x)\\ &+ \sum_{i=1}^{n} g_i(t,x,z_{\alpha(t,x)})\partial_{x_i} z(t,x) + G(t,x,z_{\alpha(t,x)})\end{aligned}$$

with the initial boundary condition of the Dirichlet type

$$(2)\qquad z(t,x) = \varphi(t,x) \quad \text{for } (t,x) \in E_0 \cup \partial_0 E.$$

We are interested in establishing a method of numerical approximation of classical solutions to (1), (2) by means of solutions of associated implicit difference schemes and in estimating the difference between the exact and approximate solutions.

Explicit difference methods for (1), (2) consist in replacing the partial derivatives ∂_t, $\partial_x = (\partial_{x_1},\dots,\partial_{x_n})$ and $\partial_{xx} = [\partial_{x_i x_j}]_{i,j=1,\dots,n}$ with difference expressions. Moreover, because equation (1) contains the functional variable $z_{\alpha(t,x)}$ which is an element of the space $C(D,\mathbb{R})$ we need interpolating operators. This leads to a difference functional equation of the Volterra type. Solutions of these equations approximate, under suitable assumptions on given functions and on the mesh, solutions of the original problem. Methods of difference inequalities or theorems on recurrent inequalities are used in the investigation of the stability of difference schemes generated by parabolic functional differential problems. The proofs of the convergence are also based on a general theorem on an error estimate for approximate solutions of functional difference equations of the Volterra type with the unknown function of several variables.

Difference methods for nonlinear parabolic differential or functional differential equations were considered by many authors under various assumptions. It is not our aim to give a full review of papers concerning convergence results for difference schemes. We only mention those which contain such reviews: [8], [11], [12], [14], [15].

Two types of assumptions are needed in theorems on convergence of explicit difference methods for (1), (2). Conditions of the first type deal with the regularity of the given functions and the parabolicity of (1). It is assumed that f, g, G are continuous on Σ and satisfy nonlinear estimates of the Perron type with respect to the functional variable. The parabolicity of (1) means that the matrix f is symmetric on Σ and

$$f_{ii}(P) > \sum_{\substack{j=1\\ j\neq i}}^{n} f_{ij}(P), \qquad i = 1, \ldots, n,$$

where $P = (t, x, w) \in \Sigma$. Note that the above condition implies that

$$\sum_{i,j=1}^{n} f_{ij}(P)\xi_i\xi_j \geq 0 \qquad \text{for } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

which means that equation (1) is parabolic in the sense of Walter [16]. The conditions of the second type are connected with the mesh. It is assumed that

$$(3) \qquad 1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} f_{ii}(P) + h_0 \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} |f_{ij}(P)| \geq 0$$

for $P \in \Sigma$, where h_0 and $h' = (h_1, \ldots, h_n)$ are steps of the mesh. This is a very restrictive assumption on the relations between h_0 and h'. The differential functional equation

$$(4) \quad \partial_t z(t,x) = \sum_{i=1}^{n} \partial_{x_i x_i} z(t,x) + \sum_{i=1}^{n} \partial_{x_i} z(t,x) g_i(t, x, z_{\alpha(t,x)}) + G(t, x, z_{\alpha(t,x)})$$

is a particular case of (1). Condition (3) for equation (4) has the form

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \geq 0.$$

The aim of the paper is to show that the restriction (3) may be omitted in the case of implicit difference schemes (9), (10).

Parabolic functional differential equations find applications in different fields of knowledge. We give a few examples. The most important classes of such problems are the Lotka–Volterra type reaction diffusion equations which include delays and integral terms [4], [5]. Some systems of delayed reaction diffusion equations have also been used in modelling genetic repression [9]. A nuclear reactor model has been described in [13] by means of a system of two parabolic equations with delays. Reaction diffusion equations with delays arise naturally in the study of climate models [6]. A mathematical description of the overall control systems may be given by a parabolic equation with a deviated time variable [17]. For an extensive bibliography on applications of parabolic functional differential equations see the monograph [18].

We give an example of equation which can be obtained from (1) by specializing given functions.

Example 1. Assume that $d_0 = 0$, $d = 0$ where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ and

$$\widetilde{f} : E \times \mathbb{R} \to M_{n\times n}, \qquad \widetilde{f} = [\widetilde{f}_{ij}]_{i,j=1,\dots,n},$$
$$\widetilde{g} : E \times \mathbb{R} \to \mathbb{R}^n, \qquad \widetilde{g} = (\widetilde{g}_1, \dots, \widetilde{g}_n), \qquad \widetilde{G} : E \times \mathbb{R} \to \mathbb{R}$$

are given functions. We define f, g, G as follows:

$$f(t,x,w) = \widetilde{f}(t,x,w(0,\mathbf{0})),$$
$$g(t,x,w) = \widetilde{g}(t,x,w(0,\mathbf{0})), \qquad G(t,x,w) = \widetilde{G}(t,x,w(0,\mathbf{0})).$$

Then (1) reduces to the differential equation with deviated variables

$$(5) \quad \partial_t z(t,x) = \sum_{i,j=1}^{n} \widetilde{f}_{ij}(t,x,z(\alpha(t,x)))\partial_{x_i x_j} z(t,x) + \sum_{i=1}^{n} \widetilde{g}_i(t,x,z(\alpha(t,x)))\partial_{x_i} z(t,x) + \widetilde{G}(t,x,z(\alpha(t,x))).$$

Example 2. Suppose that $\beta, \gamma : E \to \mathbb{R}^{1+n}$ and $\beta = (\beta_0, \beta')$, $\gamma = (\gamma_0, \gamma')$, $\beta' = (\beta_1, \dots, \beta_n)$, $\gamma' = (\gamma_1, \dots, \gamma_n)$. We assume that

$$-d_0 \le (\beta_0 - \alpha_0)(t,x) \le 0, \qquad -d_0 \le (\gamma_0 - \alpha_0)(t,x) \le 0,$$
$$-d \le (\beta' - \alpha')(t,x) \le d, \qquad -d \le (\gamma' - \alpha')(t,x) \le d,$$

where $(t,x) \in E$. Write

$$I[w](t,x) = \int_{(\beta-\alpha)(t,x)}^{(\gamma-\alpha)(t,x)} w(\tau,y)\,dy\,d\tau.$$

For the above $\widetilde{f}$, $\widetilde{g}$, $\widetilde{G}$ we put

$$f(t,x,w) = \widetilde{f}(t,x,I[w](t,x)),$$
$$g(t,x,w) = \widetilde{g}(t,x,I[w](t,x)), \qquad G(t,x,w) = \widetilde{G}(t,x,I[w](t,x)).$$

Then (1) is equivalent to the differential integral equation

$$(6) \quad \partial_t z(t,x) = \sum_{i,j=1}^{n} \widetilde{f}_{ij}\Big(t,x,\int_{\beta(t,x)}^{\gamma(t,x)} z(\tau,y)\,dy\,d\tau\Big)\partial_{x_i x_j} z(t,x) + \sum_{i=1}^{n} \widetilde{g}_i\Big(t,x,\int_{\beta(t,x)}^{\gamma(t,x)} z(\tau,y)\,dy\,d\tau\Big)\partial_{x_i} z(t,x) + \widetilde{G}\Big(t,x,\int_{(\beta)(t,x)}^{(\gamma)(t,x)} z(\tau,y)\,dy\,d\tau\Big).$$

It is clear that more complicated differential equations with deviated variables and differential integral equations can be derived from (1). Note also that equations (5), (6) cannot be obtained as particular cases of differential functional equations considered in [2], [3], [8]. The right hand sides of equations in those papers depend on the restrictions of z to the set $[t-d_0,t]\times[x-d,x+d]$. In our paper, the differential functional equations depend on the restriction of z to $[\alpha_0(t,x)-d_0,\alpha_0(t,x)]\times[\alpha'(t,x)-d,\alpha'(t,x)+d]$.

The paper is organized as follows. In Section 2 we construct a class of implicit difference schemes for (1), (2). The existence and uniqueness of approximate solutions, which is not obvious in contrast to explicit methods, are proved in Section 3. In Section 4, which is the main part of the paper, we give sufficient conditions for convergence of the implicit difference schemes. Finally, numerical examples are presented in the last part of the paper.

2. Discretization of mixed problems. We formulate a difference problem corresponding to (1), (2). We denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of natural numbers and the set of integers, respectively. Let $F(X,Y)$ denote the class of all functions defined on X and taking values in Y where X and Y are arbitrary sets. For $x\in\mathbb{R}^n$, $U\in M_{n\times n}$, where $x=(x_1,\dots,x_n)$, $U=[u_{ij}]_{i,j=1,\dots,n}$, we write

$$\|x\|=\sum_{i=1}^n|x_i|,\qquad \|U\|_*=\sum_{i,j=1}^n|u_{ij}|.$$

We define a mesh on Ω in the following way. Let (h_0,h') where $h'=(h_1,\dots,h_n)$ stand for steps of the mesh. For $h=(h_0,h')$ and $(r,m)\in\mathbb{Z}^{1+n}$ where $m=(m_1,\dots,m_n)$ we define nodal points as follows:

$$t^{(r)}=rh_0,\qquad x^{(m)}=(x_1^{(m_1)},\dots,x_n^{(m_n)})=(m_1h_1,\dots,m_nh_n).$$

Denote by H the set of all $h=(h_0,h')$ such that there exist $K_0\in\mathbb{Z}$ and $K=(K_1,\dots,K_n)\in\mathbb{Z}^n$ satisfying the conditions $K_0h_0=d_0$ and $(K_1h_1,\dots,K_nh_n)=c$. Let $N\in\mathbb{N}$ be defined by the relations $Nh_0\le a<(N+1)h_0$. For $h\in H$ we put

$$\mathbb{R}_h^{1+n}=\{(t^{(r)},x^{(m)}):(r,m)\in\mathbb{Z}^{1+n}\}$$

and

$$\begin{gathered}E_h=E\cap\mathbb{R}_h^{1+n},\ E'_h=\{(t^{(r)},x^{(m)}):0\le r\le N-1,\ -K\le m\le K\},\\ E_{0.h}=E_0\cap\mathbb{R}_h^{1+n},\qquad \partial_0E_h=\partial_0E\cap\mathbb{R}_h^{1+n},\\ \Omega_h=\Omega\cap\mathbb{R}_h^{1+n}=E_h\cup E_{0.h}\cup\partial_0E_h.\end{gathered}$$

System (1) contains the functional variable $z_{\alpha(t,x)}$ which is an element of the space $C(D,\mathbb{R})$. Thus we need an interpolating operator $T_h:F(\Omega_h,\mathbb{R})\to$

$C(\Omega,\mathbb{R})$. We define T_h in the following way. Suppose that $z \in F(\Omega_h,\mathbb{R})$ and $(t,x) \in \Omega$. We consider two possibilities.

A) Suppose that there exists a point $(t^{(r)},x^{(m)}) \in \Omega_h$ such that $(t^{(r+1)}, x^{(m+1)}) \in \Omega_h$, where $m+1 = (m_1+1,\dots,m_n+1)$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$. Put

$$\Im = \{\lambda = (\lambda_1,\dots,\lambda_n) : \lambda_i \in \{0,1\} \text{ for } 0 \le i \le n\}.$$

We define

$$\begin{aligned} T_h[z](t,x) = {} & \frac{t-t^{(r)}}{h_0} \sum_{\lambda\in\Im} z^{(r+1,m+\lambda)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda} \Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda} \\ & + \Big(1-\frac{t-t^{(r)}}{h_0}\Big) \sum_{\lambda\in\Im} z^{(r,m+\lambda)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda} \Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda} \end{aligned}$$

where

$$\Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda} = \prod_{i=1}^{n} \Big(\frac{x_i-x_i^{(m_i)}}{h_i}\Big)^{\lambda_i},$$

$$\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda} = \prod_{i=1}^{n} \Big(1-\frac{x_i-x_i^{(m_i)}}{h_i}\Big)^{1-\lambda_i}$$

and we take $0^0 = 1$ in the above formulas.

B) If $(t,x) \in \Omega$ and $t^{(N)} < t \le a$, we define $T_h[z](t,x) = T_h[z](t^{(N)},x)$.

Thus we have defined $T_h[z]$ on Ω and $T_h[z] \in C(\Omega,\mathbb{R})$.

The above interpolating operator was first proposed in [7] for the construction of explicit difference schemes related to first order partial differential functional equations.

Let $z : \Omega_h \to \mathbb{R}$ and $(t^{(r)},x^{(m)}) \in E'_h$, and define

$$\begin{aligned} \delta_i^+ z^{(r,m)} &= \frac{1}{h_i}\,(z^{(r,m+e_i)} - z^{(r,m)}), \\ \delta_i^- z^{(r,m)} &= \frac{1}{h_i}\,(z^{(r,m)} - z^{(r,m-e_i)}), \qquad 1 \le i \le n, \end{aligned}$$

and

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\,(z^{(r+1,m)} - z^{(r,m)}), \tag{7}$$

$$\delta_i z^{(r+1,m)} = \frac{1}{2}\,(\delta_i^+ z^{(r+1,m)} + \delta_i^- z^{(r+1,m)}), \qquad 1 \le i \le n. \tag{8}$$

Put $J = \{(i,j) \in \mathbb{N}^2 : 1 \le i,j \le n,\ i \ne j\}$. Suppose that a function $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ is given. We approximate solutions of (1)–(2) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[z])\delta_{ij} z^{(r+1,m)} + \sum_{i=1}^{n} g_i(P^{(r,m)}[z])\delta_i z^{(r+1,m)} + G(P^{(r,m)}[z]), \tag{9}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h, \tag{10}$$

where $P^{(r,m)}[z] = (t^{(r)}, x^{(m)}, (T_h[z])_{\alpha^{(r,m)}})$, $\alpha^{(r,m)} = \alpha(t^{(r)}, x^{(m)})$. The difference operator $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$ is defined in the following way:

$$\delta_{ii} z^{(r+1,m)} = \delta_i^+ \delta_i^- z^{(r+1,m)}, \quad 1 \le i \le n, \tag{11}$$

$$\delta_{ij} z^{(r+1,m)} = \frac{1}{2}\,(\delta_i^+ \delta_j^- z^{(r+1,m)} + \delta_i^- \delta_j^+ z^{(r+1,m)}) \quad \text{if } f_{ij}(P^{(r,m)}[z]) \le 0, \tag{12}$$

$$\delta_{ij} z^{(r+1,m)} = \frac{1}{2}\,(\delta_i^+ \delta_j^+ z^{(r+1,m)} + \delta_i^- \delta_j^- z^{(r+1,m)}) \quad \text{if } f_{ij}(P^{(r,m)}[z]) > 0. \tag{13}$$

The difference functional problem (9)–(10) with δ_0, δ, $\delta^{(2)}$ defined by (7), (8), (11)–(13) is viewed as an implicit difference method for (1)–(2). It is important in our considerations that the difference expressions δz and $\delta^{(2)} z$ appear in (9) at the point $(t^{(r+1)}, x^{(m)})$. It follows from (12), (13) that the definition of the difference expressions $\delta_{ij} z^{(r+1,m)}$ for $(i,j) \in J$ depends on the sign of $f_{ij}(P^{(r,m)}[z])$. The corresponding explicit difference scheme consists of the difference functional equation

$$\delta_0 z^{(r,m)} = \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[z])\delta_{ij} z^{(r,m)} + \sum_{i=1}^{n} g_i(P^{(r,m)}[z])\delta_i z^{(r,m)} + G(P^{(r,m)}[z]), \tag{14}$$

with the initial boundary condition (10). It is clear that there exists exactly one solution of problem (10), (14). Sufficient conditions for the convergence of the difference scheme (10), (14) can be deduced from [2].

We prove that under natural assumptions on given functions there exists exactly one solution $u_h : E_h \to \mathbb{R}$ of the implicit difference problem (9), (10). In Section 4 we prove a convergence result.

3. Solutions of difference functional problems. For a function $z : E_h \to \mathbb{R}$ and a point $(t^{(r)}, x^{(m)}) \in E_h$ we put

$$J_-^{(r,m)}[z] = \{(i,j) \in J : f_{ij}(P^{(r,m)}[z]) \le 0\}, \quad J_+^{(r,m)}[z] = J \setminus J_-^{(r,m)}[z]. \tag{15}$$

Assumption H$[f,g,\alpha]$.

1) The functions $f : \Sigma \to M[n]$ and $g : \Sigma \to \mathbb{R}^n$ are continuous on Σ and

$$-\frac{1}{2}\,|g_i(P)| + \frac{1}{h_i}\, f_{ii}(P) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_j}\,|f_{ij}(P)| \geq 0, \qquad i = 1,\ldots,n, \tag{16}$$

for $P = (t,x,w) \in \Sigma$.

2) $\alpha \in C(E,\mathbb{R}^{1+n})$, $\alpha = (\alpha_0,\alpha')$, $\alpha(t,x) \in E$ and $\alpha_0(t,x) \leq t$ for $(t,x) \in E$.

Remark 1. Suppose that

$$f_{ii}(P) - \sum_{\substack{j=1\\ j\neq i}}^{n} |f_{ij}(P)| \geq \varepsilon, \qquad P \in \Sigma, \tag{17}$$

where $\varepsilon > 0$, and $h_1 = \cdots = h_n$ are sufficiently small. Then condition (16) is satisfied.

Theorem 3.1. *If Assumption* H$[f,g,\alpha]$ *is satisfied and* $\varphi_h : \partial_0 E_h \cup E_{0.h} \to \mathbb{R}$, *then there is exactly one solution* $u_h : E_h \to \mathbb{R}$ *of problem* (9)–(10).

Proof. Suppose that $0 \leq r \leq N-1$ is fixed and that the solution u_h of (9), (10) is defined on $\Omega_h \cap ([-d_0, t^{(r)}] \times \mathbb{R}^n)$. We prove that the numbers $u_h^{(r+1,m)}$, where $(t^{(r+1)}, x^{(m)}) \in E_h$, exist and are unique. There is $Q_h > 0$ such that

$$Q_h \geq 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2}\, f_{ii}(P^{(r,m)}[u_h]) - h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j}\,|f_{ij}(P^{(r,m)}[u_h])|. \tag{18}$$

Problem (9)–(10) is equivalent to the system of equations

$$\begin{aligned} z^{(r+1,m)} = \frac{1}{Q_h+1}\Big[& Q_h z_h^{(r+1,m)} + u_h^{(r,m)} + h_0 \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[u_h])\delta_{ij} z^{(r+1,m)} \\ & + h_0 \sum_{i=1}^{n} g_i(P^{(r,m)}[u_h])\delta_i z^{(r+1,m)} + h_0 G(P^{(r,m)}[u_h])\Big], \end{aligned} \tag{19}$$

with the boundary condition

$$z^{(r+1,m)} = \varphi_h^{(r+1,m)} \qquad \text{for } (t^{(r+1)}, x^{(m)}) \in \partial_0 E_h,$$

where $z^{(r+1,m)}$, $-K \le m \le K$, are unknown. Write

$$S_h = \{x^{(m)} : -K \le m \le K\}, \qquad \partial_0 S_h = \{x^{(m)} : (t^{(r+1)}, x^{(m)}) \in \partial_0 E_h\}.$$

We consider the space $F(S_h, \mathbb{R})$. Elements of $F(S_h, \mathbb{R})$ are denoted by ξ, $\overline{\xi}$. For $\xi \in F(S_h, \mathbb{R})$ we write $\xi^{(m)} = \xi(x^{(m)})$ and

$$\delta\xi^{(m)} = (\delta_1 \xi^{(m)}, \ldots, \delta_n \xi^{(m)}), \qquad \delta^{(2)}\xi^{(m)} = [\delta_{ij}\xi^{(m)}]_{i,j=1,\ldots,n},$$

where δ_i and δ_{ij}, $1 \le i, j \le n$, are defined by (8)–(13). The norm in $F(S_h, \mathbb{R})$ is defined by

$$\|\xi\|_\infty = \max\{|\xi^{(m)}| : x^{(m)} \in S_h\}.$$

We consider the linear operator $U_h : F(S_h, \mathbb{R}) \to F(S_h, \mathbb{R})$ defined by

$$\begin{aligned} U_h[\xi]^{(m)} = \frac{1}{Q_h + 1}\Big[&Q_h \xi^{(m)} + h_0 \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[u_h])\delta_{ij}\xi^{(m)} \\ &+ h_0 \sum_{i=1}^{n} g_i(P^{(r,m)}[u_h])\delta_i \xi^{(m)}\Big] \quad \text{for } x^{(m)} \in S_h \setminus \partial_0 S_h, \end{aligned} \tag{20}$$

and

$$U_h[\xi]^{(m)} = 0 \quad \text{for } x^{(m)} \in \partial_0 E_h. \tag{21}$$

We prove that for $\xi \in F(S_h, \mathbb{R})$ we have

$$\|U_h[\xi]\|_\infty \le \frac{Q_h}{1 + Q_h}\, \|\xi\|_\infty. \tag{22}$$

Write

$$\begin{aligned} A_{i.+}^{(r,m)}[z] = {} & \frac{h_0}{2h_i} g_i(P^{(r,m)}[z]) + \frac{h_0}{h_i^2}\, f_{ii}(P^{(r,m)}[z]) \\ & - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{h_0}{h_i h_j}\, |f_{ij}(P^{(r,m)}[z])|, \end{aligned} \tag{23}$$

$$\begin{aligned} A_{i.-}^{(r,m)}[z] = {} & -\frac{h_0}{2h_i}\, g_i(P^{(r,m)}[z]) + \frac{h_0}{h_i^2}\, f_{ii}(P^{(r,m)}[z]) \\ & - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{h_0}{h_i h_j}\, |f_{ij}(P^{(r,m)}[z])|, \end{aligned} \tag{24}$$

$$A^{(r,m)}[z] = -2 \sum_{i=1}^{n} \frac{h_0}{h_i^2}\, f_{ii}(P^{(r,m)}[z]) + \sum_{(i,j) \in J} \frac{h_0}{h_i h_j}\, |f_{ij}(P^{(r,m)}[z])|, \tag{25}$$

where $1 \le i \le n$. It follows from Assumption H$[f, g, \alpha]$ that

$$|U_h[\xi]^{(m)}|(Q_h + 1)$$
$$\le |(Q_h + A^{(r,m)}[u_h])\xi^{(m)}|$$
$$+ \Big|\sum_{i=1}^{n} A_{i.+}^{(r,m)}[u_h]\xi^{(m+e_i)}\Big| + \Big|\sum_{i=1}^{n} A_{i.-}^{(r,m)}[u_h]\xi^{(m-e_i)}\Big|$$
$$+ h_0 \sum_{(i,j)\in J_+^{(r,m)}[u_h]}^{n} \frac{1}{2h_ih_j} f_{ij}(P^{(r,m)}[u_h])[|\xi^{(m+e_i+e_j)}| + |\xi^{(m-e_i-e_j)}|]$$
$$- h_0 \sum_{(i,j)\in J_-^{(r,m)}[u_h]}^{n} \frac{1}{2h_ih_j} f_{ij}(P^{(r,m)}[u_h])[|\xi^{(m+e_i-e_j)}| + |\xi^{(m-e_i+e_j)}|].$$

We conclude from Assumption H$[f, g, \alpha]$ and from (18) that

$$Q_h + A^{(r,m)}[u_h] \ge 0, \quad A_{i.+}^{(r,m)}[u_h] \ge 0, \quad A_{i.-}^{(r,m)}[u_h] \ge 0, \quad 1 \le i \le n,$$

and

$$(26) \quad A^{(r,m)}[u_h] + \sum_{i=1}^{n} A_{i.+}^{(r,m)}[u_h] + \sum_{i=1}^{n} A_{i.-}^{(r,m)}[u_h]$$
$$+h_0 \sum_{(i,j)\in J_+^{(r,m)}[u_h]}^{n} \frac{1}{h_ih_j} f_{ij}(P^{(r,m)}[u_h]) - h_0 \sum_{(i,j)\in J_-^{(r,m)}[u_h]}^{n} \frac{1}{h_ih_j} f_{ij}(P^{(r,m)}[u_h]) = 0.$$

The above considerations and (21) imply

$$|U_h[\xi]^{(m)}|(Q_h + 1) \le Q_h\|\xi\|_\infty \quad \text{for } -K \le m \le K.$$

This completes the proof of (22). It follows that the norm of the operator U_h is less than 1. Thus there exists exactly one solution of (19). Since u_h is given on the initial boundary set $\partial_0 E_h \cup E_{0.h}$, the proof is completed by induction with respect to r, $0 \le r \le N$.

4. Convergence of implicit difference schemes. Write $I = [-d_0, 0]$. We will need the operator $V : C(D, \mathbb{R}) \to C(I, \mathbb{R}_+)$ defined by

$$V[w](t) = \max\{|w(t, x)| : \ x \in [-d, d]\}, \quad t \in I.$$

Let $\|\cdot\|_D$ denote the maximum norm in the space $C(D, \mathbb{R})$. Set

$$I_h = \{t^{(r)} : \ -K_0 \le r \le 0\}, \quad A_h = \{t^{(r)} : 0 \le r \le N\}.$$

For $\xi : I_h \cup A_h \to \mathbb{R}$ we write $\xi^{(r)} = \xi(t^{(r)})$. If $\xi : I_h \cup A_h \to \mathbb{R}$ and $t^{(r)} \in A_h$ then $\xi_{[r]} : I_h \to \mathbb{R}$ is defined by

$$\xi_{[r]}(\tau) = \xi(t^{(r)} + \tau), \quad \tau \in I_h.$$

We will need the operator $T_{h_0} : F(I_h, \mathbb{R}) \to C(I, \mathbb{R})$ defined by

$$(27) \quad T_{h_0}[\xi](t) = \xi^{(r+1)} \frac{t - t^{(r)}}{h_0} + \xi^{(r)} \Big(1 - \frac{t - t^{(r)}}{h_0}\Big), \qquad t^{(r)} \le t \le t^{(r+1)},$$

where $\xi \in F(I_h, \mathbb{R})$.

It is clear that T_{h_0} is a particular case of T_h.

LEMMA 1. *Suppose that $\widetilde{z} : \Omega \to \mathbb{R}$ and*

1) *$\widetilde{z}(t, \cdot) : [-c, c] \to \mathbb{R}$ is of class C^2 for $t \in [-d_0, a]$ and $\widetilde{z}(\cdot, x) : [-d_0, a] \to \mathbb{R}$ is of class C^1 for $x \in [-c, c]$,*
2) *the constants $\widetilde{d}, \widetilde{d}_0 \in \mathbb{R}_+$ are defined by the relations*

$$(28) \qquad \begin{aligned} |\partial_{x_i x_j} \widetilde{z}(t, x)| &\le \widetilde{d} \qquad \text{for } (t, x) \in \Omega,\ i, j = 1, \ldots, n, \\ |\partial_t \widetilde{z}(t, x)| &\le \widetilde{d}_0 \qquad \text{for } (t, x) \in \Omega, \end{aligned}$$

Then

$$(29) \qquad |T_h(\widetilde{z}_h)_{[r,m]} - \widetilde{z}^{(r,m)}| \le \widetilde{d}_0 h_0 + \widetilde{d} \|h'\|^2, \qquad (t^{(r)}, x^{(m)}) \in E_h,$$

where $\widetilde{z}_h$ is the restriction of $\widetilde{z}$ to the set Ω_h.

The above lemma follows from Theorem 5.27 in [7]. The estimate (29) states that the function is approximated by $T_h[\widetilde{z}_h]$ and the error of this approximation is bounded by $\widetilde{d}_0 h_0 + \widetilde{d} \|h'\|^2$.

Now we formulate assumptions on the regularity of G, f, g with respect to the functional variables.

ASSUMPTION H$[\sigma, f, g, G]$. The functions f, g satisfy Assumption H$[f, g, \alpha]$ and

1) there is $\sigma : [0, a] \times C(I, \mathbb{R}) \to \mathbb{R}_+$ such that
   (i) σ is continuous, nondecreasing with respect to both variables and $\sigma(t, \theta) = 0$ for $t \in [0, a]$, where $\theta \in C(I, \mathbb{R}_+)$ is given by $\theta(\tau) = 0$ for $\tau \in I$,
   (ii) for each $c \ge 1$ the function $\widetilde{\omega}(t) = 0$ for $t \in I \cup [0, a]$ is the maximal solution of the Cauchy problem

$$(30) \qquad \zeta'(t) = c\sigma(t, \zeta_t), \qquad \zeta(t) = 0 \quad \text{for } t \in I,$$

2) the estimates

$$\begin{aligned} \|f(t, x, w) - f(t, x, \overline{w})\|_* &\le \sigma(t, V[w - \overline{w}]), \\ \|g(t, x, w) - g(t, x, \overline{w})\| &\le \sigma(t, V[w - \overline{w}]), \\ |G(t, x, w) - G(t, x, \overline{w})| &\le \sigma(t, V[w - \overline{w}]) \end{aligned}$$

   are satisfied on Σ.

For a function $\eta : I_h \to \mathbb{R}$ we write $\eta^{(r)} = \eta(t^{(r)})$.

Now we prove a theorem on convergence of the method (9), (10).

Theorem 4.1. *Suppose that Assumption* $\mathrm{H}[\sigma, f, g, G]$ *is satisfied and*

1) *the function* $v : \Omega \to \mathbb{R}$ *is a solution of* (1), (2) *and* v *is of class* C^2 *on* Ω,
2) *there is* $c_0 > 0$ *such that* $h_i h_j^{-1} \le c_0$, $i, j = 1, \ldots, n$,
3) *the function* $u_h : E_h \to \mathbb{R}$ *is a solution of* (9), (10) *and there is* $\gamma_0 : H \to \mathbb{R}_+$ *such that*

$$|v^{(r,m)} - u_h^{(r,m)}| \le \gamma_0(h) \quad \text{on } \partial_0 E_h \cup E_{0.h} \tag{31}$$

and $\lim_{h\to 0} \gamma_0(h) = 0$.

Then there exists a function $\alpha : H \to \mathbb{R}_+$ *such that*

$$|(u_h - v_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0 \tag{32}$$

where v_h *is the restriction of* v *to the set* E_h.

Proof. We divide the proof into a sequence of steps.

I. Write $z_h = v_h - u_h$. We construct a difference equation for z_h. Let $\Gamma_h : E'_h \to \mathbb{R}$ and $\Gamma_{0.h} : \partial_0 E_h \cup E_{0.h} \to \mathbb{R}$ be defined by the relations

$$\begin{aligned} \delta_0 v_h^{(r,m)} &= F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} && \text{on } E'_h,\\ v_h^{(r,m)} &= \varphi_h^{(r,m)} + \Gamma_{0.h}^{(r,m)} && \text{on } \partial_0 E_h \cup E_{0.h}. \end{aligned}$$

It follows from Lemma 1, condition 1) of the theorem and (31) that there is $\gamma : H \to \mathbb{R}_+$ such that

$$|\Gamma_h^{(r,m)}| \le \gamma(h) \quad \text{on } E'_h, \qquad \lim_{h\to 0} \gamma(h) = 0.$$

Then we have

$$\begin{aligned} \delta_0 z_h^{(r,m)} = {} & \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[u_h]) \delta_{ij} z_h^{(r+1,m)} + \sum_{i=1}^{n} g_i(P^{(r,m)}[u_h]) \delta_i z_h^{(r+1,m)} \\ & + \Lambda_h^{(r,m)} + \Gamma_h^{(r,m)}, \end{aligned}$$

where

$$\begin{aligned} \Lambda_h^{(r,m)} = {} & \sum_{i,j=1}^{n} [f_{ij}(P^{(r,m)}[v_h]) - f_{ij}(P^{(r,m)}[u_h])] \delta_{ij} v_h^{(r+1,m)} \\ & + \sum_{i=1}^{n} [g_i(P^{(r,m)}[v_h]) - g_i(P^{(r,m)}[u_h])] \delta_i v_h^{(r+1,m)} \\ & + G(P^{(r,m)}[v_h]) - G(P^{(r,m)}[u_h]). \end{aligned}$$

The above relations and (7), (8), (11)–(13), (23)–(25) imply

$$(33)\quad z_h^{(r+1,m)}[1 - A^{(r,m)}[u_h]]$$
$$= z_h^{(r,m)} + h_0 \sum_{i=1}^{n} A_{i.+}^{(r,m)}[u_h] z_h^{(r+1,m+e_i)} + h_0 \sum_{i=1}^{n} A_{i.-}^{(r,m)}[u_h] z_h^{(r+1,m-e_i)}$$
$$- h_0 \sum_{(i,j)\in J_-^{(r,m)}[u_h]} \frac{1}{2h_i h_j} f_{h.ij}(P^{(r,m)}[u_h])[z_h^{(r+1,m+e_i-e_j)} + z_h^{(r+1,m-e_i+e_j)}]$$
$$+ h_0 \sum_{(i,j)\in J_+^{(r,m)}[u_h]} \frac{1}{2h_i h_j} f_{h.ij}(P^{(r,m)}[u_h])[z_h^{(r+1,m+e_i+e_j)} + z_h^{(r+1,m-e_i-e_j)}]$$
$$+ h_0 \Lambda_h^{(r,m)} + h_0 \Gamma_h^{(r,m)}.$$

The above relation can be considered as a difference equation for the error z_h.

II. Let the function $\varepsilon_h^{(r)} : I_h \cup A_h \to \mathbb{R}^+$ be defined by

$$\varepsilon_h^{(r)} = \max\{|z_h^{(i,m)}| : (t^{(i)}, x^{(m)}) \in \Omega_h \cap ([-d_0, t^{(r)}] \times \mathbb{R}^n)\},\ -K_0 \le r \le N.$$

We will write a difference inequality for ε_h. We deduce from (16) that

$$(34)\quad A_{i.+}^{(r,m)}[u_h] \ge 0, \quad A_{i.-}^{(r,m)}[u_h] \ge 0, \quad 1 - A^{(r,m)}[u_h] \ge 0 \quad \text{for } 1 \le i \le n.$$

It is easy to prove by induction with respect to n that

$$\sum_{\lambda\in\Im} \left(\frac{x - x^{(m)}}{h'}\right)^{\lambda} \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} = 1 \qquad \text{for } x^{(m)} \le x \le x^{(m+1)}.$$

This gives

$$V[T_h[z_h]_{[r,m]}](\tau) \le T_{h_0}[(\varepsilon_h)_{[r]}](\tau), \qquad \tau \in I,\ 0 \le r \le N.$$

We conclude from condition 2) of Assumption H$[\sigma, f, g, G]$ that

$$(35)\quad \sum_{i,j=1}^{n} |f_{ij}(P^{(r,m)}[v_h]) - f_{ij}(P^{(r,m)}[u_h])|$$
$$= \sum_{i,j=1}^{n} |f_{ij}(t^{(r)}, x^{(m)}, (T_h[v_h])_{\alpha(r,m)}) - f_{ij}(t^{(r)}, x^{(m)}, (T_h[u_h])_{\alpha(r,m)})|$$
$$\le \sigma(t^{(r)}, T_{h_0}[(\varepsilon_h)_{[r]}])$$

In a similar way we obtain

$$(36)\qquad \|g(P^{(r,m)}[v_h]) - g(P^{(r,m)}[u_h])\| \le \sigma(t^{(r)}, T_{h_0}[(\varepsilon_h)_{[r]}]),$$

and

$$
(37)\qquad |G(t^{(r)},x^{(m)},(T_h[v_h])_{\alpha(r,m)})-G(t^{(r)},x^{(m)},(T_h[u_h])_{\alpha(r,m)})| \le \sigma(t^{(r)},T_{h_0}[(\varepsilon_h)_{[r]}]).
$$

Let $\widetilde{c}\in\mathbb{R}_+$ be a constant such that

$$
|\partial_{x_i}v(t,x)|,|\partial_{x_ix_j}v(t,x)|\le\widetilde{c},\qquad i,j=1,\dots,n,\ (t,x)\in E.
$$

It follows from (26) and (33)–(37) that the function ε_h satisfies the recurrent inequality

$$
(38)\qquad \varepsilon_h^{(r+1)}\le\varepsilon_h^{(r)}+(2\widetilde{c}+1)h_0\sigma_h(t^{(r)},T_{h_0}[(\varepsilon_h)_{t^{(r)}}])+h_0\gamma(h),\qquad 0\le r\le N-1,
$$

and

$$
(39)\qquad \varepsilon_h^{(r)}\le\gamma_0(h)\quad\text{ for }-K_0\le r\le 0.
$$

III. We prove that there is $\alpha:H\to\mathbb{R}_+$ such that $\varepsilon_h^{(r)}\le\alpha(h)$ for $0\le r\le N$ and $\lim_{h\to 0}\alpha(h)=0$. Consider the Cauchy problem

$$
(40)\qquad \xi'(t)=(2\widetilde{c}+1)\sigma(t,\xi_t+(\mu(h))_t)+\gamma(h),
$$

$$
(41)\qquad \xi(t)=\gamma_0(h)\quad\text{ for }t\in I,
$$

where $\mu:H\to(0,\infty)$, $\lim_{h\to 0}\mu(h)=0$ and $(\mu(h))_t\in C(I,\mathbb{R}_+)$ is a constant function: $(\mu(h))_t(\tau)=\mu(h)$ for $\tau\in I$. It follows from condition 1) of Assumption H$[f,g,\alpha]$ that there is $\widetilde{\varepsilon}>0$ such that for $\|h\|\le\widetilde{\varepsilon}$ the maximal solution $\omega(\cdot,h)$ of (40), (41) is defined on $I\cup[0,a]$ and

$$
(42)\qquad \lim_{h\to 0}\omega(t,h)=0\quad\text{ uniformly on }I\cup[0,a].
$$

Suppose that $\widetilde{h}\in H$ is fixed and $\|\widetilde{h}\|<\varepsilon$. Denote by $C[\widetilde{h}]$ the set of all $h\in H$ satisfying $\|h\|<\widetilde{\varepsilon}$ and $\mu(h)<\mu(\widetilde{h})$, $\gamma(h)<\gamma(\widetilde{h})$. Then the maximal solution $\omega(\cdot,h)$ of the difference equation (40), (41), where $h\in C[\widetilde{h}]$, satisfies the condition

$$
(43)\qquad \omega(t,h)\le\omega(t,\widetilde{h})\quad\text{ for }t\in I\cup[0,a].
$$

Let $\omega_{h_0}(\cdot,h)$ denote the restriction of $\omega(\cdot,h):I\cup[0,a]\to\mathbb{R}_+$ to the set $I_h\cup A_h$. It follows from (27) that

$$
(44)\qquad T_{h_0}[(\omega_{h_0}(\cdot,h))_{[r]}](\tau)-(\omega(\cdot,h))_{t^{(r)}}(\tau)\le h_0\omega'(a,h)\le h_0\omega'(a,\widetilde{h})
$$

for $\tau\in I$. There is $\overline{\varepsilon}>0$ such that for $h\in C[\widetilde{h}]$ with $\|h\|<\overline{\varepsilon}$ we have

$$
(45)\qquad \mu(\widetilde{h})>\mu(h)\ge h_0\omega'(a,\widetilde{h}).
$$

We conclude from condition 1) of Assumption H$[f,g,\alpha]$ and from (44), (45) that for $h \in C[\widetilde{h}]$, $\|h\| < \overline{\varepsilon}$ we have

$$\begin{aligned}\omega'(t^{(r)},h) &= (2\widetilde{c}+1)\sigma(t^{(r)},(\omega(\cdot,h))_{t^{(r)}}+(\mu(h))_t)+\gamma(h)\\ &= \sigma(t^{(r)},T_{h_0}[(\omega_{h_0}(\cdot,h))_{[r]}]+(\omega(\cdot,h))_{t^{(r)}}-T_{h_0}[(\omega_{h_0}(\cdot,h))_{[r]}]\\ &\quad +(\mu(h))_{t^{(r)}})+\gamma(h)\\ &\geq \sigma(t^{(r)},T_{h_0}[(\omega_{h_0}(\cdot,h))_{[r]}])-(h_0\omega'(a,h))_{(t^{(r)})}+(\mu(h))_{(t^{(r)})})+\gamma(h)\\ &\geq \sigma(t^{(r)},T_{h_0}[(\omega_{h_0}(\cdot,h))_{[r]}])+\gamma(h), \qquad 0\leq r\leq N.\end{aligned}$$

Since $\omega(\cdot,h)$ is a convex function, for $h \in C[\widetilde{h}]$ with $\|h\| < \overline{\varepsilon}$ we have the difference inequality

$$\begin{aligned}\omega_{h_0}(t^{(r+1)},h) \geq \omega_{h_0}(t^{(r)},h) &+ h_0(2\widetilde{c}+1)\sigma(t^{(r)},T_{h_0}[(\omega_{h_0}(\cdot,h))_{[r]}])\\ &+ h_0\gamma(h), \qquad 0\leq r\leq N-1.\end{aligned}$$

Since ε_h satisfies (38), (39) the above relations and (24) imply the estimate

$$\varepsilon_h^{(r)} \leq \omega(t^{(r)},h), \qquad 0\leq r\leq N.$$

where $h \in C[\widetilde{h}]$, $\|h\| < \overline{\varepsilon}$. It follows from (25), (43) that the assertion of the theorem is satisfied with $\alpha(h) = \omega(a,h)$.

Remark 2. Suppose that Assumption $\mathrm{H}[\sigma,f,g]$ is satisfied with

$$\sigma(t,w) = L\|w\|_I, \qquad (t,p)\in[0,a]\times\mathbb{R}_+ \text{ where } L\in\mathbb{R}_+.$$

Then f, g and G satisfy the Lipschitz condition with respect to the functional variable. We obtain the following error estimates:

$$|(u_h-v_h)^{(r,m)}| \leq \widetilde{\alpha}(h)e^{cLa}+\widetilde{\gamma}(h)\,\frac{e^{cLa}-1}{cL} \qquad \text{on } E_h \text{ if } L>0,$$

and

$$|(u_h-v_h)^{(r,m)}| \leq \widetilde{\alpha}(h)+a\widetilde{\gamma}(h) \qquad \text{on } E_h \text{ if } L=0.$$

The above inequalities follow from (32) with $\alpha(h) = \omega_h(a)$ where $\omega_h : [0,a]\to\mathbb{R}_+$ is a solution of the problem

$$\zeta'(t) = cL\zeta(t)+\widetilde{\gamma}(h), \qquad \zeta(0)=\alpha_0(h).$$

It is important in our considerations that equations with deviated variables appear in comparison problems. In the next lemma we give a suitable example.

Lemma 2. *If $\mu\geq\nu>1$ and $L\in\mathbb{R}_+$, $c\geq 1$ then the maximal solution of the Cauchy problem*

$$\zeta'(t) = c[\zeta(t^\mu)]^{1/\nu}+L\zeta(t), \qquad \zeta(0)=0, \tag{46}$$

is $\overline{\zeta}(t)=0$ for $t\in[0,a]$ where $a<1$.

Proof. There are $\widetilde{\varepsilon}, \widetilde{c} > 0$ such that the maximal solution $\overline{\zeta}$ of (46) satisfies

$$\overline{\zeta}(t) \le \widetilde{C}t \quad \text{for } t \in [0, \widetilde{\varepsilon}].$$

Write

$$C = \max\{c, L, \widetilde{C}\}.$$

Then $\overline{\zeta}$ satisfies the integral inequality

$$\zeta(t) \le C\Big[\int_0^t [\zeta(s^\mu)]^{1/\nu}\,ds + \int_0^t \zeta(s)\,ds\Big], \qquad t \in [0, \widetilde{\varepsilon}],$$

and $\overline{\zeta}(t) \le Ct$ for $t \in [0, \widetilde{\varepsilon}]$.

It follows from the above relations that

$$\overline{\zeta}(t) \le C^k t^k, \qquad t \in [0, \widetilde{\varepsilon}],\ k \ge 1.$$

Then there is ε_0 such that $\overline{\zeta} = 0$ for $t \in [0, \varepsilon_0]$ and consequently $\overline{\zeta}(t) = 0$ on $[0, a]$.

REMARK 3. Note that the maximal solution of (46) with $\nu > 1$ and $\mu = 1$ is positive on $(0, a]$. The above considerations and examples (5), (6) show that the class of differential functional equations which is covered by our theory is more general than the corresponding classes considered in [2], [3], [13], [14].

REMARK 4. Consider the explicit difference method (10)–(14). Then we need the following assumption on f and on the steps of the mesh ([10]):

$$1 - 2h_0 \sum_{j=1}^n \frac{1}{h_j^2} f_{jj}(P) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |f_{ij}(P)| \ge 0, \tag{47}$$

where $P \in \Sigma$. If the functions f_{ij}, $i, j = 1, \ldots, n$, are bounded on Σ then inequality (47) states relations between h_0 and $h' = (h_1, \ldots, h_n)$. It is important in our considerations that condition (47) is omitted in the convergence theorem.

REMARK 5. Suppose that the function σ has the structure

$$\sigma(t, \tau) = A(t)B(\tau),$$

where $A \in C([0, a], \mathbb{R}_+)$, $B \in C(\mathbb{R}_+, \mathbb{R}_+)$, $B(0) = 0$ and $B(\tau) > 0$ for $\tau > 0$. The Cauchy problem

$$\xi'(t) = A(t)B(\xi(t)), \qquad \xi(0) = 0,$$

has a unique solution if and only if

$$\int_0^\varepsilon \frac{d\tau}{B(\tau)} = \infty, \qquad \text{where } \varepsilon > 0.$$

Then for each $c \geq 1$ the Cauchy problem

$$\xi'(t) = cA(t)B(\xi(t)), \qquad \xi(0) = 0,$$

has the only solution $u(t) = 0$ for $t \geq 0$.

There is a comparison problem

$$\xi'(t) = \sigma(t, \xi(t)), \qquad \xi(0) = 0,$$

which has the maximal solution $u(t) = 0$ for $t \geq 0$ while the initial problem

$$\xi'(t) = c\sigma(t, \xi(t)), \qquad \xi(0) = 0,$$

for $c \geq 0$ has a positive solution on $[0, a]$. Such an example is given in [1].

5. Numerical examples

Example 1. Write

$$E = [0, 0.25] \times [-1, 1] \times [-1, 1], \qquad E_0 = \{0\} \times [-1, 1] \times [-1, 1],$$
$$\partial_0 E = [0, 0.25] \times [([-1, 1] \times [-1, 1]) \setminus ((-1, 1) \times (-1, 1))].$$

Consider the differential equation with deviated variables

$$\begin{aligned}(48) \quad \partial_t z(t,x,y) = {} & \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) \\ & + xy \sin\left[z\left(t, \frac{x+y}{2}, \frac{x-y}{2}\right) - e^{-t/2} z(t/2, x, y)\right] \partial_{xy} z(t,x,y) \\ & + \sqrt{z(t^2, x, y)} - z(t^2, x, y) + f(t,x,y) z(t,x,y)\end{aligned}$$

and the initial boundary conditions

$$(49) \qquad z(0,x,y) = 1 \qquad \text{for } (x,y) \in [-1,1] \times [-1,1],$$

$$(50) \qquad z(t,-1,y) = z(t,1,y) = e^{ty^2} \qquad \text{for } t \in [0, 0.25],\ y \in [-1,1],$$

$$(51) \qquad z(t,x,-1) = z(t,x,1) = e^{tx^2} \qquad \text{for } t \in [0, 0.25],\ x \in [-1,1].$$

where

$$f(t,x,y) = -4t - 4t^2(x^2 + y^2 + x^2y^2) + (x^2 + y^2 - 1).$$

The solution of (48)–(51) is known:

$$v(t,x,y) = e^{t(x^2+y^2-1)}.$$

We found approximate solutions of (48), (49) using both implicit and explicit numerical methods, and taking the following steps of the mesh: $h_0 = 0.0025,\ h_1 = 0.0025,\ h_2 = 0.0025$.

Let u_h denote the approximate solution of (48)–(51) which is obtained by a difference scheme.

The average errors of the method $\varepsilon_h^{(r)}$ are found with the following formula:

$$\varepsilon_h^{(r)} = \frac{1}{(2N_1+1)(2N_2+1)} \sum_{i=-N_1}^{N_1} \sum_{j=-N_2}^{N_2} |u_h^{(r,i,j)} - v_h^{(r,i,j)}|$$

where $N_1 h_1 = 1$, $N_2 h_2 = 1$ and v_h is the restriction of the function v to the mesh.

Note that the function f and the steps of the mesh do not satisfy condition (47), which is necessary for the explicit method to be convergent. In our numerical example the average errors of the explicit method exceeded 10^{17}, while the average errors ε_h for fixed $t^{(r)}$ of implicit method are given in Table 1.

Table 1. Errors (ε_h)
($h_0 = 0.0025,\ h_1 = 0.0025,\ h_2 = 0.0025$)

$t^{(r)}$	$\varepsilon_h^{(r)}$
$t = 0.05$	$33 \cdot 10^{-5}$
$t = 0.01$	$55 \cdot 10^{-5}$
$t = 0.15$	$70 \cdot 10^{-5}$
$t = 0.20$	$80 \cdot 10^{-5}$
$t = 0.25$	$86 \cdot 10^{-5}$

The differential equation (48) contains the deviated variables $(t, (x+y)/2, (x-y)/2)$ and the example has the following property: if $(t^{(r)}, x^{(m_1)}, y^{(m_2)})$ is a grid point then

$$(t^{(r)}, 0.5(x^{(m_1)} + y^{(m_2)}), 0.5(x^{(m_1)} - y^{(m_2)})),$$

in general, is not a grid point. We approximate the value $z(t^{(r)}, 0.5(x^{(m_1)} + y^{(m_2)}), 0.5(x^{(m_1)} - y^{(m_2)}))$ using the interpolating operator T_h with $n = 2$.

Example 2. Consider the integral-differential equation

$$\begin{aligned}
(52)\quad & \partial_t z(t,x,y) \\
&= \Big\{1 + \Big[2t(y^2-1)\int_{-1}^{x} sz(t,s,y)\,ds - z(t,x,y)\Big]^2\Big\}\partial_{xx} z(t,x,y) \\
&\quad + \Big\{1 + \Big[2t(x^2-1)\int_{-1}^{y} sz(t,x,s)\,ds - z(t,x,y)\Big]^2\Big\}\partial_{yy} z(t,x,y) \\
&\quad + \partial_{xy} z(t,x,y)\sin\Big[(x^2-1)(y^2-1)\int_{0}^{t} z(\tau,x,y)\,d\tau - z(t,x,y) + 1\Big] \\
&\quad + f(t,x,y)z(t,x,y)
\end{aligned}$$

and the initial boundary condition

$$z(t,x,y) = 1 \quad \text{on } \partial_0 E \cup E_0 \tag{53}$$

where E, E_0, ∂E_0 are defined as earlier and

$$f(t,x,y) = -4t(x^2+y^2-2) - 8t^2(x^2(y^2-1)^2 + y^2(x^2-1)^2).$$

The solution of (52), (53) is known:

$$v(t,x,y) = e^{t(x^2-1)(y^2-1)}.$$

As in the previous numerical example we chose the steps of the mesh which do not satisfy condition (47). In accordance with our expectations the explicit method is not convergent, and the average errors are so big that it is impossible for the personal computer to compute them, while the implicit method is convergent and gives the following average errors.

Table 2. Errors (ε_h)
($h_0 = 0.0025,\ h_1 = 0.0025,\ h_2 = 0.0025$)

$t^{(r)}$	$\varepsilon_h^{(r)}$
$t = 0.10$	$15 \cdot 10^{-4}$
$t = 0.20$	$69 \cdot 10^{-4}$
$t = 0.30$	$18 \cdot 10^{-3}$
$t = 0.40$	$38 \cdot 10^{-3}$
$t = 0.50$	$66 \cdot 10^{-3}$

The differential equation (52) contains integrals of the unknown function z. Therefore the corresponding difference equation includes the terms

$$\int_{-x^{(m_1)}}^{x^{(m_1)}} \tau z(t^{(r)},\tau,y)\,d\tau, \qquad \int_{-y^{(m_2)}}^{y^{(m_2)}} \tau z(t^{(r)},x,\tau)\,d\tau, \qquad \int_{0}^{t^{(r)}} z(\tau,x^{(m_1)},y^{(m_2)})\,d\tau$$

where $z(t^{(r)},x^{(m_1)},y^{(m_2)})$ is a grid point. The above integrals are approximated by

$$\int_{-x^{(m_1)}}^{x^{(m_1)}} \tau T_h[z_h](t^{(r)},\tau,y)\,d\tau, \qquad \int_{-y^{(m_2)}}^{y^{(m_2)}} \tau T_h[z_h](t^{(r)},x,\tau)\,d\tau,$$

$$\int_{0}^{t^{(r)}} T_h[z_h](\tau,x^{(m_1)},y^{(m_2)})\,d\tau$$

where z_h is a solution of a difference equation. The above method is equivalent to the trapezoidal rule.

The computation was performed on a PC computer. Numerical results are consistent with our mathematical theory.

Difference schemes obtained by a discretization of problem (1), (2) have the following property: a large number of previous values $z^{(r,m)}$ must be preserved, because they are needed to compute an approximate solution with $t = t^{(r+1)}$.

The above examples show that there are implicit difference schemes which are convergent, while the corresponding classical methods are not convergent. This is due to the fact that we need the relation (47) for steps of the mesh in the classical case. We do not need this condition in our implicit method. Implicit difference methods in Sections 5 and 6 have the potential for applications in the numerical solving of differential integral equations or equations with deviated variables.

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Institute of Mathematics
University of Gdańsk
Wit Stwosz St. 57
80-952 Gdańsk, Poland
E-mail: karolina.kropielnicka@math.univ.gda.pl

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