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BLOW-UP OF THE SOLUTION TO THE INITIAL-VALUE PROBLEM IN NONLINEAR THREE-DIMENSIONAL HYPERELASTICITY

Abstract. We consider the initial value problem for the nonlinear partial differential equations describing the motion of an inhomogeneous and anisotropic hyperelastic medium. We assume that the stored energy function of the hyperelastic material is a function of the point x and the nonlinear Green–St. Venant strain tensor e_{jk}. Moreover, we assume that the stored energy function is C^∞ with respect to x and e_{jk}. In our description we assume that Piola–Kirchhoff's stress tensor p_{jk} depends on the tensor e_{jk}. This means that we consider the so-called physically nonlinear hyperelasticity theory. We prove (local in time) existence and uniqueness of a smooth solution to this initial value problem. Under the assumption about the stored energy function of the hyperelastic material, we prove the blow-up of the solution in finite time.

1. Introduction. First we describe global existence results to nonlinear wave equations and nonlinear elasticity equations. Next, we show which kind of nonlinearities imply a blow up in finite time. Finally, we present a nonlinear hyperbolic system describing a hyperbolic medium in the three-dimensional space which has a physical interpretation. We assume that the nonlinear Piola–Kirchhoff stress tensor (which is nonsymmetric) depends on the nonlinear Green–St. Venant strain tensor, leading to the so-called physically nonlinear elasticity theory (cf. [4]). Our approach allows us to describe the largest class of materials considered to date. For such system of equations and nonlinearities generated physically we show that the nonlinearity

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does not satisfy the so-called "null conditions". This means that there are no global results for such nonlinearities in $\mathbb{R}^3$. Choosing some stored energy functions we prove that the solution of the initial-value problem for nonlinear hyperelasticity blows up in finite time.

We consider the nonlinear wave equation of the form

$$y_{tt} - \Delta y = f(y_t, \nabla y, \nabla y_t, \nabla^2 y) \equiv f(Dy, \nabla Dy), \tag{1.1}$$

with prescribed initial data

$$y(t=0) = y_0, \qquad y_t(t=0) = y_1, \tag{1.2}$$

where $y = y(t,x) \in \mathbb{R}$, $t \geq 0$, $x \in \mathbb{R}^n$, $n \in \mathbb{N}$ arbitrary, $\Delta = \sum_{i=1}^n \partial_i^2$, $\partial_i = \partial/\partial x_i$, $i = 1, \ldots, n$; $\partial_t = \partial/\partial_t$, $D = \binom{\partial_t}{\nabla}$, $\nabla = (\partial_1, \ldots, \partial_n)$. Let $u := (Dy) = (\partial_t y, \partial_1 y, \ldots, \partial_n y)$, $\nabla u = \nabla Dy = (\partial_1 u, \ldots, \partial_n u)$, $u_0 := (y_1, Dy_0)$.

Existence of global solutions to problem (1.1), (1.2) for small data is shown in [12], [13].

Let us recall the following result (see [22]).

Lemma 1.1. *Assume that the nonlinear function f satisfies*

$$\begin{aligned} &f \in C^\infty(\mathbb{R}^{(n+1)^2}, \mathbb{R}), \\ &f(u, \nabla u) = o(|u| + |\nabla u|_{\alpha+1}) \quad \text{as } |u| + |\nabla u| \to 0, \end{aligned} \tag{1.3}$$

where α satisfies the inequality

$$\frac{1}{\alpha}\left(1 + \frac{1}{\alpha}\right) < \frac{n-1}{2}. \tag{1.4}$$

Then there is a unique global solution of the initial value problem for the nonlinear wave equation (1.1).

The lemma describes a relation between α and n.

The case $\alpha = 1$ implies that $n \geq 6$. This case in not physical.

If $\alpha = 2$ then $n \geq 4$, which is also not physical. If $\alpha = 3, 4$ then $n \geq 2$, which is also nonphysical because physical nonlinearities are quadratic.

The *quadratic nonlinearities in* $\mathbb{R}^3$ in general tend to develop *singularities in finite time*, i.e. blow up in finite time (see [12], [22]).

However, the global existence of small solutions is proved in the case of quadratic nonlinearities in $\mathbb{R}^3$ satisfying the null condition (cf. [13]–[15]).

The null condition for the function

$$F = F(y, w) = Q(w) + O(|(y, w)|^3), \tag{1.5}$$

where

$$w = (Dy, D^2 y), \qquad Q(w) = O(|w|^2) \quad \text{as } |w| \to 0 \tag{1.6}$$

reads

$$\tag{1.7}
\begin{gathered}
\frac{\partial^2 Q(w)}{\partial(\partial_a y)(\partial(\partial_b y))}\,\xi_a\xi_b = 0,\\
\frac{\partial^2 Q(w)}{\partial(\partial_a y)\partial(\partial_b\partial_c y)}\,\xi_a\xi_b\xi_c = 0,\\
\frac{\partial^2 Q(w)}{\partial(\partial_a\partial_b y)\partial(\partial_c\partial_d y)}\,\xi_a\xi_b\xi_c\xi_d = 0,
\end{gathered}$$

for all

$$\tag{1.8} \xi = (\xi_0,\xi_1,\xi_2,\xi_3) \in \mathbb{R}\times\mathbb{R}^3 \quad \text{with} \quad \xi_0^2 = \xi_1^2+\xi_2^2+\xi_3^2$$

and arbitrary w, and the summation convention over all repeated indices from 0 to 3 is assumed.

A typical example of a nonlinearity F satisfying the null conditions is

$$\tag{1.9} F = |\nabla y|^2 - y_t^2,$$

which was studied in [19]. For formulations of the null condition see also F. John [14] and W. Strauss [24].

The null condition is a sufficient condition for global existence of solutions for (1.1), (1.2) with quadratic nonlinearities in $\mathbb{R}^3$ (see D. Christodoulou [1]). The method of invariant norms with respect to the Lorentz group has also been applied with appropriate modifications to the Klein–Gordon equation by S. Klainerman in [17], to the Schrödinger equation by P. Constantin [3], and to the equation of elasticity by F. John [14].

It is known that there exists an almost global solution of the nonlinear elasticity equations in $\mathbb{R}^3$ of the form

$$\tag{1.10} \partial_t^2 u_i - \sum_{m,j,l=1}^{3} c_{imjk}(0)\partial_n\partial_k u_j = f_i(\nabla u,\nabla^2 u), \qquad i=1,2,3,$$

with initial conditions

$$\tag{1.11} u(t=0)=u^0, \qquad u_t(t=0)=u^1,$$

where

$$\tag{1.12} c_{imjk}(0) = (c_1^2-2c_2^2)\delta_{im}\delta_{jk} + c_2^2(\delta_{ij}\delta_{km}+\delta_{jm}\delta_{ik}), \qquad i,m,j,k=1,2,3,$$

if the nonlinearity $F=F(w)$ behaves like

$$\tag{1.13} F(w) = O(|w|^3) \quad \text{as } |w|\to 0.$$

Essentially, F consists of the terms f_i, $i=1,2,3$, where f_i is given by (1.10).

For *cubic nonlinearities* the existence of global solutions of nonlinear elasticity *equations* can be proved.

But cubic nonlinearities are not in accord with physical intuition. In the case of the so-called geometrically nonlinear elasticity and also in the case of physically nonlinear elasticity as in the case of hyperelastic materials we have a *quadratic nonlinearity* (cf. [4], [8], [9]).

F. John demonstrated that in the general quadratic case, more precisely in the so-called "genuinely nonlinear" case, solutions will develop singularities in finite time (cf. [12] for radially symmetric solutions and also [13] plane-wave solutions).

John [12] investigated the life span time T_∞ of local solutions for the quadratic case and proved a lower bound for T_∞ in analogy to the situation known for scalar nonlinear wave equations (cf. [13]) which is the so-called "almost global existence", by using the method of invariant norms adapted to the equation of elasticity.

John [14] proved that there exists an "almost global solution" to the following initial value problem for nonlinear elasticity:

$$\Box u = C^{rs}(Du) D_r D_s u, \tag{1.14}$$

$$u = \varepsilon f(x), \quad D_0 u = \varepsilon g(x) \quad \text{for } t = 0,\ x \in \mathbb{R}^3, \tag{1.15}$$

where $D_0 \equiv \partial_t$, $D_n \equiv \partial_n$,

$$\Box_{ik} = \delta_{ik} D_0^2 - (c_2^2 \delta_{rs} \delta_{ik} + (c_1^2 - c_2^2)\delta_{ri}\delta_{sk})\partial_{ri}\partial_{sk}, \quad i,k,r,s = 1,2. \tag{1.16}$$

Here $C^{rs}(Du)$ are matrices with element C^{rs}_{in} depending on the space gradient ∇u and satisfying

$$C^{rs}_{ik}(0) = 0, \quad C^{rs}_{ik}(\nabla u) = C^{sr}_{ki}(\nabla u),$$

where $C^{rs}_{ik}(\nabla u)$ are of class C^∞ and have bounded derivatives in $\mathbb{R}^3$, ∇u is restricted to a small neighbourhoud of the orgin, ε is a positive constant, the vectors f and g belong to $C_0^\infty(\mathbb{R}^3)$, $c_1^2 = (\lambda + 2\mu)/\varrho$, $c_2^2 = \mu/\varrho$, λ, μ are Lamé's constant, ϱ is the density and δ_{rs} denotes Kronecker's symbol.

John obtained the following results:

There exist positive constants A, B, ε_0 (depending on f, g, C^{rs}_{ik} but not on ε) such that a C^∞ solution $u(t,x)$ of (1.14), (1.16) exists for

$$0 \le t \le B \exp(1/A\varepsilon), \quad x \in \mathbb{R}, \tag{1.17}$$

provided $\varepsilon < \varepsilon_0$. The time interval described by (1.17) is exponentially large for small ε. Hence we say that u exists almost globally.

The aim of our paper is to consider a system more general than (1.14)–(1.16). It is known (cf. [4]) that there exist two nonlinear theories of elasticity:

(I) geometrically nonlinear elasticity theory, where we assume that the stress tensor depends linearly on the nonlinear strain tensor;

(II) physically nonlinear elasticity theory, where we assume that the stress tensor depends nonlinearly on the linear strain tensor.

We consider the nonlinear system of equations which describes hyperelastic materials. In our approach we assume that the stored energy function for the hyperelastic medium is a function of the point x and *full nonlinear Green–St. Venant* tensor e. In the equation of motion of an inhomogeneous and anisotropic hyperelastic medium the Piola–Kirchhoff first stress tensor appears, which is nonsymmetric. Such a system of equations describes a wider class of materials. For such a system we consider the initial-value problem in $\mathbb{R}^3$. Next, we prove the existence (local in time) and uniqueness of a solution to the initial-value problem for a nonlinear hyperelastic medium. Finally, assuming the special form of the stored energy function we prove that the solution of the above initial-value problem blows up in finite time.

It is worth mentioning that the nonlinearity that appears on the right hand side of (1.14) has a physical interpretation.

In Section 2 we present the problem and discuss its connection with hyperelastic media. Section 3 is devoted to the local (in time) existence and uniquenes for the solution of the initial-value problem for a nonlinear hyperelastic medium. Finally, in Section 4 we prove the blow-up of the solution to the initial-value problem for the nonlinear system of equations of hyperelasticity theory formulated in Section 2.

2. Statement of the problem. First, we will give the definition of a hyperelastic material (cf. [21]). An elastic material (cf. [21]) is *hypereleastic* if there exists a function

$$\sigma : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{R}, \tag{2.1}$$

differentiable with respect to the variable $e \in \mathbb{M}^3_+$ for each $x \in \overline{\Omega}$, such that the first Piola–Kirchhoff stress tensor p can be represented as follows:

$$p_{jk} = (\delta_{jl} + \partial_l u_j)\,\frac{\partial \sigma}{\partial e_{kl}}, \tag{2.2}$$

where Ω is a bounded open subset in $\mathbb{R}^3$, $\mathbb{M}^3_+ = \{F \in \mathbb{M}^3 : \det F > 0\}$, $\mathbb{M}^3$ the set of 3×3 matrices, $F = \nabla\varphi$ the deformation gradient, and e the nonlinear strain Green–St. Venant tensor with components e_{jk}.

The function σ is called the *stored energy function*. Naturally, if the material is homogeneous, it is a function of $e \in \mathbb{M}^3_+$ only.

Remark 2.1.

(i) The stored energy function is sometimes called the strain energy function.

(ii) The definition of a hyperelastic material given above has a mechanical interpretation. More precisely, it can be shown that an elastic material is hyperelastic if and only if the work is performed in closed processes as in Gurtin [10], [11] (see also Margues [21] for a related result).

The equations of motion for a hyperelastic material with density ϱ form a system of three nonlinear partial differential equations of the following form (cf. [8]):

$$\varrho\partial_t^2 u_j = \partial_k p_{jk} + \varrho f_j, \tag{2.3}$$

$$p_{jk} = (\delta_{jl} + \partial_l u_j)\frac{\partial\sigma}{\partial e_{kl}}, \quad j,k,l = 1,2,3, \tag{2.4}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the displacement vector of the medium and it is a function of time t and position $x = (x_1, x_2, x_3)$, $\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial x_k$, f_j are the components of the body force f, ϱ the density of the medium, p_{jk} the components of Piola–Kirchhoff's stress tensor, and σ the stored energy function of the hyperelastic material which is a function of the point x and Green–St. Venant strain tensor e and is of class C^∞ with respect to x and e.

The Green–St. Venant strain has the following form:

$$e_{jk} = \tfrac{1}{2}(\partial_j u_k + \partial_k u_j + (\partial_j u_s)(\partial_k u_s)), \quad j,k,s = 1,2,3. \tag{2.5}$$

We assume that $\varrho = 1$.

We associate the following initial conditions to the system (2.3), (2.4):

$$u(0,x) = u^0(x), \tag{2.6}$$

$$\partial_t u(0,x) = u^1(x), \tag{2.7}$$

for $x \in \mathbb{R}^3$ and $t \in I = [0,T]$, $T < \infty$. We are looking for solutions of (2.3), (2.4) in $\mathbb{R}^3$ for $t \in I$ satisfying the conditions (2.6), (2.7) and we examine the behaviour of the solution to the above initial value problem in finite time.

In the next section we prove the local (in time) existence and uniqueness of solutions of the initial-value problem (2.3)–(2.7).

Finally, in the last section we prove that the solution of the initial-value problem (2.3)–(2.7) blows up in finite time.

3. Local (in time) existence and uniqueness theorem. In this section, we sketch the proof of the existence and uniqueness (local in time) of a solution to the initial value problem for the nonlinear system (2.3)–(2.7):

THEOREM 3.1. *Let*

$$u^0 \in H^{s+1}(\mathbb{R}^3), \quad u^1 \in H^s(\mathbb{R}^3), \quad f \in C^1(0,T; H^s(\mathbb{R}^3)) \tag{3.1}$$

and $u^0(x), u^1(x) \in G_1$, $G_1 \subset G$ *(*G *is an open set in* $\mathbb{R}^3$*) where* $s > 7/2$. *Then there exists a time interval* $[0,T]$ *with* $T > 0$ *such that the problem*

(1.1)–(1.3) *has a unique classical solution*

$$u \in C^2([0,T] \times \mathbb{R}^3) \tag{3.2}$$

with the properties

$$u(t,x) \in G_2, \qquad \overline{G}_2 \subset G_1 \qquad \textit{for } (t,x) \in [0,T] \times \mathbb{R}^3. \tag{3.3}$$

Proof (for details cf. [8, 9]). From formulae (2.4), we get

$$p_{jk} = a_{jkmn}(x, \partial u)\partial_n u_m + \frac{\partial \sigma}{\partial e_{jk}}, \tag{3.4}$$

where

$$a_{jkmn}(x, \partial u) = \delta_{jm}\,\frac{\partial \sigma}{\partial e_{kn}}, \qquad j,k,m,n = 1,2,3. \tag{3.5}$$

Differentiating (3.4) with respect to t and x_h (for $h = 1,2,3$) we obtain

$$(3.6)_1 \qquad \partial_t p_{jk} = (b_{jkmn}(x,\partial u) + c_{jkmn}(x,\partial u) + d_{jkmn}(x,\partial u))\partial_t \partial_n u_m,$$

$$\begin{aligned}(3.6)_2 \qquad \partial_h p_{jk} = {} & (b_{jkmn}(x,\partial u) + c_{jkmn}(x,\partial u) + d_{jkmn}(x,\partial u))\partial_h \partial_n u_m \\ & + a^h_{jkmn}(x,\partial u)\partial_n u_m + \partial^2\sigma/\partial x_l e_{jk},\ j,k,m,n{=}1,2,3,\end{aligned}$$

where

$$\begin{aligned}
a_{jkmn}(x,\partial u) &= \delta_{jm}\,\frac{\partial\sigma}{\partial e_{kn}},\\
b_{jkmn}(x,\partial u) &= \frac{\partial^2\sigma}{\partial e_{jk}\partial e_{mn}} + \delta_{jm}\,\frac{\partial^2\sigma}{\partial e_{kn}},\\
c_{jkmn}(x,\partial u) &= \frac{\partial^2\sigma}{\partial e_{jk}\partial e_{sn}}\,\partial_s u_m + \frac{\partial^2\sigma}{\partial e_{ks}\partial e_{mn}}\,\partial_s u_j,\\
d_{jkmn}(x,\partial u) &= \frac{\partial^2\sigma}{\partial e_{sk}\partial e_{sn}}\,\partial_l u_j \partial_s u_m,\\
a^h_{jkmn}(x,\partial u) &= \delta_{jm}\,\frac{\partial^2\sigma}{\partial x_k \partial e_{km}},
\end{aligned} \tag{3.7}$$

and $j,k,m,n,l,\ s = 1,2,3$. It is easy to see that the tensors $b_{jkmn}(x,\partial u)$, $c_{jkmn}(x,\partial u)$, $d_{jkmn}(x,\partial u)$ satisfy the following symmetry conditions:

$$b_{jkmn}(x,\partial u) = b_{mnjk}(x,\partial u), \tag{3.8}$$

$$c_{jkmn}(x,\partial u) = c_{mnjk}(x,\partial u), \tag{3.9}$$

$$d_{jkmn}(x,\partial u) = d_{mnjk}(x,\partial u), \tag{3.10}$$

for all 3×3 matrices ∂u and $x \in \mathbb{R}^3$. Putting

$$\begin{aligned} u &= (u_1, u_2, u_3),\\ p &= (p_{11}, p_{21}, p_{31}, p_{12}, p_{22}, p_{32}, p_{13}, p_{23}, p_{33}) \end{aligned} \tag{3.11}$$

and introducing the matrix partial differential operator

$$D = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_2 \\ \partial_3 & 0 & 0 \\ 0 & \partial_3 & 0 \\ 0 & 0 & \partial_3 \end{bmatrix} \tag{3.12}$$

which is called the *generalized gradient* we can express the relations (3.4), $(3.6)_1$, $(3.6)_2$ as follows:

$$p = a(x,\partial u)Du + E(x,\partial u), \tag{3.13}$$

$$\partial_t p = A(x,\partial u)D(\partial_t u), \tag{3.14}$$

$$\partial_h p = D(x,\partial u)D(\partial_h u) + a^h(x,\partial u)Du + E^h(x,\partial u), \qquad h = 1,2,3, \tag{3.15}$$

where $a(x,\partial u)$, $A(x,\partial u)$, $E(x,\partial u)$, $a^h(x,\partial u)$, $E^h(x,\partial u)$ are the following matrices:

$$a(x,\partial u) = (a_{jkmn}(x,\partial u) : k,j = 1,\dots,n;\ m = 1,2,3), \tag{3.16}$$

$$A(x,\partial u) = b(x,\partial u) + c(x,\partial u) + d(x,\partial u), \tag{3.17}$$

$$b(x,\partial u) = (b_{jkmn}(x,\partial u)), \tag{3.18}$$

$$c(x,\partial u) = (c_{jkmn}(x,\partial u)), \tag{3.19}$$

$$d(x,\partial u) = (d_{jkmn}(x,\partial u)), \tag{3.20}$$

with $j,k,m,n \in \{1,2,3\}$, and

$$\begin{aligned} E(x,\partial u) = \left(\frac{\partial\sigma}{\partial e_{jk}}\right) = \bigg(&\frac{\partial\sigma}{\partial e_{11}}, \frac{\partial\sigma}{\partial e_{21}}, \frac{\partial\sigma}{\partial e_{31}}, \frac{\partial\sigma}{\partial e_{12}}, \frac{\partial\sigma}{\partial e_{22}}, \frac{\partial\sigma}{\partial e_{32}}, \\ &\frac{\partial\sigma}{\partial e_{13}}, \frac{\partial\sigma}{\partial e_{23}}, \frac{\partial\sigma}{\partial e_{33}}\bigg), \end{aligned} \tag{3.21}$$

$$a^h(x,\partial u) = (a^h_{jkmn}(x,\partial u) : j,k,m,n,h = 1,2,3), \tag{3.22}$$

$$\begin{aligned} E^h(x,\partial u) = \bigg(&\frac{\partial^2\sigma}{\partial x_h\partial e_{11}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{21}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{31}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{12}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{22}}, \\ &\frac{\partial^2\sigma}{\partial x_h\partial e_{32}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{13}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{23}}, \frac{\partial^2\sigma}{\partial x_h\partial e_{33}}\bigg), \qquad h = 1,2,3. \end{aligned} \tag{3.23}$$

Therefore we can write the system (2.3) in matrix form as follows:

$$\partial_t^2 u = D^* p + f, \tag{3.24}$$

where the growth of p and f with respect to $|u|+|Du|$ is at most cubic, and D^* is the transposed matrix to D.

We assume that the matrix $A(x,\partial u)$ is positive definite, i.e.

$$\langle A(x,\theta)\xi,\xi\rangle \geq c|\xi|^2, \qquad c>0,$$

for any $x\in\mathbb{R}^3$, $\xi\in\mathbb{R}^9$, $\theta\in G_3\subset\mathbb{R}^9$, where G_3 is an open set (cf. [20]).

Remark 3.1. If we assume that

$$\sigma(x,e)=\tfrac{1}{2}e_{khlm}(x)e_{kh}e_{lm},$$

i.e. we consider the so-called geometrically nonlinear elasticity, we obtain a system (cf. [9]) with quadratic nonlinearities on the right side, which suggests that the solution should develop singularities in finite time.

From (3.6) it follows that the right hand side of (3.24) also contains quadratic nonlinearities, which gives the blow-up of the solution to the initial value problem (2.3)–(2.7).

Using the modified Sommerfeld method (cf. [23]), i.e. introducing the notation

$$v=\partial_t u, \qquad u_h=\partial_h u, \qquad p_h=\partial_h p, \qquad h=1,2,3, \tag{3.25}$$

and the vector U with sixty components of the form

$$U=(u,u_1,u_2,u_3,p,p_1,p_2,p_3,v,v_1,v_2,v_3), \tag{3.26}$$

we can convert the initial value problem (1.1)–(1.3) to an equivalent initial value problem for a quasilinear hyperbolic system of first order:

$$A^0(x,U)\partial_t U=A^j\partial_j U+B(x,U)+F, \tag{3.27}$$

with initial condition

$$U(0,x)=U^0(x), \tag{3.28}$$

where

(3.29)

$$A^0(x,\partial u)=\begin{bmatrix} E & 0 & 0 & 0 & 0 & 0\\ 0 & A^{-1}(x,\partial u) & 0 & 0 & 0 & 0\\ 0 & 0 & A^{-1}(x,\partial u) & 0 & 0 & 0\\ 0 & 0 & 0 & A^{-1}(x,\partial u) & 0 & 0\\ 0 & 0 & 0 & 0 & A^{-1}(x,\partial u) & 0\\ 0 & 0 & 0 & 0 & 0 & E \end{bmatrix},$$

(3.30)
$$A^j\partial_j=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&D&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&D&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&D&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&D\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&D^*&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&D^*&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&D^*&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&D^*&0&0&0&0&0&0&0&0
\end{bmatrix},$$

(3.31) $B(x,U)=$
$$\begin{bmatrix}
0&0&0&0&E&0&0&0\\
0&0&0&0&0&E&0&0\\
0&0&0&0&0&0&E&0\\
0&0&0&0&0&0&0&E\\
0&0&0&0&0&0&0&0\\
0&\begin{matrix}A^{-1}(a_t^1-A_t\\ \times A^{-1}a^1)\end{matrix}&0&A^{-1}A_tA^{-1}&0&0&0&\begin{matrix}A^{-1}(a^1\\ +\kappa_1-\lambda_1)\end{matrix}\\
0&\begin{matrix}A^{-1}(a_t^2-A_t\\ \times A^{-1}a^2)\end{matrix}&0&0&A^{-1}A_tA^{-1}&0&0&\begin{matrix}A^{-1}(a^2\\ +\kappa_2-\lambda_2)\end{matrix}\\
0&\begin{matrix}A^{-1}(a_t^3-A_t\\ \times A^{-1}a^3)\end{matrix}&0&0&0&A^{-1}A_tA^{-1}&0&\begin{matrix}A^{-1}(a^3\\ +\kappa_3-\lambda_3)\end{matrix}\\
0&0&0&0&0&0&0&0
\end{bmatrix},$$

(3.32) $$F=(0,f,f_1,f_2,f_3)^*,\qquad f_h=\partial_h f,\qquad h=1,2,3.$$

In (3.29)–(3.31), 0 denotes the null matrix and E the unit matrix, and D denotes the so-called generalized gradient of the form (3.12).

The matrices A_t, a_t^h, κ_k, λ_k, $k=1,2,3$, appearing in (3.29)–(3.32) have the following form:

(3.33) $$A_t(x,\partial u)=\partial_t(x,\partial u),$$

(3.34) $$a_t^h(x,\partial u)=\partial_t a^h(x,\partial u),$$

where

(3.35) $$A(x,\partial u)=b(x,\partial u)+c(x,\partial u)+d(x,\partial u)$$

and

$$\kappa_n = (\kappa^h_{jkmn}(x, \partial u)), \quad j, k = 1, 2, 3, 4;\ m, n, h = 1, 2, 3; \tag{3.36}$$

with

$$\kappa^h_{jkmn} = \frac{\partial^3 \sigma}{\partial x_h \partial e_{jn} \partial e_{mn}} + \frac{\partial^3 \sigma}{\partial x_h \partial e_{jk} \partial e_{sn}}\, \partial_s u_n, \tag{3.37}$$

$$\lambda_k = (\lambda_{jk\alpha\beta}(x, \partial u)), \quad j, k, \alpha, \beta, h = 1, 2, 3, \tag{3.38}$$

$$\lambda^h_{jk\alpha\beta} = \lambda^{jkmn}_{\alpha\beta} \eta^h_{mn} \tag{3.39}$$

with $(\eta^h_{mn} : m, n = 1, 2, 3) \equiv A^{-1} E^h(x, \partial u)$ and

$$\begin{aligned}
\lambda^{jkmn}_{\alpha\beta} &= \frac{\partial^3 \sigma}{\partial e_{jk} \partial e_{mn} \partial e_{\alpha\beta}} + \frac{\partial^3 \sigma}{\partial e_{jk} \partial e_{mn} \partial e_{\gamma\beta}}\, \partial_\gamma u_\alpha \\
&+ \delta_{jm} \Big(\frac{\partial^2 \sigma}{\partial e_{kn} \partial e_{\alpha\beta}} + \frac{\partial^2 \sigma}{\partial e_{kn} \partial e_{\gamma\beta}} \partial_\gamma u_\alpha \Big) \\
&+ \frac{\partial^3 \sigma}{\partial e_{jk} \partial e_{sn} \partial e_{\alpha\beta}}\, \partial_s u_m + \frac{\partial^3 \sigma}{\partial e_{jk} \partial e_{sn} \partial e_{\gamma\beta}}\, \partial_s u_m \partial_\gamma u_\alpha \\
&+ \delta_{m\alpha} \frac{\partial^2 \sigma}{\partial e_{jk} \partial e_{\beta n}} + \frac{\partial^3 \sigma}{\partial e_{ks} \partial e_{mn} \partial e_{\alpha\beta}}\, \partial_s u_j \\
&+ \frac{\partial^3 \sigma}{\partial e_{ks} \partial e_{mn} \partial e_{\gamma\beta}}\, \partial_s u_j \partial_\gamma u_\alpha + \delta_{j\alpha}\, \frac{\partial^2 \sigma}{\partial e_{\beta k} \partial e_{mn}} \\
&+ \frac{\partial^3 \sigma}{\partial e_{kl} \partial e_{sn} \partial e_{\alpha\beta}}\, (\partial_l u_j)(\partial_s u_m) \\
&+ \frac{\partial^3 \sigma}{\partial e_{kl} \partial e_{sn} \partial e_{\gamma\beta}}\, (\partial_l u_j)(\partial_s u_m)(\partial_\gamma u_\alpha) \\
&+ \delta_{j\alpha}\, \frac{\partial^2 \sigma}{\partial e_{\beta k} \partial e_{sn}}\, \partial_s u_m + \delta_{m\alpha}\, \frac{\partial^2 \sigma}{\partial e_{kl} \partial e_{\beta n}}\, \partial_l u_j.
\end{aligned} \tag{3.40}$$

The initial data U^0 has the form

$$U^0 = (u^0, u^0_1, u^0_2, u^0_3, p^0, p^0_1, p^0_2, p^0_3, u^1, u^1_1, u^1_2, u^1_3),$$

where

$$p^0 = a(x, \partial u^0) D u^0 + E(x, \partial u^0), \tag{3.41}$$

$$p^0_h = A(x, \partial u^0) D u^0_h + a_h(x, \partial u^0) D u^0 + E_h(x, \partial u^0). \tag{3.42}$$

So, from (3.25)–(3.30) it follows that the matrix $A^0(x, U)$ is symmetric and positive definite and that the matrices A^1, A^2, A^3 are also symmetric. Hence (3.27)–(3.28) is a quasilinear symmetric hyperbolic system of first order.

Applying Egorov's approach (cf. [5, p. 320, 323]) to the initial value problem (3.25)–(3.28) and next using Klainerman's theorem (cf. [16, p. 95, Th. 1]) and Majda's theorem (cf. [20, p. 30, Th. 3.1]), we get Theorem 3.1.

4. Blow-up of the solution to the initial value problem (2.3)–(2.7). In this section we prove that the solution in the form of plane wave, i.e. of the initial value problem (2.3)–(2.7), *blows up in finite time* under some assumption about the stored energy function $\sigma(x, e)$. We start with the formulation of the blow-up theorem.

THEOREM 4.1. *If the stored energy function $\sigma(x, e)$ has the following nonlinear form:*

$$\sigma(x, e) = 2(e_{13} \cdot e_{21})^2 + 8(e_{12} \cdot e_{13} \cdot e_{31})^2 + e_{11} \tag{4.1}$$

and if the initial data u^0, u^1 have compact support and are nonvanishing and sufficiently small, i.e.

$$\sup_{x_1 \in \mathbb{R}} |\partial_1(\partial_1 u_1^0, \partial_1 u_2^0, \partial_1 u_3^0, u_1^1, u_2^1, u_3^1)(x_1)| < \delta,$$
$$(or \sup_{x_1 \in \mathbb{R}} |\partial_1^2(\partial_1 u_1^0, \partial_1 u_2^0, \partial_1 u_3^0, u_1^1, u_2^1, u_3^1)(x_1)| < \delta),$$

where δ is sufficiently small, then the plane wave solution of the initial value problem (1.1)–(1.3) *in $\mathbb{R}^3$ cannot be of class C^2 (resp. C^3) for all positive t.*

Proof. We are looking for a solution of the problem (1.1)–(1.3) under the assumption (4.1) in the form of plane waves:

$$u(t, x) = U(t, \tau x), \qquad \tau \in \mathbb{R}^3 \text{ fixed.} \tag{4.2}$$

For simplicity we choose

$$\tau = [1,\, 0,\, 0]^*, \tag{4.3}$$

that is, U becomes a function of x_1 and t only. We may decompose u into

$$u = (0, u_2, u_3)^* + (u_1, 0, 0)^* = \overline{U}^s + \overline{U}^p, \tag{4.4}$$

where $\overline{U}^s, \overline{U}^p$ satisfy

$$\operatorname{div} \overline{U}^s = 0, \qquad \operatorname{rot} \overline{U}^p = 0. \tag{4.5}$$

In this sense it is a decomposition into divergence-free, respectively curl-free parts. In view of $(3.6)_2$ we get (since the derivatives with respect to x_2 and x_3 of the function u vanish)

$$\partial_1 p_{j1} = (b_{j1m1} + c_{j1m1} + d_{j1m1})\partial_1\partial_1 u_m. \tag{4.6}$$

Then for $k = 1$, $n = 1$, b_{j1m1} has the form

$$b_{j1m1} = \frac{\partial^2 \sigma}{\partial e_{j1} \partial e_{m1}} + \delta_{jm} \frac{\partial \sigma}{\partial e_{11}}, \tag{4.7}$$

and

$$\begin{aligned} b_{1111} &= 1, & b_{2121} &= 4(e_{13})^2 + 1, \\ b_{3131} &= 16(e_{12}e_{13})^2 + 1, & b_{j1m1} &= 0 \quad \text{for } j \neq m. \end{aligned} \tag{4.8}$$

For $m = 1$, c_{j1m1} has the form

$$c_{j1m1} = \frac{\partial^2 \sigma}{\partial e_{j1} \partial e_{11}} \partial_1 u_m + \frac{\partial \sigma}{\partial e_{11} \partial e_{m1}} \partial_1 u_j. \tag{4.9}$$

Then $c_{j1m1} = 0$ for $j, m = 1, 2, 3$. Taking into account (4.1) we get

$$d_{j1m1} = \frac{\partial^2 \sigma}{\partial e_{11} \partial e_{11}} \partial_1 u_j \partial_1 u_m \quad \text{for } m = 1,\ k = 1, \tag{4.10}$$

so

$$d_{j1m1} = 0 \quad \text{for } j, m = 1, 2, 3 \tag{4.11}$$

and finally

$$a^k_{j1m1} = 0. \tag{4.12}$$

Taking into account (4.6)–(4.12) we can write the system (2.3), (2.4) as follows:

$$\partial_t^2 u - A(e_{12}, e_{13}) \partial_1^2 u = 0, \tag{4.13}$$

where $u = [u_1, u_2, u_3]^*$ and

$$A(e_{12}, e_{13}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4(e_{13})^2 + 1 & 0 \\ 0 & 0 & 16(e_{12})^2 (e_{13})^2 + 1 \end{bmatrix}, \tag{4.14}$$

or in the form

$$\partial_t^2 u - A(\partial_1 u_2, \partial_1 u_3) \partial_1^2 u = 0, \tag{4.15}$$

where the matrix $A(\partial_1 u_2, \partial_1 u_3)$ has the form

$$A(\partial_1 u_2, \partial_1 u_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\partial_1 u_3)^2 + 1 & 0 \\ 0 & 0 & (\partial_1 u_2)^2 (\partial_1 u_3)^2 + 1 \end{bmatrix}. \tag{4.16}$$

The system (4.15) can be written in the form

$$\partial_t^2 u_1 - \partial_1^2 u_1 = 0, \tag{4.17}$$

$$\partial_t^2 \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} (\partial_1 u_3)^2 + 1 & 0 \\ 0 & (\partial_1 u_2)^2 (\partial_1 u_3)^2 + 1 \end{bmatrix} \partial_1^2 \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = 0. \tag{4.18}$$

With the system (4.17)–(4.18) we associate the initial data from (4.4),

$$u(0, x) = u^0(x), \tag{4.19}$$

$$\partial_t u(0, x) = u^1(x). \tag{4.20}$$

Now, we consider the quasilinear system (4.18) (equation (4.17) is a wave equation for u_1). Introducing the vector

$$V = [U_2, U_3, V_2, V_3], \tag{4.21}$$

where

$$U_2 = \partial_t u_2, \quad U_3 = \partial_t u_3, \quad V_2 = \partial_1 u_2, \quad V_3 = \partial_1 u_3, \tag{4.22}$$

we can write the system (4.18) as follows:

$$\partial_t V - B(V_2, V_3)\partial_1 V = 0, \tag{4.23}$$

with the initial conditions

$$V(0,x) = [u_2^1(x_1), u_3^1(x_1), \partial_1 u_2^0(x_1), \partial_1 u_3^0(x_1)], \tag{4.24}$$

where

$$B(V_2, V_3) = \begin{bmatrix} 0 & 0 & V_3^2+1 & 0 \\ 0 & 0 & 0 & V_3^2V_2^2+1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4.25}$$

It is easy to see that under the assumption

$$V_3^2(V_2^2 - 1) > 0, \tag{4.26}$$

the matrix $B(V_2, V_3)$ has four real eigenvalues

$$\begin{aligned} \lambda_1 &= \sqrt{V_3^2V_2^2+1}, & \lambda_2 &= \sqrt{V_3^2+1}, \\ \lambda_3 &= -\sqrt{V_3^2V_2^2+1}, & \lambda_4 &= -\sqrt{V_3^2+1}. \end{aligned} \tag{4.27}$$

So, the system (4.23) is *strictly hyperbolic.*

We have four right eigenvectors r_1, r_2, r_3, r_4 corresponding to the eigenvectors λ_1, λ_2, λ_3, λ_4,

$$\begin{aligned} r_1 &= [0,\ \sqrt{V_3^2V_2^2+1},\ 0,\ 1], & r_2 &= [\sqrt{V_3^2+1},\ 0,\ 1,\ 0], \\ r_3 &= [0,\ -\sqrt{V_3^2V_2^2+1},\ 0,\ 1], & r_4 &= [-\sqrt{V_3^2+1},\ 0,\ 1,\ 0]. \end{aligned} \tag{4.28}$$

From (4.27) and (4.28) we get

$$\nabla_V \lambda_1 \cdot r_1 = \frac{2V_2^2V_3}{\sqrt{V_3^2V_2^2+1}}, \tag{4.29}$$

$$\nabla_V \lambda_3 \cdot r_3 = \frac{-2V_2^2V_3}{\sqrt{V_3^2V_2^2+1}}, \tag{4.30}$$

$$\nabla_V \lambda_2 \cdot r_2 = 0 \quad \text{and} \quad \nabla_V \lambda_4 \cdot r_4 = 0. \tag{4.31}$$

If $V_2 \neq 0$ and $V_3 \neq 0$, then from (4.29) and (4.30) it follows that

$$\nabla_V \lambda_1 \cdot r_1 \neq 0 \quad \text{and} \quad \nabla_V \lambda_3 \cdot r_3 \neq 0. \tag{4.32}$$

This means that (4.15) is a *genuinely nonlinear strictly hyperbolic system* (cf. [5]).

Hence, from the general results of T.-P. Liu [19] (or F. John [12] or A. Majda [20]) it follows that for compactly supported (in x_1) nonvanishing smooth data (4.24) which are sufficiently small a plane wave solution (4.2) cannot be of class C^2 (respectively C^3) for all positive time. This means that the solution of the problem (2.3)–(2.7) under the assumption (4.1) blows up in finite time. ∎

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